EXPLODING DOTS
CHAPTER 4

SUBTRACTION

Let's keep working with the $1 \leftarrow 10$ machine.

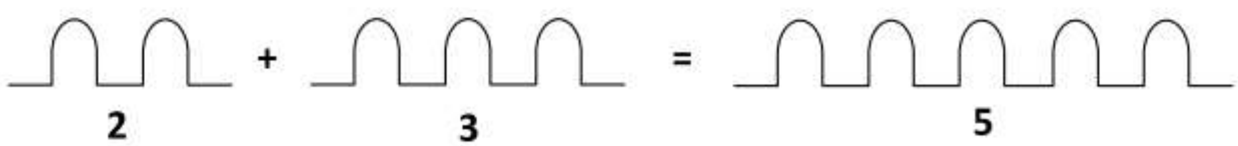
So far we've made sense of addition and multiplication. But we skipped over subtraction. Why? Because I don't believe in subtraction! To me, subtraction is just the addition of the opposite.

Let's explore what I mean by this.

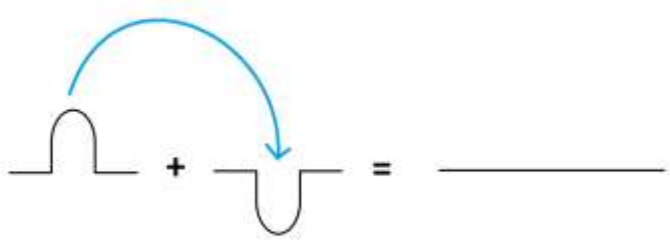
NEGATIVE NUMBERS

My disbelief in subtraction comes from another story that isn't true. Briefly, it goes as follows.

> As a young child I used to regularly play in a sandbox. And there I discovered the positive counting numbers as piles of sand: one pile, two piles, and so on. And I also discovered the addition of positive numbers simply by lining up piles. For example, I saw that two plus three equals five simply by lining up piles like this.

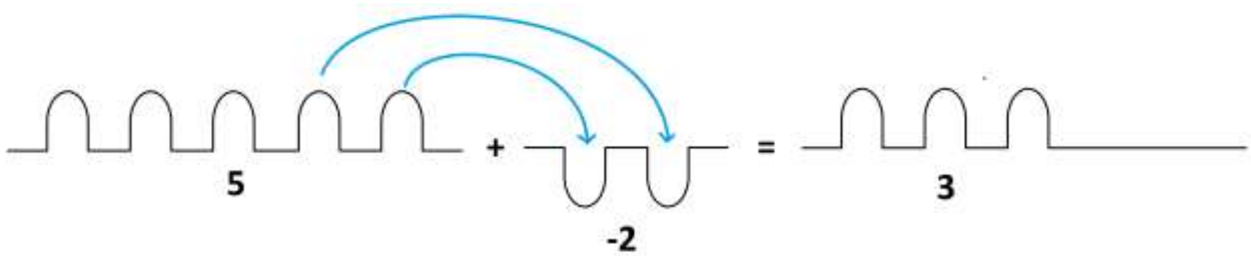


> I had hours of fun counting and lining up piles to explore addition.
>
> But then one day I had an astounding flash on insight! Instead of making piles of sand, I realized I could also make holes. And I saw right away that a hole is the opposite of a pile: place a pile and a hole together and they cancel each other out. Whoa!



> Later in school I was taught to call a hole "-1", and two holes "-2," and so on and was told to do this thing called "subtraction." But I never really believed in subtraction. My colleagues

would read $5-2$, say, as ”five take away two,” but I was thinking of five piles and the addition of two holes. A picture shows that the answer is three piles.



Yes. This gives the same answer as my peers, of course: the two holes “took away” two of the piles. But I had an advantage. For example, my colleagues would say that $7-10$ has no answer. I saw that it did.

$7-10$ = seven piles and ten holes
= three holes
$=-3$

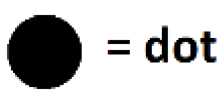
Easy!

Subtraction is just the addition of the opposite.

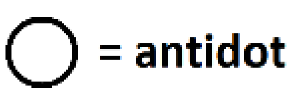
(By the way, I will happily write $7-10$ as “$7 + -10$.” This makes the thinking more obvious.)

Let’s go back now to our dots-and-boxes machines, the $1 \leftarrow 10$ machine in particular.

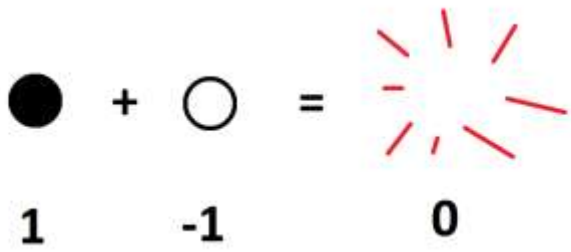
There we work with dots, which I’ve been drawing as solid dots.



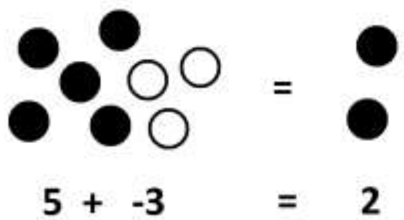
We now need the notion of the opposite of dot, like a hole is the opposite of a pile. I’ll draw the opposite of a dot as a hollow circle and call it an antidot.

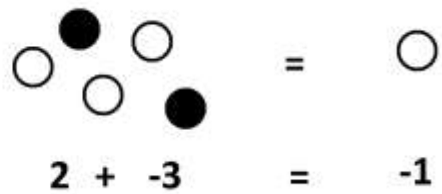


Like matter and antimatter, and like 1 and -1 in piles and holes, which each annihilate one another when brought together, a dot and an antidot should also annihilate too – POOF! – when brought together to leave nothing behind.



And we can conduct basic arithmetic with dots and antidots, just like we did with piles and holes.





Aside: By the way, some students prefer to call the opposite of a dot a *tod*. Can you guess what made them think of that name?

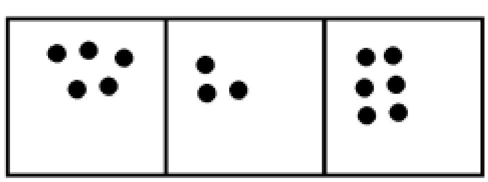
SUBTRACTION

Consider this subtraction problem.

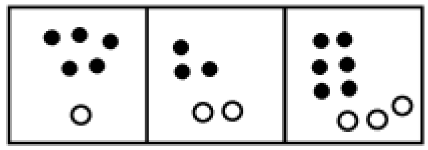
$$\begin{array}{r} 536 \\ -\ \underline{123} \end{array}$$

To me, this is 536 plus the opposite of 123.

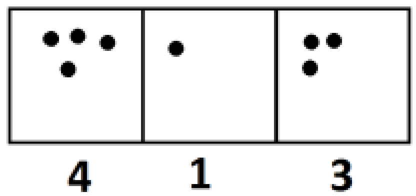
The first number, 536, looks like this in a $1 \leftarrow 10$ machine: five dots, three dots, six dots.



To this we are adding the opposite of 123. That is, we're adding one anti-hundred, two anti-tens, and three anti-ones.



And now there are a lot of annihilations.: POOF!; POOF POOF!; POOF POOF POOF!



We see the answer 413 appear.

And notice, we get this answer as though we just work left to right and say

5 take away 1 is 4,
3 take away 2 is 1,
and
6 take away 3 is 3.

Yes! Left to right again!

$$\begin{array}{r} 536 \\ -\ 123 \\ \hline 413 \end{array}$$

All right. That example was too nice. How about $512-347$?

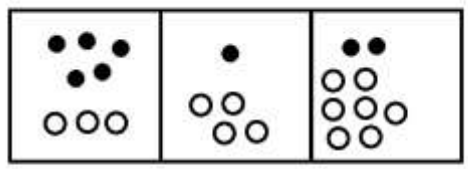
$$\begin{array}{r} 5\ 1\ 2 \\ -3\ 4\ 7 \\ \hline \end{array}$$

Going from left to right, we get: 5 take away 3 is 2, 1 take away 4 is -3, and 2 take away 7 is -5. The answer is two-hundred negative-three-ty negative-five.

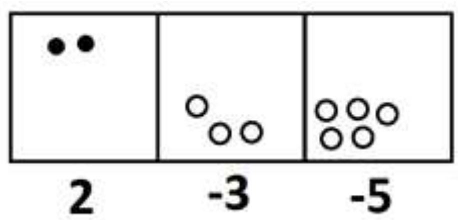
$$\begin{array}{r} 5\ 1\ 2 \\ -3\ 4\ 7 \\ \hline 2|{-3}|{-5} \end{array}$$

And this answer is absolutely mathematically correct! The picture shows it is.

Here’s five hundreds, one ten and two ones together with three anti-hundreds, four anti-tens, and seven anti-ones.



And after lots of annihilations we get are left with two actual hundreds, three anti-tens, and five anti-ones.

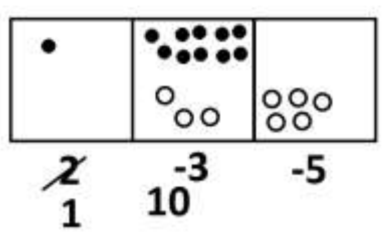


The answer really is two-hundred negative-three-ty negative-five!

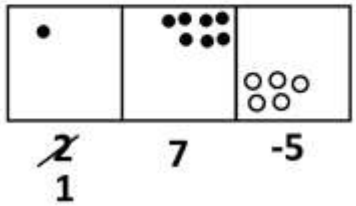
But, of course, saying the answer to our subtraction problem this way seems mighty weird to society. Can we fix up this mathematically correct answer for society’s sake?

It takes a flash of insight to realize that we can “unexplode” dots: any dot in a box to the left must have come from ten dots in the box just to its right.

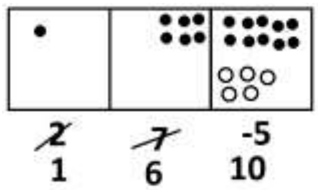
Okay. Let’s unexplode one of the two dots we have in the leftmost box. Doing so gives this picture.



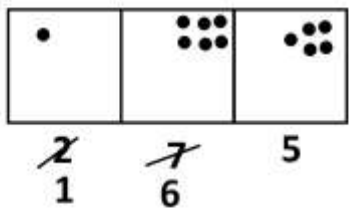
After annihilations, we see we now have the answer one-hundred seventy negative-five. Beautiful!



Let’s unexplode again.



And with some more annihilations we see an answer society can understand: one-hundred sixty five.



THE TRADITIONAL ALGORITHM

How does this dots-and-boxes approach compare with the standard algorithm?

Consider again $512-347$.

$$\begin{array}{r} 5\,1\,2 \\ -3\,4\,7 \\ \hline \end{array}$$

The standard algorithm has you start at the right to first look at “2 take away 7,” which you can’t do. (Well you can do it, it is -5, but you are not to write that for this algorithm.)

So what do you do?

You “borrow one.” That is, you take a dot from the tens column and unexplode it to make ten ones. That leaves zero dots in the tens column. We should write ten ones to go with the two in the ones column.

$$\begin{array}{rccc} & & 0 & 10 \\ & 5 & \cancel{1} & 2 \\ - & 3 & 4 & 7 \\ \hline \end{array}$$

But we are a bit clever here and just write 12 rather than $10+2$. (That is, we put a 1 in front of the 2 to make it look like twelve.)

$$\begin{array}{rccc} & & 0 & \\ & 5 & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline \end{array}$$

Then we say “twelve take seven is five” and write that answer.

$$\begin{array}{rccc} & & 0 & \\ & 5 & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline & & & 5 \end{array}$$

The rightmost column is complete. Shift now to the middle column.

We see “zero take away four,” which can’t be done. So perform another unexplosion, that is, another “borrow,” to see $10-4$ in that column. We write the answer 6.

We then move to the last remaining column where we have $4-3$, which is 1.

$$\begin{array}{rccc} & 4 & {}^{1}0 & \\ & \cancel{5} & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline & & 6 & 5 \end{array} \qquad \begin{array}{rccc} & 4 & {}^{1}0 & \\ & \cancel{5} & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline & 1 & 6 & 5 \end{array}$$

Phew!

Here is a question for you to try, if you like. My answer to it is at the end of this chapter.

1. Compute each of the following two ways: the dots-and-boxes way (and fixing the answer for society to read) and then with the traditional algorithm. The answers should be the same.

$$\begin{array}{r} 6328 \\ -\ 4469 \\ \hline \end{array} \qquad \begin{array}{r} 78390231 \\ -\ 32495846 \\ \hline \end{array}$$

Thinking question along the way: As you fix up your answers for society, does it seem easier to unexplode from left to right, or from right to left?

Additional question: Do you think you could become just as speedy the dots-and-boxes way as you currently are with the traditional approach?

Again. All correct approaches to mathematics are correct, and it is just a matter of style as to which approach you like best for subtraction. The traditional algorithm has you work from right to left and do all the unexplosions as you go along. The dots-and-boxes approach has you "just do it!" and conduct all the unexplosions at the end. Both methods are fine and correct.

WILD EXPLORATIONS

Here are some “big question” investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: IS THERE ANOTHER WAY TO INTERPRET THE DOTS-AND-BOXES ANSWERS?

When Sunil saw

$$\begin{array}{r} 5\ 1\ 2 \\ -3\ 4\ 7 \\ \hline 2 \mid -3 \mid -5 \end{array}$$

he wrote on his paper the following lines:

$$\begin{array}{r} 200 \\ -30 \\ -5 \end{array}$$

He then said that the answer has to be 165.

a) Can you explain what he is seeing and thinking?

b) What would Sunil likely write on the page for $7109-3384$?

EXPLORATION 2: WHAT ABOUT NEGATIVE ANSWERS?

How might you handle and interpret this subtraction problem?

$$\begin{array}{r} 148 \\ -\ 677 \\ \hline \end{array}$$

SOLUTIONS

As promised, here are my solutions to the question posed.

1.

$$6328-4469 = 2\,|-1\,|-4\,|-1 = 1\,|\,9\,|-4\,|-1 = 1\,|\,8\,|\,6\,|-1 = 1\,|\,8\,|\,5\,|\,9 = 1859$$

$$\begin{aligned}78390231-32495846 &= 4\,|\,6\,|-1\,|\,0\,|-5\,|-6\,|-1\,|-5\\ &= 4\,|\,5\,|\,9\,|\,0\,|-5\,|-6\,|-1\,|-5\\ &= 4\,|\,5\,|\,8\,|\,10\,|-5\,|-6\,|-1\,|-5\\ &= 4\,|\,5\,|\,8\,|\,9\,|\,5\,|-6\,|-1\,|-5\\ &= 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,4\,|-1\,|-5\\ &= 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,3\,|\,9\,|-5\\ &= 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,3\,|\,8\,|\,5 \qquad = 45894385\end{aligned}$$

I personally find it much easier to do the unexplosions from left to right.