EXPLODING DOTS
CHAPTER 3

ADDITION AND MULTIPLICATION

Society loves working in base ten. So let's stay with a $1 \leftarrow 10$ machine for a while and make good sense of all the arithmetic we typically learn in school.

We have just seen how to write numbers. Next students usually learn how to add them.

Let's start by exploring addition, and then go on from there.

ADDITION

Here's an addition problem: Compute $251+124$. Such a problem is usually set up this way.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline \end{array}$$

This addition problem is easy to compute: $2+1$ is 3, $5+2$ is 7, and $1+4$ is 5. The answer 375 appears.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline 375 \end{array}$$

But did you notice something curious just then? I worked from left to right just as I was taught to read. This is usually opposite to what most people are taught to do in a mathematics class: go right to left. Even though we went the opposite direction, our answer 375 is correct. (Check: Do you get the same answer if we work the other way?)

So why are we taught to work right to left in mathematics classes?

Many people suggest that the problem we just did is "too nice." We should do a more awkward addition problem, one like $358+287$.

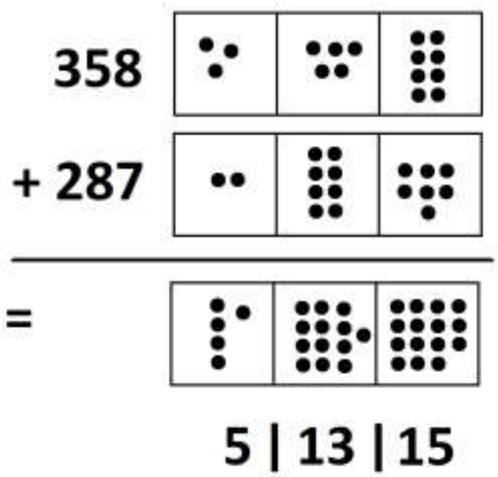
$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

Okay. Let’s do it!

If we go from left to right again we get $3+2$ is 5; $5+8$ is 13; and $8+7$ is 15. The answer “five-hundred thirteenty fifteen” appears. (Remember, “ty” is short for *ten*.)

$$\begin{array}{r} 358 \\ +\,287 \\ \hline 5\,|\,13\,|\,15 \end{array}$$

And this answer is absolutely mathematically correct! You can see it is in a $1 \leftarrow 10$ machine. Here are 358 and 287.



Adding 3 hundreds and 2 hundreds really does give 5 hundreds.
Adding 5 tens and 8 tens really does give 13 tens.
Adding 8 ones and 7 ones really does give 15 ones.

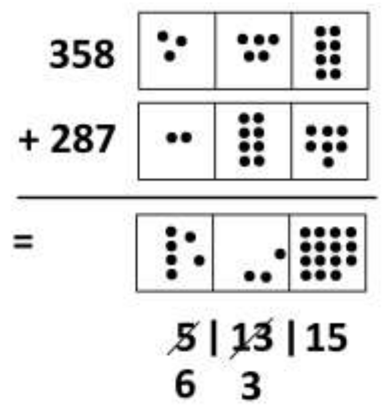
“Five-hundred thirteenty fifteen” is absolutely correct as an answer – and I even said it correctly. We really do have 5 hundreds, 13 tens, and 15 ones. There is nothing mathematically wrong with this answer. It just sounds weird. Society prefers us not to say numbers this way.

So the question is now:

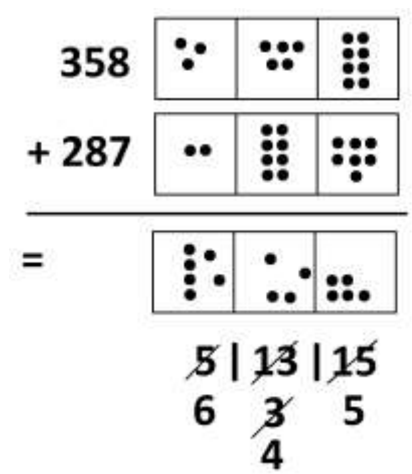
Can we fix up this answer for society’s sake – not mathematics’ sake – just for society’s sake?

The answer is yes! We can do some explosions. (This is a $1 \leftarrow 10$ machine, after all.)

Let’s explode ten dots in the middle box and replace them with one dot, one place to the left.



The answer “six hundred three-ty fifteen” now appears. This is still a lovely, mathematically correct answer. But society at large might not agree. Let’s do another explosion: ten dots in the rightmost box.



Now we see the answer “six hundred four-ty five,” which is one that society understands. (Although, in English, “four-ty” is usually spelled *forty*.)

Here are some practice problems you might or might not want to try. My solutions to them appear at the end of the chapter.

1. Write down the answers to the following addition problems working left to right and not worrying about what society thinks! Then, do some explosions to translate each answer into something society understands.

$$\begin{array}{r} 148 \\ +\ 323 \\ \hline = \end{array} \qquad \begin{array}{r} 567 \\ +\ 271 \\ \hline = \end{array} \qquad \begin{array}{r} 377 \\ +\ 188 \\ \hline = \end{array} \qquad \begin{array}{r} 582 \\ +\ 714 \\ \hline = \end{array}$$

$$\begin{array}{r} 310462872 \\ +\ 389107123 \\ \hline = \end{array} \qquad \begin{array}{r} 87263716381 \\ +\ 18778274824 \\ \hline = \end{array}$$

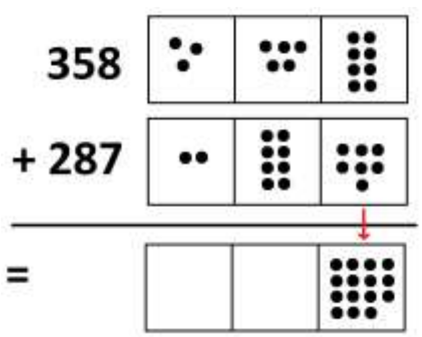
THE TRADITIONAL ALGORITHM

How does this dots-and-boxes approach to addition compare to the standard algorithm most people know?

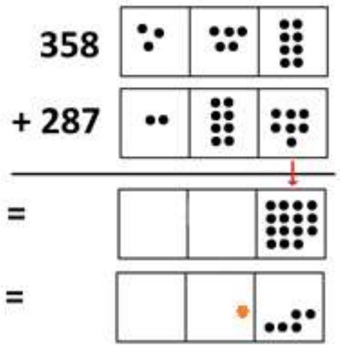
Let's go back to the example $358+287$. Most people are surprised (maybe even perturbed) by the straightforward left-to-right answer $5\,|\,13\,|\,15$.

$$\begin{array}{r} 358 \\ +\ 287 \\ \hline 5\,|\,13\,|\,15 \end{array}$$

This is because the traditional algorithm has us work from right to left, looking at $8+7$ first.

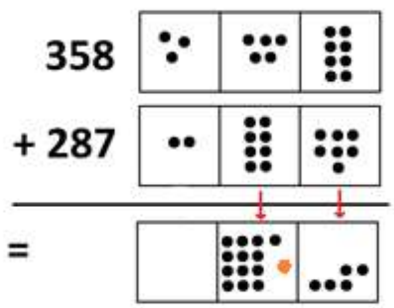


But in the algorithm we don't write down the answer 15. Instead, we explode ten dots right away and write on paper a 5 in the answer line together with a small 1 tacked on to the middle column. People call this *carrying the one* and it – correctly – corresponds to adding an extra dot in the tens position.

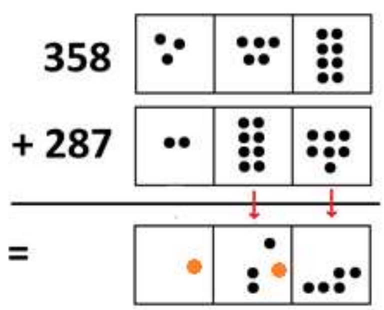


$$\begin{array}{r} {}^{1} \\ 358 \\ +\,287 \\ \hline 5 \end{array}$$

Now we attend to the middle boxes. Adding gives 14 dots in the tens box ($5+8$ gives thirteen dots, plus the extra dot from the previous explosion).



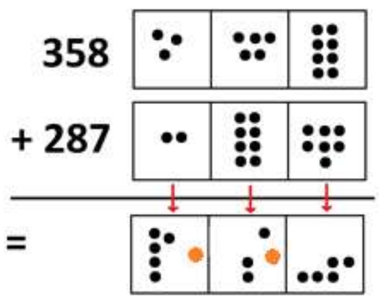
And we perform another explosion.



$$\begin{array}{r} {}^{1}\;{}^{1} \\ 358 \\ +\,287 \\ \hline 45 \end{array}$$

On paper, one writes "4" in the tens position of the answer line, with another little "1" placed in the next column over. This matches the idea of the dots-and-boxes picture precisely.

And now we finish the problem by adding the dots in the hundreds position.



$$\begin{array}{r} {}^{1}\;{}^{1}\; \\ 358 \\ +\,287 \\ \hline 645 \end{array}$$

So the traditional algorithm works right to left and does explosions ("carries") as one goes along. On paper it is swift and compact and this might be why it has been the favored way of doing long addition for centuries.

The Exploding Dots approach works left to right, just as we are taught to read in English, and leaves all the explosions to the end. It is easy to understand and kind of fun.

Both approaches, of course, are good and correct. It is just a matter of taste and personal style which one you choose to do. (And feel free to come up with your own new, and correct, approach too!)

MULTIPLICATION

Let's keep playing with the $1 \leftarrow 10$ machine. And let's do a multiplication problem … right now!

You've got less than three seconds to write down an absolutely correct speedy answer to this multiplication problem. What's a good answer?

26417 x 3

Can you see that $6\,|\,18\,|\,12\,|\,3\,|\,21$, that is, "six ten thousand, eighteen thousand, twelve hundred and threety twenty-one," is correct and does the speedy trick?

Here's what's going on.

Let's start with a picture of 26417 in a $1 \leftarrow 10$ machine. (Is it okay if I just write numbers rather than draw dots?)

2	6	4	1	7

We're being asked to triple this number.

2	6	4	1	7

x 3

Right now we have 2 ten-thousands. If we triple this, we'd have 6 ten-thousands.
Right now we have 6 thousands, and tripling would make this 18 thousands.
Also, 4 hundreds becomes 12 hundreds; 1 ten becomes 3 tens; and 7 ones becomes 21 ones.

6	18	12	3	21

We see the answer "sixty eighteen thousand, twelve hundred and threety twenty-one." Absolutely solid and mathematically correct!

Now, how can we fix up this answer for society?

Do some explosions of course!

We can explode in any order we like it seems. Can you follow this chain of events?

$$6\,|\,18\,|\,12\,|\,3\,|\,21 \;=\; 6\,|\,19\,|\,2\,|\,3\,|\,21 \;=\; 6\,|\,19\,|\,2\,|\,5\,|\,1 \;=\; 7\,|\,9\,|\,2\,|\,5\,|\,1$$

The answer 79251 appears.

2. Compute each of the following: 26417×4, 26417×5, and 26417×9.

Compute 26417×10 and explain why the answer has to be 264170. (This answer looks like the original number with the digit zero tacked on to its end.)

Extra: Care to compute 26417×11 and 26417×12 too? (The answer could be "No! I do not care to do this!)

MULTIPLYING BY TEN

Actually, let's answer one of the practice questions here. *Why must the answer to* 26417×10 *look like the original number with a zero tacked on to its end?*

I remember being taught this rule in school: to multiply by ten tack on a zero. For example,

$$37 \times 10 = 370$$
$$98989 \times 10 = 989890$$
$$100000 \times 10 = 1000000$$

and so on.

This observation makes perfect sense in the dots-and-boxes thinking.

Here's the number 26417 again in a $1 \leftarrow 10$ machine.

2	6	4	1	7

Here's 26417×10.

20	60	40	10	70

Now let's perform the explosions, one at a time. (We'll need an extra box to the left.)

We have that 2 groups of ten explode to give 2 dots one place to the left, and 6 groups of ten explode to give 6 dots one place to the left, and 4 groups of ten explode to give 4 dots one place to the left, and so on. The digits we work with stay the same. In fact, the net effect of what we see is all digits shifting one place to the left to leave zero dots in the ones place.

	20	60	40	10	70

2	0	60	40	10	70

2	6	0	40	10	70

2	6	4	0	10	70

2	6	4	1	0	70

2	6	4	1	7	0

Indeed it looks like we just tacked on a zero to the right end of 26417. (But this is really because of a whole lot of explosions.)

3. a) What must be the answer to 476×10 ? To 476×100 ?

b) What must be the answer to $9190 \div 10$? To $3310000 \div 100$?

OPTIONAL: LONG MULTIPLICATION

Is it possible to do, say, 37×23, with dots and boxes?

Here we are being asked to multiply three tens by 23 and seven ones by 23. If you are good with your multiples of 23, this must give $3 \times 23 = 69$ tens and $7 \times 23 = 161$ ones. The answer is thus $69 | 161$. With explosions this becomes 851.

But this approach seems hard! It requires you to know multiples of 23 .

Thinking Exercise:

Suzzy thought about 37×23 for a little while, she eventually drew the following diagram.

	6	14	0
+		9	21
=	6	23	21

She then said that $37 \times 23 = 6 | 23 | 21 = 8 | 3 | 21 = 851$.

a) Can you work out what Suzzy was thinking?

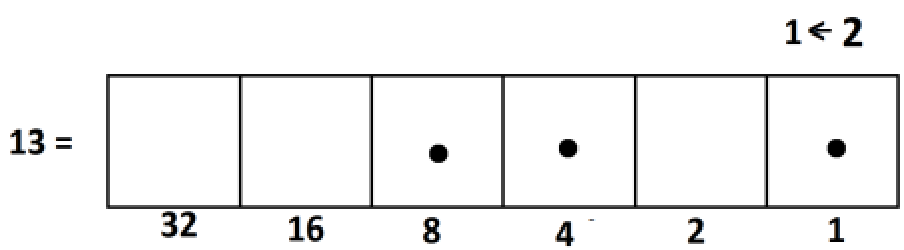
b) What diagram do you think Suzzy might draw for 236×34 (and what answer will she get from it)?

c) Using Suzzy's approach do 37×23 and 23×37 give the same answer? Is it obvious as you go through the process that they will? Do 236×34 and 34×236 give the same answer in Suzzy's approach?

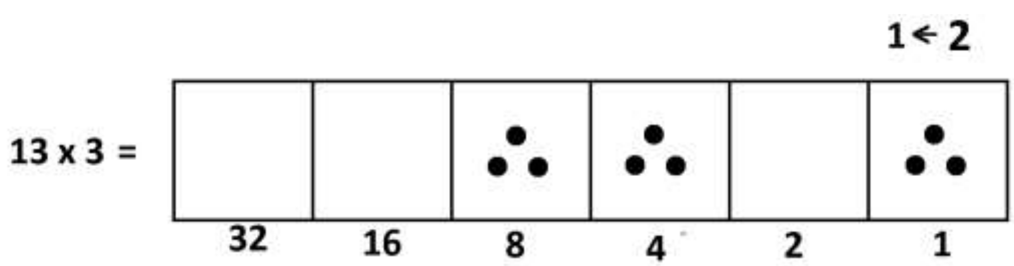
Here's another fun way to think about multiplication. Let's work with a $1 \leftarrow 2$ machine this time.

Let's compute 13×3.

Here's what 13 looks like in a $1 \leftarrow 2$ machine.

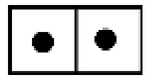


We're being asked to triple everything. So each dot we see is to be replaced by three dots.



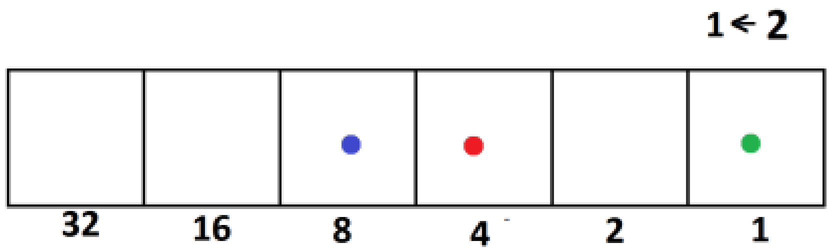
And now we can do some explosions to see the answer 39 appear (which is 100111 in the $1 \leftarrow 2$ machine).

Alternatively, we can notice that three dots in a $1 \leftarrow 2$ machine actually look like this.

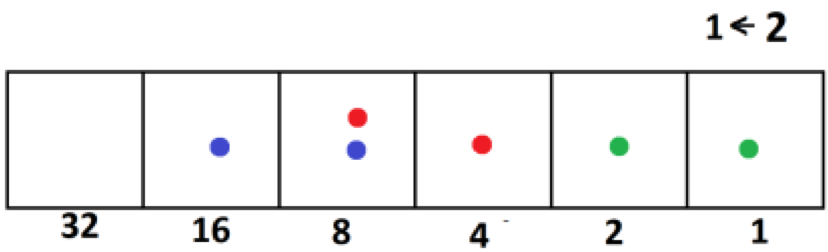


So we can replace each dot in our picture of 13 instead by one dot and a second dot one place to the left. (I've added some color to the picture to help.)

13 =



13 x 3 =



Now with less explosions to do, we see the answer 100111 appear.

WILDS EXPLORATIONS

Here are some "big question" investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: THERE IS NOTHING SPECIAL ABOUT BASE TEN FOR ADDITION

Here is an addition problem in a $1 \leftarrow 5$ machine. (That is, it is a problem in base five.) This is not a $1 \leftarrow 10$ machine addition.

$$\begin{array}{r} 20413 \\ +\ 13244 \\ \hline \end{array}$$

a) What is the $1 \leftarrow 5$ machine answer?

b) What number has code 20413 in a $1 \leftarrow 5$ machine? What number has code 13244 in a $1 \leftarrow 5$ machine? What is the sum of those two numbers and what is the code for that sum in a $1 \leftarrow 5$ machine?

[Here are the answers so that you can check your clever thinking.

The sum, as a $1 \leftarrow 5$ machine problem, is

$$20413 + 13244 = 3|3|6|5|7 = 3|4|1|5|7 = 3|4|2|0|7 = 3|4|2|1|2 = 34212.$$

In a $1 \leftarrow 5$ machine, 20413 is two 625s, four 25s, one 5, and three 1s, and so is the number 1358 in base ten; 13244 is the number 1074 in base ten; and 34212 is the number 2432 in base ten. We have just worked out $1358 + 1074 = 2432$.]

EXPLORATION 2: THERE IS NOTHING SPECIAL ABOUT BASE TEN FOR MULTIPLICATION

Let's work with a $1 \leftarrow 3$ machine.

a) Find 111×3 as a base three problem. Also, what are 1202×3 and 2002×3? Can you explain what you notice?

Let's now work with a $1 \leftarrow 4$ machine.

b) What is 133×4 as a base four problem? What is 2011×4? What is 22×4? Can you explain what you notice?

In general, if we are working with a $1 \leftarrow b$ machine, can you explain why multiplying a number in base b by b returns the original number with a zero tacked on to its right?

SOLUTIONS

As promised, here are my solutions to the questions posed.

1.

$$148+323=4|6|11=471$$

$$567+271=7|13|8=838$$

$$377+188=4|15|15=5|5|15=565$$

$$582+714=12|9|6=1|2|9|6=1296$$

$$310462872+389107123=6|9|9|5|6|9|9|9|5=699569995$$

$$87263716381+18778274824=9|15|9|13|11|9|8|10|11|10|5$$
$$=\ldots\ =106041991205$$

2.

We have

$$26417\times 4=8|24|16|4|28=10|4|16|4|28=1|0|4|16|4|28=1|0|5|6|4|28=105668$$

$$26417\times 5=10|30|20|5|35=10|30|20|8|5=10|32|0|8|5=13|2|0|8|5=132085$$
$$26417\times 9=18|54|36|9|63=18|54|36|15|3=\ldots=237753$$
$$26417\times 11=22|66|44|11|77=\ldots=290587$$
$$26417\times 12=24|72|48|12|84=\ldots=317004$$

Read on in the chapter to see why 26417×10 is 264170.

3.

a) 476×10 is 4760. Since 476×100 is "476 times ten times ten" the answer will be 47600.

b) We have that 9190 is the answer to 919×10. This means that $9190\div 10$ must be 919.

And 3310000 is the answer to 33100×100, and so $3310000\div 100$ must be 33100.