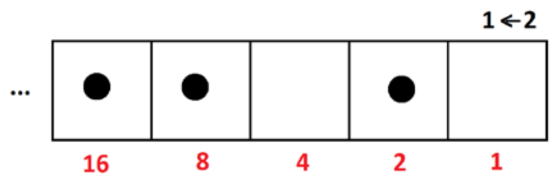
VISUAL GRAPHS OF BINARY REPRESENTATIONS
with *Exploding Dots*

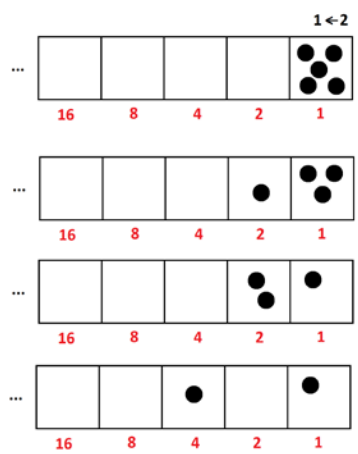
In the global phenomenon that is the story of *Exploding Dots*[1] a $1 \leftarrow 2$ machine is simply a row of boxes that extends as far to the left as one could ever desire.



26 = 16 + 8 + 2 = 1|1|0|1|0

Dots in each box are worth a power of two and, as such, in any distribution of dots among the boxes, any two dots in any one box can be erased – they explode! kaboom! – and replaced by one dot, one place to the left and not affect the total value of the dots. And conversely, any dot in a box can "unexplode" and be replaced by two dots one box to its right.

There are four ways to represent the number five in a $1 \leftarrow 2$ machine, namely as five dots in the rightmost box, 5, or as one dot and then three dots in the last two boxes, the result of a first explosion, $1|3$, or as $2|1$, the result of a second explosion, or, after one more explosion as $1|0|1$. A representation of a number in a $1 \leftarrow 2$ machine with at most one dot per box is the usual *binary representation* of the number. ("Five" in base two is 101.)



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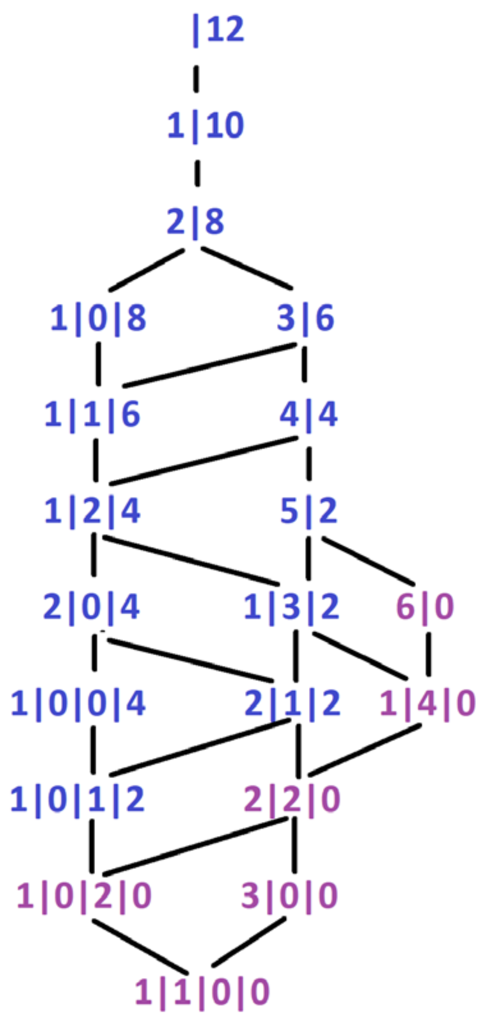
[1] See http://gdaymath.com/courses/ and the blog post about Global Math Week 2017 https://medium.com/@jamestanton/the-global-math-project-uplifting-mathematics-for-all-9253d383e5e6.

Let $B(N)$ be the count of ways to represent the counting number N in a $1 \leftarrow 2$ machine. (We have, for instance, $B(5) = 4$.) This represents the count of ways to write N in base two with no restrictions on the size of the digits allowed, or, equivalently, the number of ways to write N as a sum of powers of two with repeats allowed and not worrying about the order of those terms. (So we might was well order the terms from largest to smallest, non-increasing order.)

One checks that $B(1)=1$, $B(2)=2$, $B(3)=2$, and $B(4)=4$. (Check this!)

Question 1: Explain why $B(N+1) = B(N)$ if N is even. (Why must the final box in a $1 \leftarrow 2$ machine contain one persistent dot for all representations of a given odd value? What if you just pretend that dot is not there?)

Here's a diagram of all the ways to represent the number twelve in a $1 \leftarrow 2$ machine.



We start with twelve dots in the rightmost box of the machine and draw a branching diagram of all the results that can appear from explosions. As each explosion erases two dots but only draws back in one dot, the total count of dots in a distribution decreases by one with each explosion. Thus each "level" we draw in the diagram has the same total number of dots in each distribution of that level, one less than the that count for the previous level.

We see that $B(12)=20$.

> **Question 2:** Draw the analogous diagrams for $N=2,4,6,8,$ and 10 and find the values of $B(6),\ B(7),\ldots,\ B(10),\ B(11)$.
>
> **Question 3:** Look at your diagrams for $N=10$ and for $N=6$. Look at the blue and the purple of the diagram for $N=12$. What do you notice?

The blue and the purple in our diagram for $N=12$ correspond to those representations of 12 that, respectively, have a non-zero number of dots in the final box and those that have zero dots in the final box.

If we delete two from final value in all the blue representations, then we have all the representations of $N=10$. If we ignore the final zero of all the purple representations, then we have all the representations of $N=6$. It looks like $B(12)=B(10)+B(6)$.

Let's explore this more generally.

First observe that if $a\,|\,b\,|\,c\,|\cdots|\,d\,|\,e$ is a representation of n in a $1\leftarrow 2$ machine, then $a\,|\,b\,|\,c\,|\cdots|\,d\,|\,e\,|\,0$ is a representation of $N=2n$.

Also note that a representation of an even number $N=2n$ in a $1\leftarrow 2$ machine must have an even number of dots in the final box. (Why?) As such, the representations of N fall into two types:

> Type Blue: Those that have a non-zero even count of dots in the final box: $a\,|\,b\,|\,c\,|\cdots|\,d\,|\,e\,|\,f$ with $f\geq 2$.
>
> Type Purple: Those that have zero dots in the final box: $a\,|\,b\,|\,c\,|\cdots|\,d\,|\,e\,|\,0$.

Each representation of $N=2n$ of type blue matches a representation of $N-2$ by erasing two dots from its final box. In fact, there is a perfect matching between type blue representations of N and representations of $N-2$.

$$a\,|\,b\,|\,c\,|\cdots|\,d\,|\,e\,|\,f \leftrightarrow a\,|\,b\,|\,c\,|\cdots|\,d\,|\,e\,|\,f-2$$

Thus the count of type blue representations is $B(N-2)$.

Each representation of $N = 2n$ of type purple matches a representation of n by ignoring the final empty box. In fact, there is a perfect matching between type purple representations of $N = 2n$ and representations of n .

$$a \mid b \mid c \mid \cdots \mid d \mid e \mid 0 \leftrightarrow a \mid b \mid c \mid \cdots \mid d \mid e$$

Thus the count of type purple representations is $B(n)$.

We have

$$B(N) = B(N-2) + B\left(\frac{N}{2}\right) \qquad \text{if } N \text{ is even}$$

and from before

$$B(N) = B(N-1) \qquad \text{if } N \text{ is odd}$$

This gives

$$\begin{aligned}
&B(2) && = 2 \\
&B(4) = B(2) + B(2) && = 4 \\
&B(6) = B(3) + B(4) = B(2) + B(4) && = 6 \\
&B(8) = B(4) + B(6) && = 10 \\
&B(10) = B(5) + B(8) = B(4) + B(8) && = 14 \\
&B(12) = B(6) + B(10) && = 20 \\
&B(14) = B(7) + B(12) = B(6) + B(12) && = 26 \\
&B(16) = B(8) + B(14) && = 36
\end{aligned}$$

and so on.

The sequence of values for $B(N)$ begins:

$1,\ 2,\ 2, 4, 4, 6, 6, 10, 10, 14, 14, 20, 20, 26, 26, 36, 36, 46, 46, 60, 60, 74, 74, \ldots$.

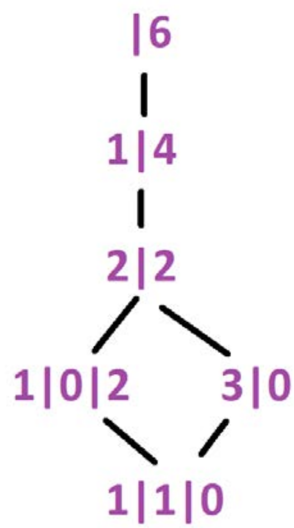
Question 4: Let $B_k(N)$ be the number of ways to represent N in a $1 \leftarrow 2$ machine using precisely k dots. (Thus $B_k(N)$ is the number of distributions at the k th level of a diagram of distributions we draw for N .) Show that $B_k(N) = B_{k-2}(N-2) + B_k\left(\frac{N}{2}\right)$ if N is even. What is a formula for $B_k(N)$ if N is odd?

GOING FOR AN EXPLOSION WALK

Here's a question:

> *Starting with N dots in the rightmost box of a $1 \leftarrow 2$ machine, is it possible to perform a series of explosions and unexplosions so that each and every possible representation of N appears as you play with no repeat representations?*

Here is the diagram of distributions for $N = 6$.



Starting with the distribution $|\,6$ and ending at $3\,|\,0$ we can see how to trace a path along the edges of your diagram that visits each and every distribution in the diagram exactly once. Each movement along an edge corresponds to performing an explosion or an unexplosion.

Question 5:

a) Look at your diagrams for $N = 2$, 4, 8, and 10. Find an analogous path in each diagram that starts at $|\,N$ and ends at $\frac{N}{2}\,|\,0$.

b) Look at the blue and the purple in our diagram for $N = 12$. Copy the path you found for $N = 10$ in the blue part of the diagram and the path for $N = 6$ in the purple part of the diagram.

c) Explain why, for N even, there is sure to be a path in its diagram of distributions following the edges that starts at $|\,N$ and ends at $\frac{N}{2}\,|\,0$ and visits each distribution exactly once.

d) Explain why a similar path is sure to exist in a diagram for N odd as well. (At which distribution does one end?)

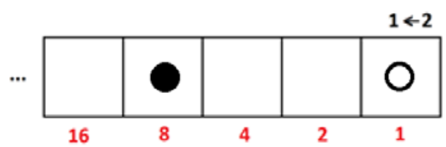
A VARIATION

$B(N)$ counts the number of ways to write N as a sum of powers of two, with repetitions allowed and order of terms considered immaterial, and $B_k(N)$ is the count of those representations using precisely k terms.

Let $\widetilde{B}(N)$ be the count of ways to write N as a sum of the (non-zero) numbers one less than a power of two ($1,3,7,15,31,\ldots$), with repetitions allowed and order of terms considered immaterial, and $\widetilde{B}_k(N)$ the count of those using precisely k terms.

For example, we can write 9 as sum of three numbers, each one less than as a power of two, as $3+3+3$ and as $7+1+1$, and no other ways. Thus $\widetilde{B}_3(9)=2$. In general, we can also write 9 as $3+3+1+1+1$, $3+1+1+1+1+1+1$, and $1+1+1+1+1+1+1+1+1$ and $\widetilde{B}(9)=5$.

Each number one less than a power of two can be represented in a $1\leftarrow 2$ machine as a single dot in a box away from the rightmost box and an antidot in the rightmost box.



7 = 8 - 1

If $a|b|c|\cdots|d|e$ is a representation of n in a $1\leftarrow 2$ machine using k dots (that is, a representation of n as a sum of powers of two, k terms in all), then $a|b|c|\cdots|d|e|0$ is a representation of $2n$ in a $1\leftarrow 2$ machine using k dots (that is, a representation of n as a sum of powers of two, k terms in all), and $a|b|c|\cdots|d|e|-k$ is a representation of $2n-k$ in a $1\leftarrow 2$ machine, which corresponds to a representation of $2n-k$ as a sum of numbers each one less than a power of two, k terms in all.

We have $\widetilde{B}_k(2n-k)=B_k(n)$, assuming $k<2n$.

> **Question 6:** Explain why $\widetilde{B}_k(N)$ is zero if N is even and k is odd, or vice versa. If, on the other hand, $N+k$ is even, explain why $\widetilde{B}_k(N)=B_k\left(\frac{N+k}{2}\right)$.

> **Research:** Develop some more result about the values of $\widetilde{B}_k(N)$ and the values of $\widetilde{B}(N)=\widetilde{B}_1(N)+\widetilde{B}_2(N)+\widetilde{B}_3(N)+\cdots$.

ARML POWER QUESTON 2018
Partitions

Question 7: Solve the 2018 ARML Power Question.

Power Question 2018: Partitions

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Binary Partitions

A *binary partition* of a positive integer n is an ordered n-tuple of non-increasing integers, each of which is either 0 or a power of 2, whose sum is n. Each of the integers in the n-tuple is called a *part* of the partition. Each binary partition of n has n parts. Let $p_2(n)$ denote the number of binary partitions of n. For example, $p_2(3) = 2$ because of the two ordered triples $(2, 1, 0)$ and $(1, 1, 1)$.

1. Compute $p_2(n)$ for $n = 4$, 5, 6, and 7. [4 pts]

2. **a.** Show that $p_2(n) \leq p_2(n+1)$ for all positive integers n. [3 pts]

b. Is the inequality strict for sufficiently large n? Justify your answer. [3 pts]

3. **a.** Prove that if n is even and $n \geq 4$, then $p_2(n) = p_2(\frac{n}{2}) + p_2(n-2)$. [2 pts]

b. Find the least $n > 1$ such that $p_2(n)$ is odd, or prove that no such n exists. [2 pts]

Partial Orderings

A *partial ordering* on a set S is a relation, usually denoted $\preceq$, such that all of the following conditions are true:

- $a \preceq a$ for all $a \in S$ **(reflexivity property)**,
- $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ for all $a, b, c \in S$ **(transitivity property)**, and
- $a \preceq b$ and $b \preceq a$ implies $a = b$ for all $a, b \in S$ **(antisymmetry property)**.

The word "partial" refers to the possibility that some two elements, a and b, may be *incomparable*, i.e., neither $a \preceq b$ nor $b \preceq a$ (these negative relations on $\preceq$ are sometimes written as $a \npreceq b$ and $b \npreceq a$, respectively). This Power Question largely defines and explores a partial ordering on the set of binary partitions of n.

The notation $a \prec b$ means that $a \preceq b$ and $a \neq b$. The symbols $\succ$ and $\succeq$ may also be used, and they are defined in the following way: $a \succ b$ means $b \prec a$, and $a \succeq b$ means $b \preceq a$.

If a and b are elements of S with $a \prec b$, and if there is no element $c \in S$ for which $a \prec c \prec b$, then it is said that b *covers* a.

Let a denote the binary partition $(a_1, a_2, \ldots, a_n)$. Similarly, let b denote the binary partition $(b_1, b_2, \ldots, b_n)$. In this Power Question, define $a \prec b$ if it is possible to obtain a from b by a sequence of replacing one 2^k by two 2^{k-1}s (and deleting a 0). For example, $(4,1,1,1,1,0,0,0) \prec (4,4,0,0,0,0,0,0)$ because $(4,1,1,1,1,0,0,0) \prec (4,2,1,1,0,0,0,0) \prec (4,2,2,0,0,0,0,0) \prec (4,4,0,0,0,0,0,0)$. This Power Question will use this partial ordering on binary partitions.

4. **a.** Show that the binary partitions of 5 are totally ordered; i.e., if p and p' are two different binary partitions of 5, then either $p \prec p'$ or $p' \prec p$. [2 pts]

   b. Show that the binary partitions of 8 are not totally ordered, i.e., find two binary partitions of 8 – call them q and q' – such that $q \nprec q'$ and $q' \nprec q$. [3 pts]

5. **a.** Find the smallest binary partition of n, using this partial ordering. That is, find the binary partition p such that for all other binary partitions p', $p \prec p'$. [2 pts]

   b. Find the largest binary partition of n, using this partial ordering. That is, find the binary partition P such that for all other binary partitions P', $P \succ P'$. [3 pts]

Hasse Diagrams

Suppose that a set S has a partial ordering $\preceq$. Then a *Hasse diagram* can be used to display the covering relation in a graphical way. The Hasse diagram is a graph whose vertices are the elements of S and where edges are drawn between two elements x and y if $x \prec y$ or $y \prec x$ and if there is no element z for which $x \prec z$ and $z \prec y$ or for which $y \prec z$ and $z \prec x$. Also, y appears "above" x if $x \prec y$. Note that it is possible for more than one element to appear on the same level of a Hasse diagram. For example, the partially ordered set of divisors of 12, ordered by divisibility, is shown in Figure 1. In this diagram, the number 1 is said to be at Level 0 because it is the least divisor in the partial ordering shown. The numbers 2 and 3 are said to be on Level 1 because $2 \succ 1$ and $3 \succ 1$ and there is no number n such that $2 \succ n$ and $n \succ 1$ or $3 \succ n$ and $n \succ 1$. Other Levels are similarly defined. The number 12 is on Level 3.

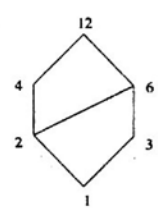


Figure 1

Hasse diagrams can be drawn to show the ordering of partitions such as the ones from Problems 4 and 5. For convenience, rather than labeling the vertices in the Hasse diagram with the partition itself, like $(8,0,0,0,0,0,0,0)$, it is common to label the vertex with its nonzero parts only, using exponents to indicate parts within the partition with multiplicity greater than 1. For example, the partition $(8,0,0,0,0,0,0,0)$ would be labeled 8, the partition $(4,2,2,0,0,0,0,0)$ would be labeled 42^2, and the partition $(2,2,1,1,1,1,0,0)$ would be labeled 2^21^4.

6. **a.** Draw the Hasse diagram for the binary partitions of 8. Label each vertex. [1 pt]

b. List a path through the Hasse diagram for the binary partitions of 8 that begins at the bottom vertex (that is, the vertex at Level 0) and, traveling only along edges, passes through every other vertex exactly once. Such a path is called a *Hamiltonian path.* [1 pt]

c. Let n be an even integer with $n > 4$. Let S be the set of binary partitions of n. Let S_1 be the set of binary partitions of $\frac{n}{2}$. Let S_2 be the set of binary partitions of $n-2$. Prove that there is a bijection B (i.e., a one-to-one correspondence) from the set S to the set $S_1 \cup S_2$. Prove that this bijection B preserves order; that is, given that $p \prec p'$ for binary partitions $p, p' \in S$, then either $B(p) \prec B(p')$ or $B(p)$ and $B(p')$ are incomparable. [3 pts]

d. Prove that for each positive integer n, the Hasse diagram of the binary partitions of n has a Hamiltonian path that begins with the vertex at Level 0. [5 pts]

Let $f_L(n)$ represent the number of elements at level L in the Hasse diagram of the binary partitions of n.

7. Prove that the value of $f_L(n)$ is the number of binary partitions of n that have $n-L$ nonzero parts. [3 pts]

.. Not all partitions are binary partitions. Some partitions have parts of the form $2^j - 1$ where j is a nonnegative integer. Such partitions will be called *s-partitions*, and their parts are written in nonincreasing order. Two s-partitions of 5 are $(3,1,1,0,0)$ and $(1,1,1,1,1)$.

a. List two partitions of 7, one that is an s-partition and one that is neither an s-partition nor a binary partition. Make sure to identify which is which. [2 pts]

b. Prove that if $n \geq 2L$, the value of $f_L(n)$ is equal to the number of s-partitions of L. [3 pts]

Trinary Partitions

A *trinary partition* of a positive integer n is an ordered n-tuple of non-increasing integers, each of which is either 0 or a power of 3, whose sum is n. Let $p_3(n)$ denote the number of trinary partitions of n. For example, $p_3(4) = 2$ because of the two ordered quadruples $(3,1,0,0)$ and $(1,1,1,1)$.

As with binary partitions, one can define partial orderings for trinary partitions. Let c denote the trinary partition $(c_1, c_2, \ldots, c_n)$. Similarly, let d denote the trinary partition $(d_1, d_2, \ldots, d_n)$. Define $c \prec d$ if it is possible to obtain c from d by a sequence of replacing one 3^k by three 3^{k-1}s (and deleting two 0s).

9. Draw the Hasse diagram for the trinary partitions of 12. Label each vertex. [2 pts]

10. State a value of n less than 23 for which the Hasse diagram of the trinary partitions of n does **not** contain a Hamiltonian path. Prove your claim. (Recall that a Hamiltonian path is defined in Problem 6b.) [6 pts]