

EXPLODING DOTS CHAPTER 8

DECIMALS

All our machines, so far, have had boxes going to the left as far as we please. But that seems awfully lopsided! Why can't we have boxes going infinitely far to the right as well?

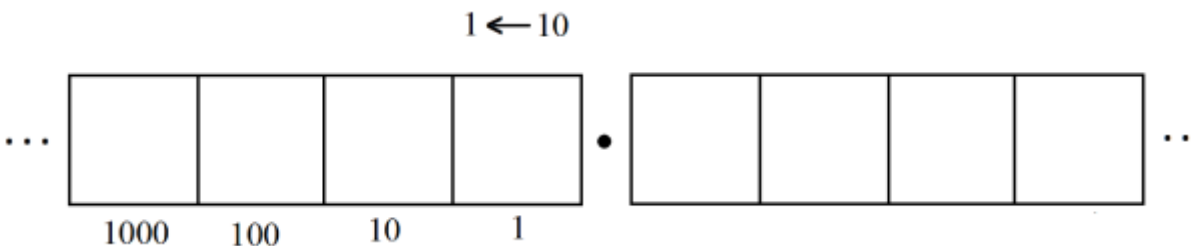
Mathematicians like symmetry and so let's follow suit and now make all our machines symmetrical. Let's have boxes going to the left and to the right.

But the challenge now is to figure out what those boxes to the right mean.

DISCOVERING DECIMALS

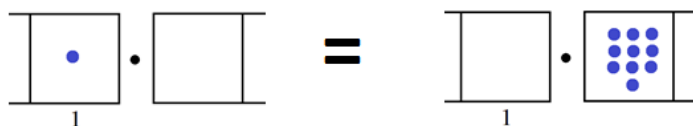
Let's go back to working with a $1 \leftarrow 10$ machine and see what boxes to the right could mean in that machine. (Then it should become clear what they mean in other machines as well.)

To keep the left and right boxes visibly distinct, we'll separate them with a point. (Society calls this point, for a $1 \leftarrow 10$ machine at least, a *decimal point*.)

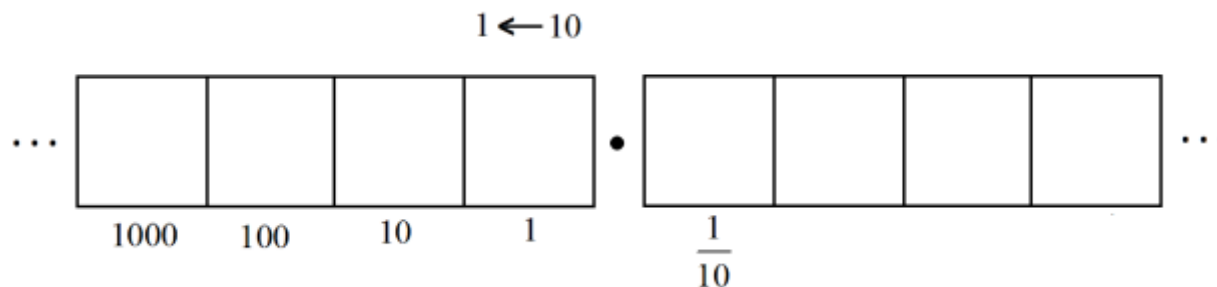


So, what does it mean to have dots in the right boxes? What are the values of dots in those boxes?

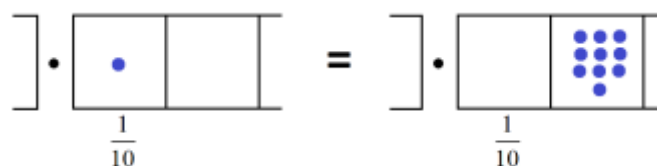
Since this is a $1 \leftarrow 10$ machine, we do know that ten dots in any one box explode to make one dot one place to the left. So ten dots in the box just to the right of the decimal point are equivalent to one dot in the 1s box. Each dot in that box must be worth one-tenth.



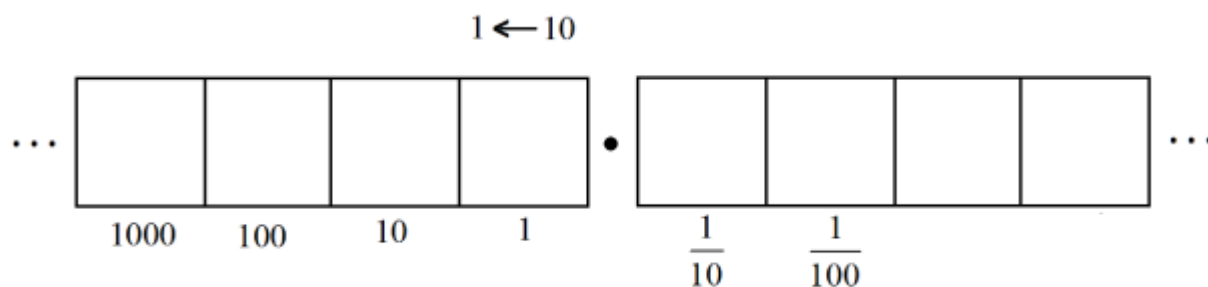
We have our first place value to the right of the decimal point.



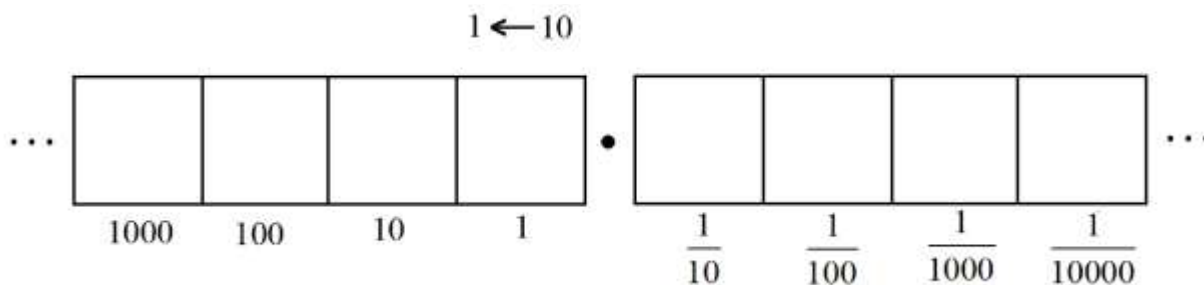
In the same way, ten dots in the next box over are worth one-tenth. And so each dot in that next box must be worth one-hundredth.



We now have two place values to the right of the decimal point.



And ten one-thousandths make a hundredth, and ten ten-thousandths make a thousandth, and so on.

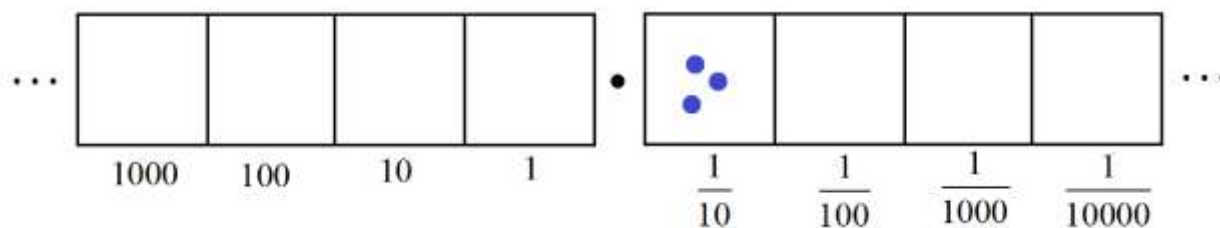


We see that the boxes to the left of the decimal point represent place values as given by the powers of ten, and the boxes to the right of the decimal point represent place values given by the reciprocals of the powers of ten.

We have just discovered decimals!

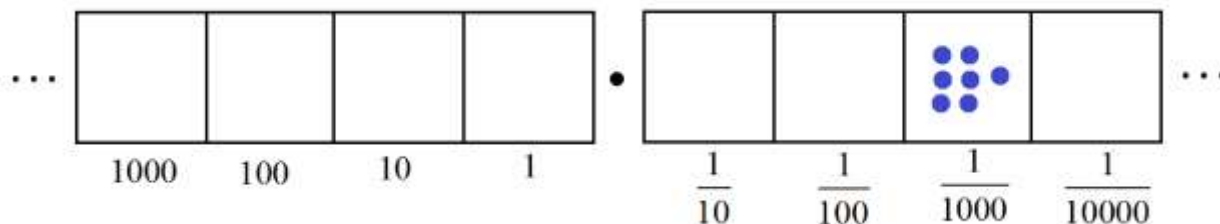
Question: What does the prefix *deci* mean in English? Would we talk about “decimals” in a $1 \leftarrow 2$ machine, for instance? (And why isn’t December the tenth month of the year? What happened in the history of the western calendar?)

When people write 0.3, for example, in base ten, they mean the value of placing three dots in the first box after the decimal point.



We see that 0.3 equals three tenths: $0.3 = \frac{3}{10}$.

Seven dots in the third box after the decimal point is seven thousandths: $0.007 = \frac{7}{1000}$.

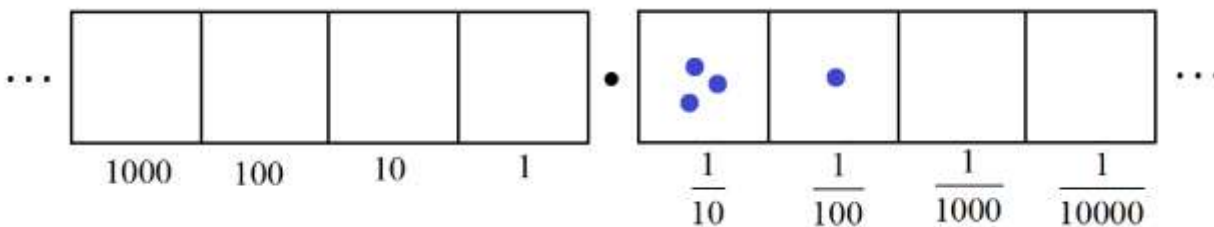


Comment: Some people might leave off the beginning zero and just write $.007 = \frac{7}{1000}$. It's just a matter of personal taste.

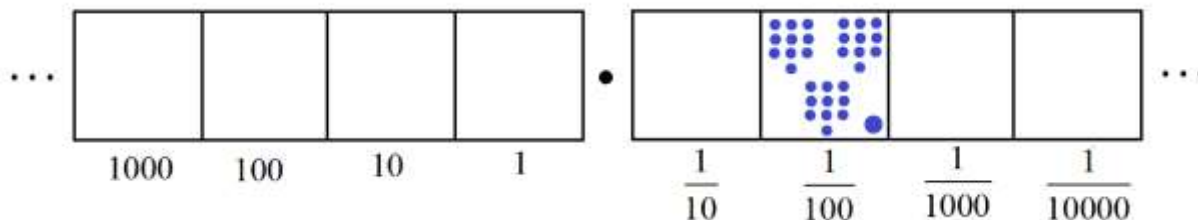
Here's a question. I've included my answer to it, and the answers to all the other questions I sprinkle throughout this chapter, at the end of this lesson.

1. Some people read 0.6 out loud as "point six" and others read it out loud as "six tenths." Which is more helpful for understanding what the number really is?

There is a possible source of confusion with a decimal such as 0.31. This is technically three tenths and one hundredth: $0.31 = \frac{3}{10} + \frac{1}{100}$.



But some people read 0.31 out loud as “thirty-one hundredths,” which looks like this.



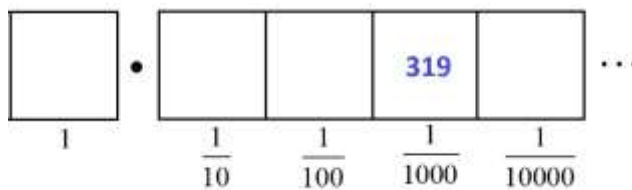
Are these the same thing?

Well, yes! With three explosions we see that thirty-one hundredths becomes three tenths and one hundredth.

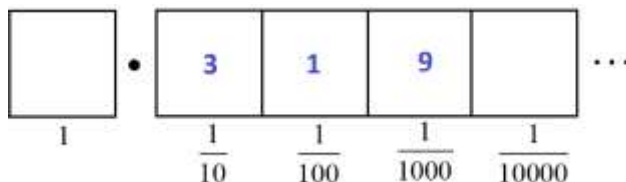
Comment: You can also show that $\frac{3}{10} + \frac{1}{100}$ and $\frac{31}{100}$ are the same with the arithmetic of adding fractions. We have $\frac{3}{10} + \frac{1}{100} = \frac{30}{100} + \frac{1}{100} = \frac{31}{100}$. (Do you see that this is really the result of performing three unexplosions in a picture of $\frac{3}{10} + \frac{1}{100}$?)

2. A teacher asked his students to each draw a $1 \leftarrow 10$ machine picture of the fraction $\frac{319}{1000}$.

JinJin drew:



Subra drew:



The teacher marked both students as correct. Are each of these solutions indeed valid? Explain your thinking. (By the way, the teacher doesn't mind if students just write numbers instead of drawing dots.)

3. Multiple choice!

a) The decimal 0.23 equals: (A) $\frac{23}{10}$ (B) $\frac{23}{100}$ (C) $\frac{23}{1000}$ (D) $\frac{23}{10000}$?

b) The decimal 0.0409 equals: (A) $\frac{409}{100}$ (B) $\frac{409}{1000}$ (C) $\frac{409}{10000}$ (D) $\frac{409}{100000}$?

SIMPLE FRACTIONS AS DECIMALS

Some decimals give fractions that simplify further. For example,

$$0.5 = \frac{5}{10} = \frac{1}{2}$$

and

$$0.04 = \frac{4}{100} = \frac{1}{25}.$$

Conversely, if a fraction can be rewritten to have a denominator that is a power of ten, then we can easily write it as a decimal. For example,

$$\frac{3}{5} = \frac{6}{10} \text{ and so } \frac{3}{5} = 0.6$$

and

$$\frac{13}{20} = \frac{13 \times 5}{20 \times 5} = \frac{65}{100} = 0.65.$$

Here are some more problems. Do whichever ones seem interesting to you.

4. What fractions (in simplest terms) do the following decimals represent?

0.05 0.2 0.8 0.004

5. Write each of the following fractions as a decimal.

$$\frac{2}{5} \quad \frac{1}{25} \quad \frac{1}{20} \quad \frac{1}{200} \quad \frac{2}{2500}$$

6. MULTIPLE CHOICE!

- a) The decimal 0.050 equals (A) $\frac{50}{100}$ (B) $\frac{1}{20}$ (C) $\frac{1}{200}$ (D) None of these?
- b) The decimal 0.000208 equals (A) $\frac{52}{250}$ (B) $\frac{52}{2500}$ (C) $\frac{52}{25000}$ (D) $\frac{52}{250000}$?

7. Write each of the following fractions as decimals.

$$\frac{7}{20} \quad \frac{16}{25} \quad \frac{301}{500} \quad \frac{17}{50} \quad \frac{3}{4}$$

8. CHALLENGE

- a) What fraction does the decimal 2.3 represent?
- b) What fraction does 17.04 represent?
- c) What fraction does 1003.1003 represent?

- 9. Let's explore the question: Do 0.19 and 0.190 represent the same number or different numbers?**

Here are two dots and boxes pictures for the decimal 0.19:

$$0.19 = \begin{array}{|c|c|} \hline & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline 1 & 9 & \\ \hline \end{array}$$

$\frac{1}{10}$ $\frac{1}{100}$ $\frac{1}{1000}$

$$0.19 = \begin{array}{|c|c|} \hline & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline & 19 & \\ \hline \end{array}$$

$\frac{1}{10}$ $\frac{1}{100}$ $\frac{1}{1000}$

Here are two dots and boxes picture for the decimal 0.190

$$0.190 = \begin{array}{|c|c|} \hline & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline 1 & 9 & 0 \\ \hline \end{array}$$

$\begin{array}{cc} 10 & 1 \\ \hline \end{array}$
 $\begin{array}{ccc} \frac{1}{10} & \frac{1}{100} & \frac{1}{1000} \\ \hline \end{array}$

$$0.190 = \begin{array}{|c|c|} \hline & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline & & 190 \\ \hline \end{array}$$

$\begin{array}{cc} 10 & 1 \\ \hline \end{array}$
 $\begin{array}{ccc} \frac{1}{10} & \frac{1}{100} & \frac{1}{1000} \\ \hline \end{array}$

- Explain how one “unexplosion” establishes that the first picture of 0.19 is equivalent to the second picture of 0.19.
- Explain how several unexplosions establishes that the first picture of 0.190 is equivalent to the second picture of 0.190.
- Explain how explosions and unexplosions in fact establish that all four pictures are equivalent to each other.
- In conclusion then: Does 0.190 represent the same number as 0.19?

To a mathematician, the expressions 0.19 and 0.190 represent exactly the same numeric quantity. But you may have noticed in science class that scientists will often write down what seems like unnecessary zeros when recording measurements. This is because scientists want to impart more information to the reader than just a numeric value.

For example, suppose a botanist measures the length of a stalk. By writing the measurement as 0.190 meters in her paper, the scientist is saying to the reader that she measured the length of the stalk to the nearest one thousandth of a meter and that she got 1 tenth, 9 hundredths, and 0 thousandths of a meter. Thus we are being told that the true length of the stalk is somewhere in the range of 0.1895 and 0.1905 meters.

If she wrote in her paper, instead, just 0.19 meters, then we would have to assume she measured the length of the stalk only to the nearest tenth of a meter and so its true length lies somewhere between 0.185 and 0.195 meters.

MIXED NUMBERS AS DECIMALS

How does $12\frac{3}{4}$, for example, appear as a decimal?

Well, $12\frac{3}{4} = 12 + \frac{3}{4}$ and we can certainly write the fractional part as a decimal. (The non-fractional part is already in the $1 \leftarrow 10$ machine format!) We have

$$12\frac{3}{4} = 12 + \frac{75}{100}$$

and so we see

$$12\frac{3}{4} = 12.75.$$

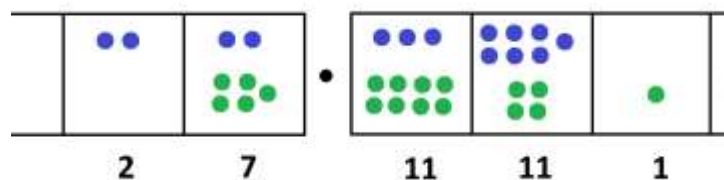
10. Write each of the following numbers in decimal notation.

$$5\frac{3}{10} \quad 7\frac{1}{5} \quad 13\frac{1}{2} \quad 106\frac{3}{20} \quad \frac{78}{25} \quad \frac{9}{4} \quad \frac{131}{40}$$

ADDING AND SUBTRACTING DECIMALS

Performing addition and subtraction with decimals in a $1 \leftarrow 10$ machine is no different from performing addition and subtraction in a $1 \leftarrow 10$ machine without decimals.

For example, here is a picture of $22.37 + 5.841$.

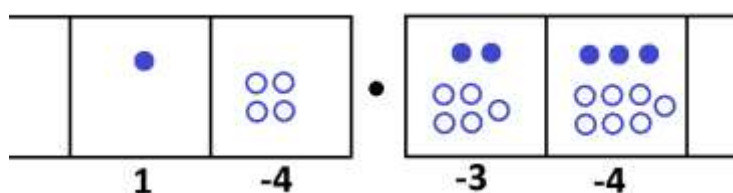


We see the answer $2|7.11|11|1$. Just add left to right and don't worry about explosions!

$$\begin{array}{r}
 22.37 \\
 + 5.841 \\
 \hline
 2|7.11|11|1
 \end{array}$$

With explosions we obtain an answer that society understand. We get 28.211 . (Check this!)

And here is $10.23 - 4.57$.



We see the answer, after some annihilations, as $1|-4.-3|-4$. (Do you see this?)

$$\begin{array}{r}
 10.23 \\
 - 4.57 \\
 \hline
 1|-4.-3|-4
 \end{array}$$

Now with some unexplosions, we can fix up this answer for society to read. We get $10.23 - 4.57 = 5.66$. (Check that you follow this.)

$$\begin{array}{r}
 10.23 \\
 - 4.57 \\
 \hline
 1|-4.-3|-4 \\
 \quad \cancel{6}.-3|-4 \\
 \quad \quad 5.\cancel{7}|-4 \\
 \quad \quad \quad 5.6|6
 \end{array}$$

Here are some practice questions you might, or might now, want to do.

11. What is $0.05 + 0.006$?

What is $0.05 - 0.006$?

Can you see the answer with dots-and-boxes?

12. Agatha says that computing $0.0348 + 0.0057$ is essentially the same work as computing $348 + 57$ in whole numbers. Do you agree?

Percy says that computing $0.0852 + 0.037$ is essentially the same work as computing $852 + 37$ in whole numbers. Do you agree?

MULTIPLYING AND DIVIDING DECIMALS

Even though the addition and subtraction of decimals is straightforward in a $1 \leftarrow 10$ machine, performing multiplication and division on them can be awkward. The trouble is that the $1 \leftarrow 10$ machine is based on whole dots, not parts of dots, and so it is tricky to make easy sense of pictures. But there are ways to play with dots-and-boxes nonetheless.

MULTIPLICATION

We can certainly do some basic multiplication. For example, a picture of dots and boxes shows that

$$2.615 \times 7$$

equals $14.42 \overline{) 7} \overline{) 35}$.

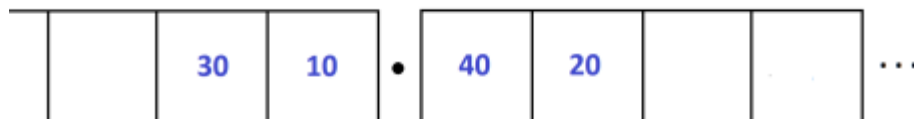


With explosions, this becomes 18.305 . (Check this!)

We can also explain a rule that students are taught to memorize:

If you multiply a decimal by 10, just shift the decimal point one place to the right.

For example, 31.42×10 gives the answer $30 \overline{) 10} . 40 \overline{) 20}$. With explosions, this becomes 314.20 and the answer looks as though we just shifted the decimal point. (Most people leave off the final zero.)



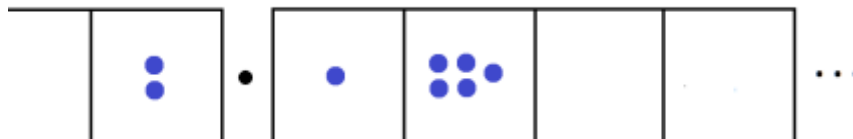
By the way, if you multiply a number a little larger than 31 by ten, you should get an answer a little larger than 310. Memorizing the direction of travel of a decimal point is unnecessary.

- 13.** Explain why multiplying a decimal by 100 has the same effect as shifting the decimal point two places? (Do you need to memorize the direction of the shift?)

We can also multiply decimals by decimals, but now the work is getting less fun. We need to follow the same approach as we did when we explored long multiplication in chapter 3.

Let's do an example. Consider 2.15×0.3 .

Here's what 2.15 looks like.



Here's what 0.3 looks like: it's three dots to the right of where one dot should be.



So to compute 2.15×0.3 we need to replace each dot that appears in the picture of 2.15 with three dots to the right of where that dot should be. Doing so gives this picture.



We see the answer $0.6 \overline{)3 \mid 15}$, or 0.645 after an explosion.

So multiplication with decimals is doable, but awkward. It is actually easier, in practice, to go back to basic arithmetic and work with decimals as fractions. For example,

$$0.2 \times 0.4 = \frac{2}{10} \times \frac{4}{10} = \frac{8}{100} = 0.08$$

and

$$0.05 \times 0.006 = \frac{5}{100} \times \frac{6}{1000} = \frac{30}{100000} = \frac{3}{10000} = 0.0003$$

and

$$2.15 \times 0.3 = \left(2 + \frac{15}{100}\right) \times \frac{3}{10} = \frac{6}{10} + \frac{45}{1000} = 0.6 + 0.045 = 0.645.$$

We also see the effect of multiplying a decimal by ten this way. For example

$$10 \times 3.142 = 10 \times \left(3 + \frac{1}{10} + \frac{4}{100} + \frac{2}{1000}\right) = 30 + 1 + \frac{4}{10} + \frac{2}{100} = 31.42.$$

14. Compute 0.04×0.5 and compute 1000×0.0385 .

DIVISION

Consider $0.08 \div 0.005$.

To do this problem I could try drawing a picture of 0.08 in a $1 \leftarrow 10$ machine and attempt to make sense of finding groups of 0.005 in that picture, that is groups of five dots three places to the right of where they should be. Although possible to do, it seems hard to keep straight in my head and it makes my brain hurt!

Here's a piece of advice from a mathematician: **Avoid hard work!** Mathematicians will, in fact, work very hard to avoid hard work!

Since converting decimals to fractions makes the multiplication of fractions easier, let's do the same thing for division.

Here are some examples.

Example: Examine $0.08 \div 0.005$.

A fraction is a number that is an answer to a division problem. And, conversely, we can think of a division problem as a fraction. For example, the quantity $0.08 \div 0.005$ is really this "fraction"

$$\frac{0.08}{0.005}$$

(It is okay to have non-whole numbers as numerators and denominators of fractions.)

To make this fraction look friendlier, let's multiply the top and bottom by factors of ten.

$$\frac{0.08 \times 10 \times 10 \times 10}{0.005 \times 10 \times 10 \times 10} = \frac{80}{5}$$

The division problem $\frac{80}{5}$ is much friendlier. It has the answer 16. (Use a $1 \leftarrow 10$ machine now to compute it if you like!)

Example: Examine $\frac{8.5}{100}$.

Dividing by 100 is the same as multiplying by $\frac{1}{100}$. So, $\frac{8.5}{100}$, for example, can be thought of as

$$\frac{1}{100} \times 8.5 = \frac{1}{100} \left(8 + \frac{5}{10} \right) = \frac{8}{100} + \frac{5}{1000} = 0.085.$$

Of course, we can also think of this as $\frac{8.5 \times 10}{100 \times 10} = \frac{85}{1000}$, which seems more direct. (All correct paths in mathematics are good and correct!)

Example: Examine $1.51 \div 0.07$.

If I was asked to compute $1.51 \div 0.07$, I would personally do this problem instead:

$$\frac{1.51 \times 10 \times 10}{0.07 \times 10 \times 10} = \frac{151}{7}.$$

I see the answer $21\frac{4}{7}$. If that same person asked me to give the answer as a decimal, then I would have to convert the fraction $\frac{4}{7}$ into a decimal. And that is possible. The next section explains how!

- 15.** Explain why the answer to $0.9 \div 10$ must be 0.09.
 Explain why the answer to $2.34 \div 1000$ must be 0.00234.
 Explain why the answer to 40.04×0.01 must be .4004.

16. Compute $\frac{0.75}{25}$.

- 17.** Let's put it all together. Here are some unpleasant computations. Do you want to try evaluating them?

a) $0.3 \times (5.37 - 2.07)$

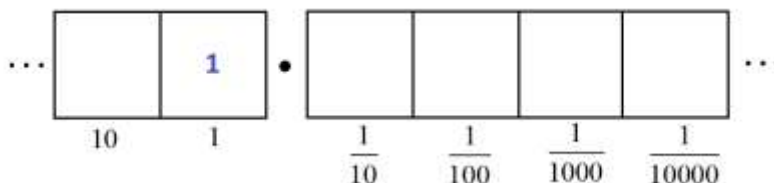
b) $\frac{0.1 + (1.01 - 0.1)}{0.11 + 0.09}$

c) $\frac{(0.002 + 0.2 \times 2.02)(2.2 - 0.22)}{2.22 - 0.22}$

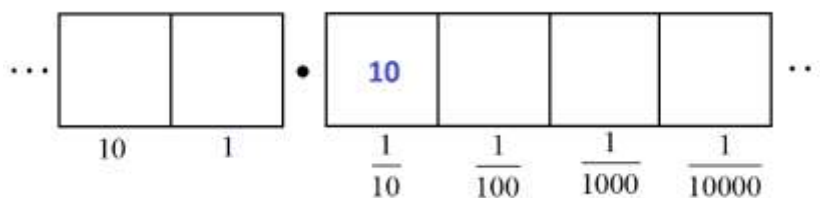
CONVERTING FRACTIONS INTO DECIMALS

A fraction is a number that is an answer to a division problem. For example, the fraction $\frac{1}{8}$ is the result of dividing 1 by 8. And we can compute $1 \div 8$ in a $1 \leftarrow 10$ machine by making use of decimals. The method is exactly the same as for division without decimals.

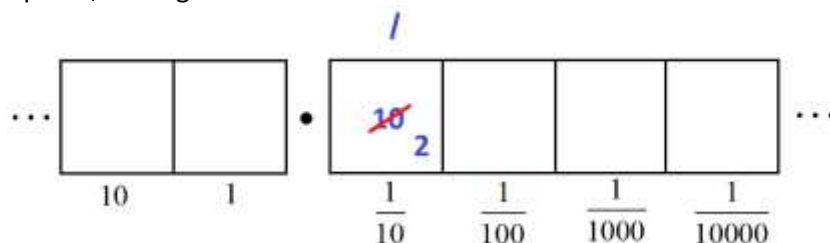
For $1 \div 8$ we seek groups of eight in the following picture.



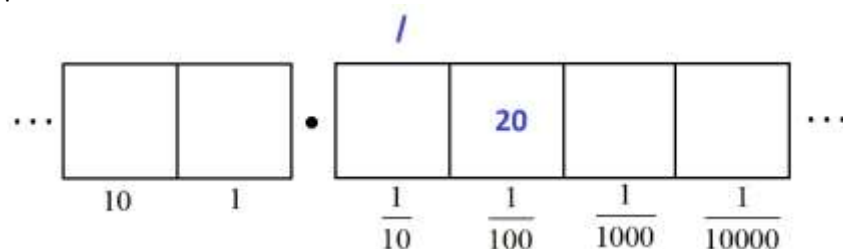
None are to be found right away, so let's unexplode.



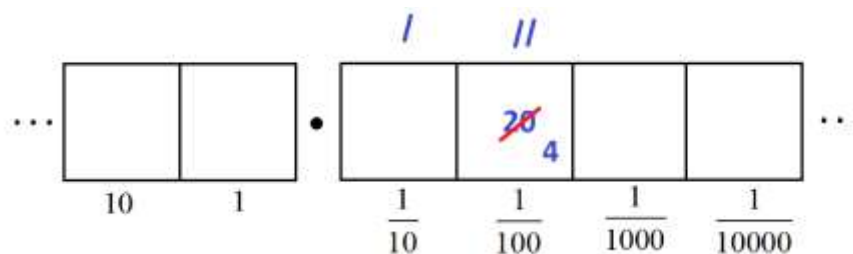
We have one group of 8, leaving two behind.



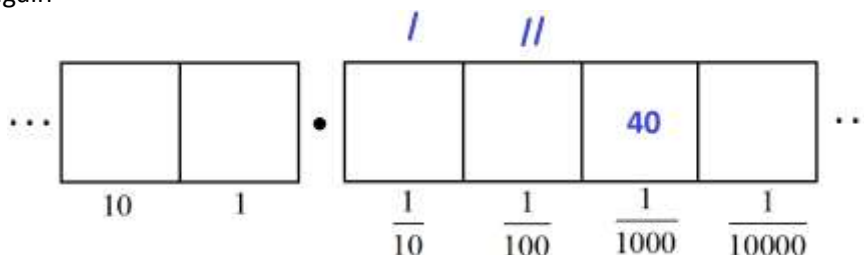
Two more unexploding.



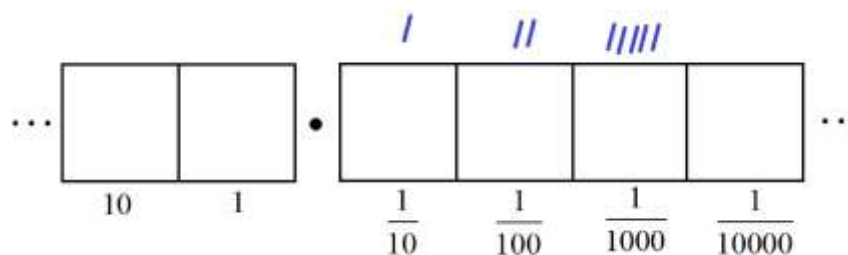
This gives two more groups of 8 leaving four behind.



Unexploding again



reveals five more groups of 8 leaving no remainders.



We see that, as a decimal, $\frac{1}{8}$ turns out to be 0.125. And as a check we have

$$0.125 = \frac{125}{1000} = \frac{25}{200} = \frac{5}{40} = \frac{1}{8}.$$

Here are some practice problems for you to pick and choose from.

18. Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{1}{4}$, as a decimal, is 0.25.

19. Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{1}{2}$, as a decimal, is 0.5.

20. Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{3}{5}$, as a decimal, is 0.6.

21. Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{3}{16}$, as a decimal, is 0.1875.

22. In simplest terms, what fraction is represented by each of these decimals?

0.75 0.625 0.16 0.85 0.0625

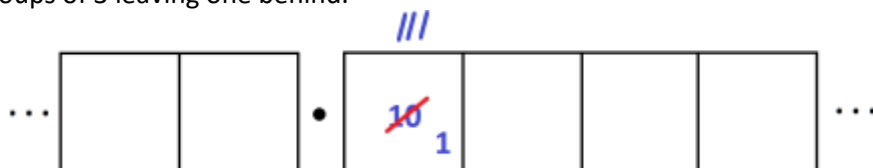
Not all fractions lead to simple decimal representations. For example, consider the fraction $\frac{1}{3}$. To compute it, we seek groups of three in the following picture.



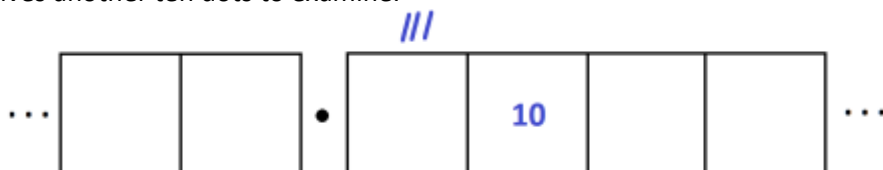
Let's unexplode.



We see three groups of 3 leaving one behind.



Unexploding gives another ten dots to examine.



We find another three groups of 3 leaving one behind.



And so on. We are caught in an infinitely repeating cycle.



This puts us in a philosophically interesting position. As human beings we cannot conduct this, or any, activity for an infinite amount of time. But it seems very tempting to write

$$\frac{1}{3} = 0.33333\dots$$

with the ellipsis representing the instruction “keep going with this pattern forever.” In our minds we can almost imagine what this means. But as a practical human being it is beyond our abilities: one cannot actually write down those infinitely many 3s represented by the ellipses.

Nonetheless, many people choose not to contemplate what an infinite statement like this means and just carry on to say that some decimals are infinitely long and not be worried by it. In which case, the fraction $\frac{1}{3}$ is one of those fractions whose decimal expansion goes on forever.

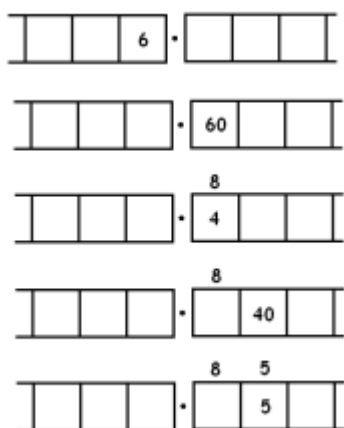
Notation: Many people make use of a *vinculum* (a horizontal bar) to represent infinitely long repeating decimals. For example, $0.\bar{3}$ means “repeat the 3 forever”

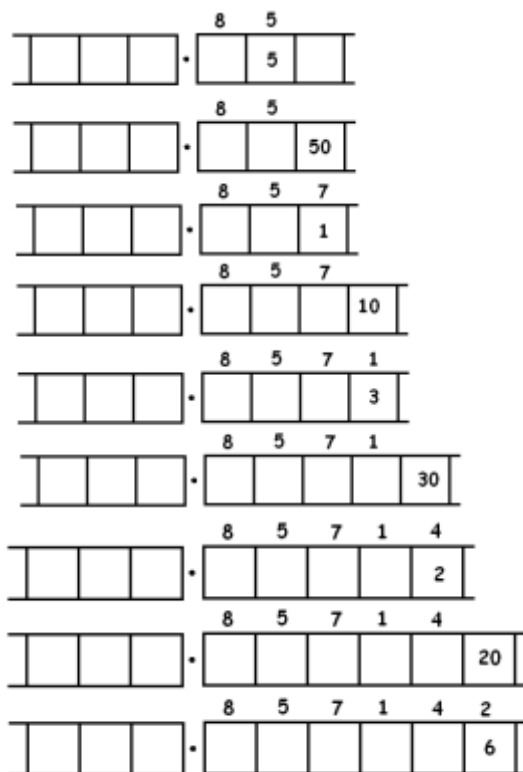
$$0.\bar{3} = 0.3333\dots$$

and $0.38\overline{142}$ means “repeat the group 142 forever” after the beginning “38” hiccup:

$$0.38\overline{142} = 0.38142142142142\dots$$

As another (complicated) example, here is the work that converts the fraction $\frac{6}{7}$ to an infinitely long repeating decimal. Make sure to understand the steps one line to the next.





Do you see, with this 6 in the final right-most box that we have returned to the very beginning of the problem? This means that we shall simply repeat the work we have done and obtain the same sequence 857142 of answers, and then again, and then again. We have

$$\frac{6}{7} = 0.857142857142857142857142\dots$$

Some more practice questions if you like.

23. Compute $\frac{4}{7}$ as an infinitely long repeating decimal.

24. Compute $\frac{1}{11}$ as an infinitely long repeating decimal.

25. Which of the following fractions give infinitely long decimal expansions?

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10}$$

26. Use a $1 \leftarrow 10$ machine to compute $133 \div 6$, writing the answer as a decimal.

27. Use a $1 \leftarrow 10$ machine to compute $255 \div 11$, writing the answer as a decimal.

IRRATIONAL NUMBERS

We have seen that fractions can possess finitely long decimal expansions. For example,

$$\frac{1}{8} = 0.125 \text{ and } \frac{1}{2} = 0.5.$$

And we have seen that fractions can possess infinitely long decimal expansion. For example,

$$\frac{1}{3} = 0.3333\dots \text{ and } \frac{6}{7} = 0.857142057142857142\dots$$

All the examples of fractions with infinitely long decimal expansions we've seen so far fall into a repeating pattern. This is curious.

We can even say that our finite examples eventually fall into a repeating pattern too, a repeating pattern of zeros after an initial start.

$$\frac{1}{8} = 0.12500000\dots = 0.125\bar{0}$$

$$\frac{1}{2} = 0.50000\dots = 0.5\bar{0}$$

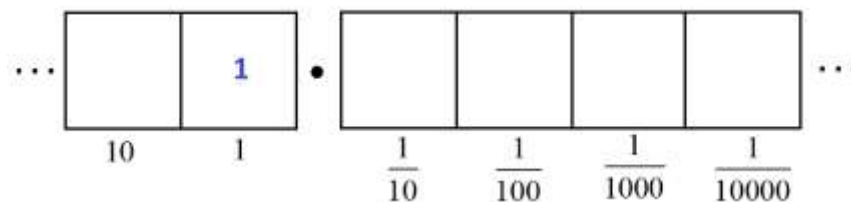
$$\frac{1}{3} = 0.\bar{3}$$

$$\frac{6}{7} = 0.\overline{857142}$$

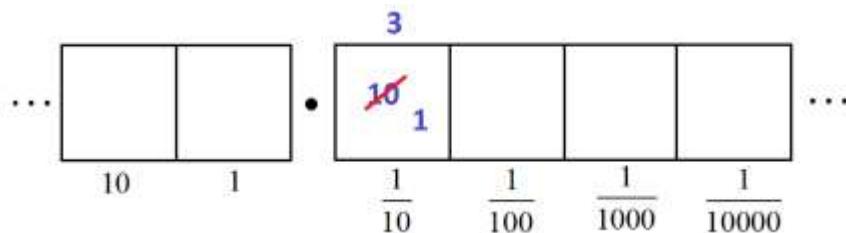
This begs the question: *Does every fraction have a decimal representation that eventually repeats?*

The answer to this question is YES and our method of division explains why.

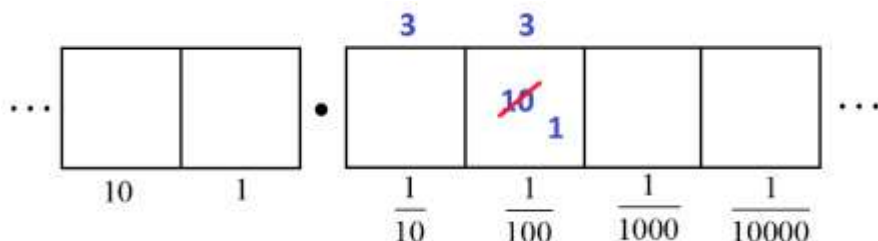
Let's go through the division process again, slowly, first with a familiar example. Let's compute the decimal expansion of $\frac{1}{3}$ again in a $1 \leftarrow 10$ machine. We think of $\frac{1}{3}$ as the answer to the division problem $1 \div 3$, and so we need to find groups of three within a diagram of one dot.



We unexplode the single dot to make ten dots in the tenths position. There we find three groups of three leaving a remainder of 1 in that box.

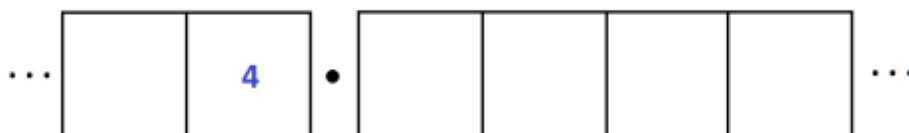


Now we can unexplode that single dot in the tenths box and write ten dots in the hundredths box. There we find three more groups of three, again leaving a single dot behind.

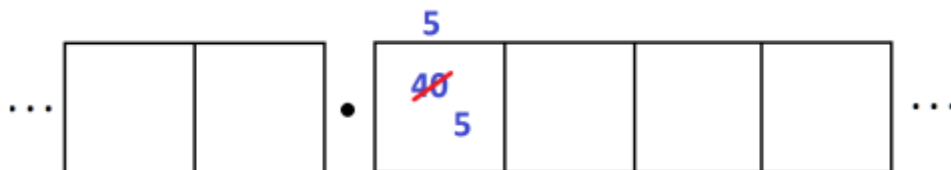


And so on. We are caught in a cycle of having the same remainder of one dot from cell to cell, meaning that the same pattern repeats. Thus we conclude $\frac{1}{3} = 0.333\dots$. The key point is that the same remainder of a single dot kept appearing.

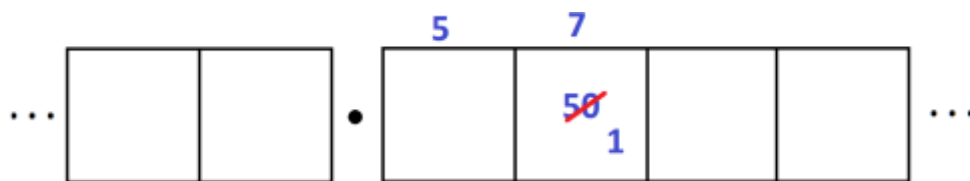
Here's a more complicated example. Let's compute the decimal expansion of $\frac{4}{7}$ in the $1 \leftarrow 10$ machine. That is, let's compute $4 \div 7$.



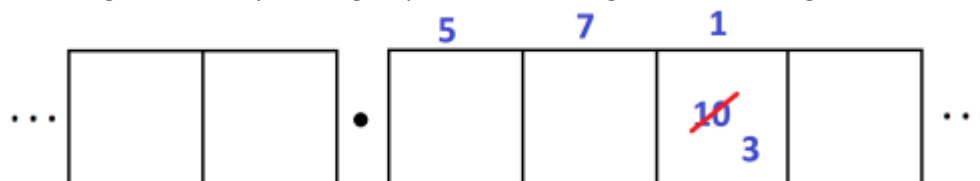
We start by unexploding the four dots to give 40 dots in the tenths cell. There we find 5 groups of seven, leaving five dots over.



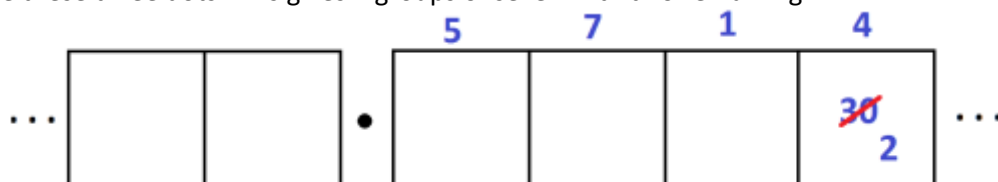
Now unexplode those five dots to make 50 dots in the hundredths position. There we find 7 groups of seven, leaving one dot over.



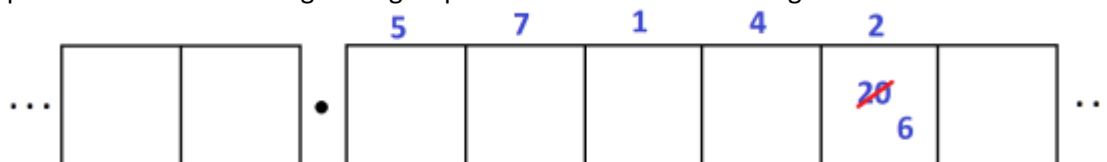
Unexplode this single dot. This yields 1 group of seven leaving three remaining.



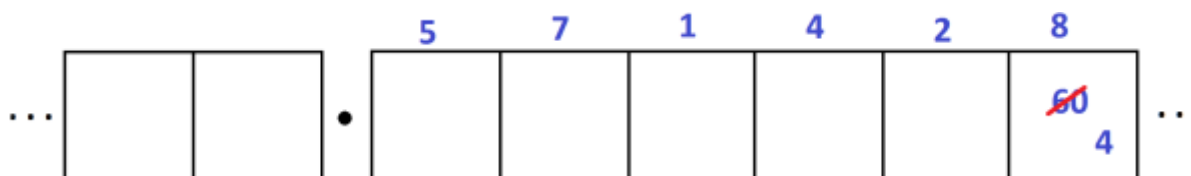
Unexplode these three dots. This gives 4 groups of seven with two remaining.



Unexplode the two dots. This gives 2 groups of seven with six remaining.



Unexplode the six dots. This gives 8 groups of seven with four remaining.



But this is predicament we started with: four dots in a box!

So now we are going to repeat the pattern and produce a cycle in the decimal representation. We have

$$\frac{4}{7} = 0.571428 \ 571428 \ 571428 \ \dots$$

Stepping back from the specifics of this problem, it is clear now that one must be forced into a repeating pattern. In dividing a quantity by seven, there are only seven possible numbers for a remainder number of dots in a cell - 0, 1, 2, 3, 4, 5, or 6 – and there is no option but to eventually repeat a remainder and so enter a cycle.

In the same way, the decimal expansion of $\frac{18}{37}$ must also cycle. In doing the division, there are only thirty-seven possible remainders for dots in a cell (0, 1, 2, ..., 36). As we complete the division computation, we must eventually repeat a remainder and again fall into a cycle.

We have just established a very interesting fact.

ALL FRACTIONS HAVE A REPEATING DECIMAL REPRESENTATION.

(A repeating pattern of zeros is possible. In fact, as a check, conduct the division procedure for the fraction $\frac{1}{8}$. Make sure to understand where the cycle of repeated remainders commences.)

This now opens up a curious idea:

A quantity given by a decimal expansion that does not repeat cannot be a fraction!

For example, the quantity

0.10 1100 111000 11110000 11111000000...

is designed not to repeat (though there is a pattern to this decimal expansion) and so represents a number that is not a fraction.

A number that is equivalent to the ratio of two whole numbers (that is, a fraction) is called a *rational number*. A number that cannot be represented this way is called an *irrational number*.

It looks like we have just proved that irrational numbers exist! Not all numbers are fractions.

In fact, we can now invent all sorts of numbers that can't be fractions! For example

0.102030405060708090100110120130140150...

and

0.3030030003000030000030000003...

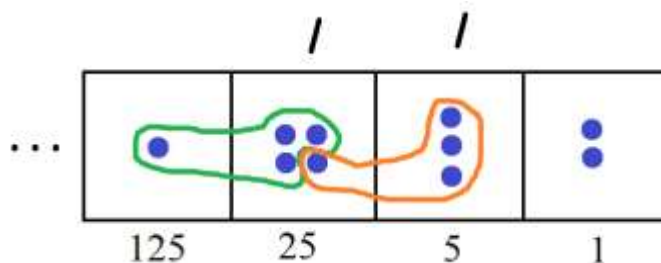
are irrational numbers.

Of course, we have all heard that numbers like $\sqrt{2}$ and π are irrational numbers. It is not at all obvious why they are and how you would go about proving that they are. (In fact, it took mathematicians about 2000 years to finally establish that π is an irrational number. Swiss mathematician Johann Lambert finally proved it so in 1761.) But if you are willing to believe that these numbers are irrational, then you can say for sure that their decimal expansions possess no repeating patterns!

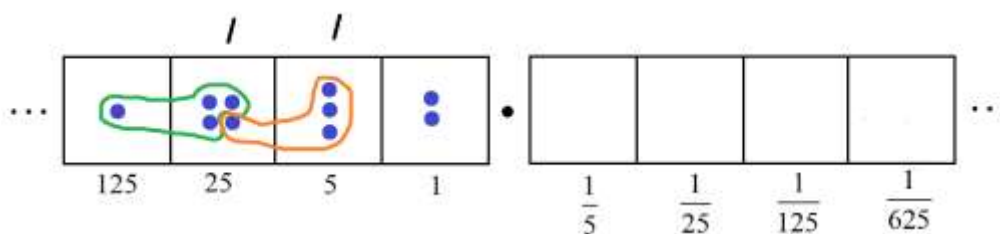
DECIMALS IN OTHER BASES

Who said we need to stay with a $1 \leftarrow 10$ machine?

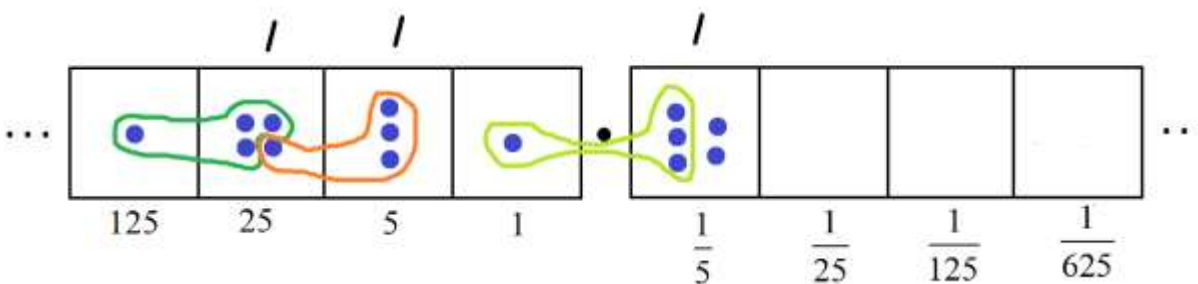
The following picture shows that $1432 \div 13 = 110 R 2$ in a $1 \leftarrow 5$ machine.



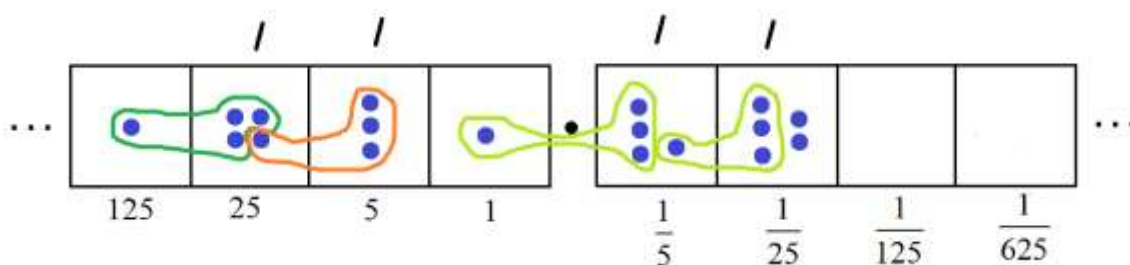
If we work with reciprocals of powers of five we can keep unexploding dots and continue the division process.



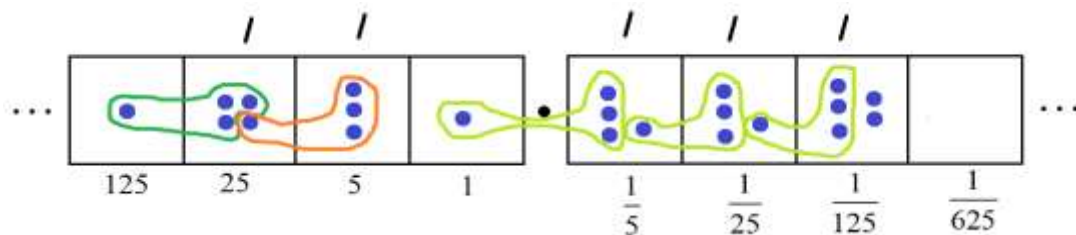
Here goes!



(This reads as $1432 \div 13 = 110.1 R 0.2$ if you like.)



(This reads as $1432 \div 13 = 110.11 R 0.02$ if you like.)



And so on. We get, as a statement of base 5 arithmetic,

$$1432 \div 13 = 110.1111\dots$$

To translate this to ordinary arithmetic we have that

$$1432 \text{ in base five is } 1 \times 125 + 4 \times 25 + 3 \times 5 + 1 \times 1 = 242,$$

$$13 \text{ in base five is } 1 \times 5 + 3 \times 1 = 8,$$

$$110.111\dots \text{ in base five is } 30 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots,$$

so we are claiming, in ordinary arithmetic, that

$$242 \div 8 = 30 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots$$

Whoa!

Question: What does the geometric series formula from chapter 7 say about the sum $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots$? Does it equal a quarter? (To be clear, in chapter 7 we showed that

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}.$$

Multiplying through by x then gives

$$x + x^2 + x^3 + x^4 + \dots = \frac{x}{1-x}.$$

Perhaps it is this second version of the geometric formula we need to work with now.)

Here are some (challenging) practice questions if you are up for them.

28. Compute $8 \div 3$ in a base 10 machine and show that it yields the answer $2.666\dots$.

29. Compute $1 \div 11$ as a problem in base 3 and show that it yields the answer $0.0202020202\dots$.
 (In base three, "11" is the number four, and so this question establishes that the fraction $\frac{1}{4}$ written in base three is $0.0202020202\dots$.)
30. Show that the fraction $\frac{2}{5}$, written here in base ten, has the "decimal" representation $0.121212\dots$ in base four. (That is, compute $2 \div 5$ in a $1 \leftarrow 4$ machine.)
31. **CHALLENGE:** What fraction has decimal expansion $0.3333\dots$ in base 7? Is it possible to answer this question by calling this number x and multiplying both sides by 10? (Does "10" represent ten?)
32. Use an $1 \leftarrow x$ machine and x -mals to show that $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$.



WILDS EXPLORATIONS

Here's are some "big question" investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: WHICH FRACTIONS GIVE FINITE DECIMAL EXPANSIONS?

We've seen that $\frac{1}{2} = 0.5$ and $\frac{1}{4} = 0.25$ and $\frac{1}{8} = 0.125$ each have finite decimal expansions. (We're ignoring infinite repeating zeros now.)

Of course, all finite decimal expansions give fractions with finite decimal expansions! For example, 0.37 is the fraction $\frac{37}{100}$, showing that $\frac{37}{100}$ has a finite decimal expansion.

What must be true about the integers a and b (or true just about a or just about b) for the fraction $\frac{a}{b}$ to have a finite decimal expansion?

EXPLORATION 2: BACKWARDS? ARE REPEATING DECIMALS FRACTIONS?

We have seen that $\frac{1}{3} = 0.\overline{3}$ and $\frac{4}{7} = 0.\overline{571428}$, for example, and that every fraction gives a decimal expansion that (eventually) repeats, perhaps with repeating zeros.

Is the converse true? Does every infinitely repeating decimal fraction correspond to a number that is a fraction?

Is $0.\overline{17}$ a fraction? If so, which fraction is it?

Is $0.4\overline{50}$ a fraction? If so, which fraction is it?

Is $0.322222\dots = 0.3\overline{2}$ a fraction? Is $0.17\overline{023}$ a fraction?

Indeed, does every repeating decimal have a value that is a fraction?



SOLUTIONS

As promised, here are my solutions to the questions posed.

1. "Six tenths" says exactly what the number is.
2. Indeed both students are correct. Performing some explosions on Subra's picture gives JinJin's picture. (And performing some unexploding on JinJin's picture gives Subra's!)
3. a) B b) C
4. We have, in order, $\frac{1}{20}$, $\frac{1}{5}$, $\frac{4}{5}$, and $\frac{1}{250}$.
5. We have, in order, $\frac{4}{10} = 0.4$, $\frac{4}{100} = 0.04$, $\frac{5}{100} = 0.05$, $\frac{5}{1000} = 0.005$, and $\frac{8}{10000} = 0.0008$.
6. a) B b) D
7. In order, we have 0.35, 0.64, 0.602, 0.34, and 0.75.
8. a) As a mixed number it is $2\frac{3}{10}$. Or we could write $2 + \frac{3}{10} = \frac{20}{10} + \frac{3}{10} = \frac{23}{10}$.
 b) $17 + \frac{4}{100} = 17\frac{1}{25}$.
 c) $1003\frac{1003}{10000}$.
9. a) Unexploding one dot in the tenths box does "add" ten dots to the hundredths box.
 b) Unexploding one dot in the tenths box does now give us 19 dots in the hundredths box. Unexploding these gives 190 dots in the thousandths box.
 c) There isn't much to say here. One can see that one can convert any one picture to any other by explosions and/or unexploding.
 d) Yes! All the pictures are the same, so 0.19 and 0.190 are the same number.
10. a) 5.3
 b) $7 + \frac{2}{10} = 7.2$

c) 13.5

$$d) 106 + \frac{15}{100} = 106.15$$

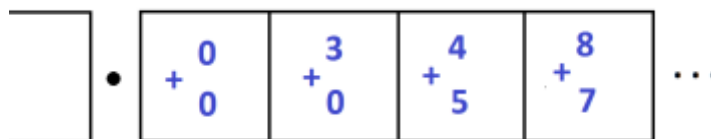
$$e) 3 + \frac{3}{25} = 3 + \frac{12}{100} = 3.12$$

$$f) 2 + \frac{1}{4} = 2 + \frac{25}{100} = 2.25$$

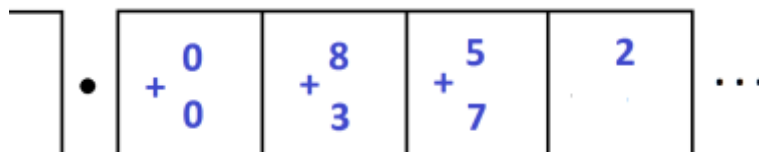
$$g) 3 + \frac{11}{40} = 3 + \frac{275}{1000} = 3.275$$

11. $0.05 + 0.006 = 0.056$ and $0.05 - 0.006 = 0.044$. Drawing the dots and boxes makes these answers clear.

12. If I draw dots and boxes, then I agree with Agatha.



But I don't agree with Percy.



13. Multiplying by 100 is the same as multiplying by ten and then multiplying by ten again.

$$14. \text{ We have } 0.04 \times 0.5 = \frac{4}{100} \times \frac{5}{10} = \frac{20}{1000} = \frac{2}{100} = 0.02.$$

(We could have seen this right away too if we noted that 0.5 is a half, and one half of 0.04 is 0.02.)

$$\text{We have } 1000 \times 0.0385 = 1000 \times \left(\frac{3}{100} + \frac{8}{1000} + \frac{5}{10000} \right) = 30 + 8 + \frac{5}{10} = 38.5.$$

$$15. \text{ We have } \frac{0.9}{10} = \frac{9}{100} = 0.09.$$

$$\text{We have } \frac{2.34}{1000} = \frac{1}{1000} \times \left(2 + \frac{3}{10} + \frac{4}{100} \right) = \frac{2}{1000} + \frac{3}{10000} + \frac{4}{100000} = 0.00234 .$$

$$\text{We have } \left(40 + \frac{4}{100} \right) \times \frac{1}{100} = \frac{4}{10} + \frac{4}{10000} = 0.4004 .$$

$$16. \quad \frac{0.75}{25} = \frac{75}{2500} = \frac{3}{100} = 0.03 .$$

17. This is technically a yes/no question, and the smart answer is no.

For those who answered yes, here are my approaches to the problems.

$$a) \quad 0.3 \times (5.37 - 2.07) = 0.3 \times (3.3) = \frac{3}{10} \times \left(3 + \frac{3}{10} \right) = \frac{9}{10} + \frac{9}{100} = 0.99$$

$$b) \quad \frac{0.1 + (1.01 - 0.1)}{0.11 + 0.09} = \frac{0.1 + (0.91)}{0.2} = \frac{1.01}{0.2} = \frac{101}{20} = \frac{505}{100} = 5.05 .$$

$$\begin{aligned} c) \quad & \frac{(0.002 + 0.2 \times 2.02)(2.2 - 0.22)}{2.22 - 0.22} = \frac{(0.002 + 0.404)(1.98)}{2} \\ & = \frac{(0.406)(1.98)}{2} \\ & = \frac{\left(\frac{4}{10} + \frac{6}{1000} \right) \left(1 + \frac{9}{10} + \frac{8}{100} \right) \times 1000 \times 100}{2 \times 1000 \times 100} \\ & = \frac{(400 + 6)(100 + 90 + 8)}{2 \times 1000 \times 100} \\ & = \frac{406 \times 198}{2 \times 1000 \times 100} \\ & = \frac{80388}{2 \times 100000} \\ & = \frac{40194}{100000} = 0.40194 \end{aligned}$$

Whoa!

18. Do it!

19. Do it too!

20. Yep. Do it!

21. You guessed it. Do it!

22. In order, we have $\frac{3}{4}$, $\frac{5}{8}$, $\frac{4}{25}$, $\frac{17}{20}$, and $\frac{5}{80}$.

23. We get $\frac{4}{7} = 0.571428\ 571428\ 571428\ 571428\ \dots$.

24. We get $\frac{1}{11} = 0.09\ 09\ 09\ 09\ \dots$.

25. The fractions $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, and $\frac{1}{9}$ have infinitely long decimal expansions.

26. $\frac{133}{6} = 22.166666\dots$.

27. $\frac{255}{11} = 23.18181818\dots$.

28. Do it!

29. Do it too!

30. Yep, do it!

31. Let's do all our writing in a $1 \leftarrow 7$ machine. So the number seven is 10 and one less than this number is 6 . Also, notice that multiplying a number by seven in a $1 \leftarrow 7$ machine has the effect of adding a zero at the end of the number. (Can you see why? It is the same reason why multiplying by ten in a $1 \leftarrow 10$ machine has the effect of adding a zero.) Multiplying by seven also "shifts the decimal point" one place.

Set $0.333\dots = x$.

Multiply each side by seven (which looks like 10). This gives

$$3.3333\dots = 10x.$$

The left side is $3 + 0.3333\dots$, that is, 3 plus the original number.

$$3 + x = 10x$$

Subtract x from both sides.

$$3 = 6x$$

So x must equal a half!

32. Can you see how to do it?