EXPLODING DOTS
CHAPTER 2

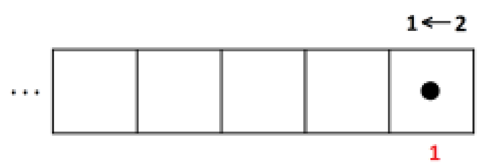
WHAT ARE THESE MACHINES REALLY DOING?

All right. It's time to explain what the machines from the previous chapter are really doing. (Did you already figure it all out? Did you play with the final explorations of that chatpter?)

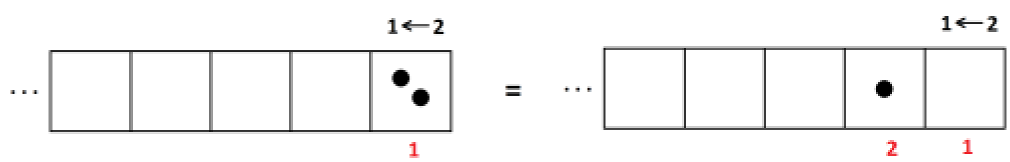
Let's go back to the $1 \leftarrow 2$ machine and first make sense of that curious device. Recall that it follows the rule

> *Whenever there are two dots in any one box they "explode," that is, disappear, and are replaced by one dot, one place to their left.*

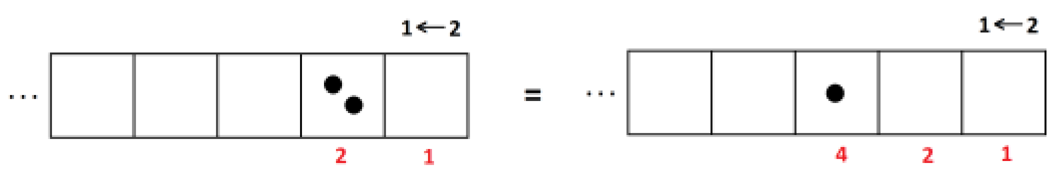
And this machine is set up so that dots in the rightmost box are always worth one.



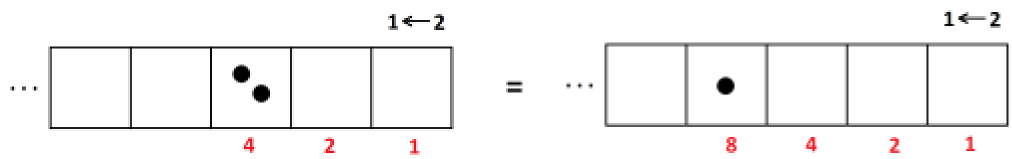
With an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1s, that is, 2.



And two dots in this second box is equivalent to one dot one place to the left. Such a dot must be worth two 2s, that is, worth 4.



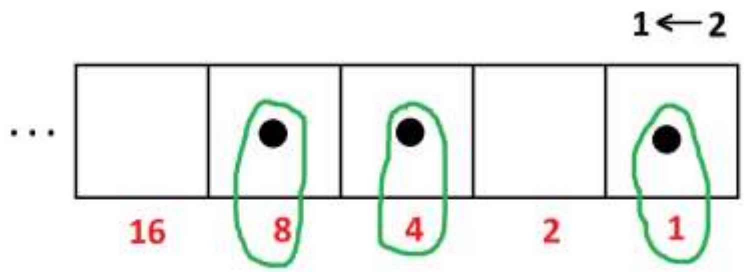
And two 4s makes 8 for the value of a dot the next box over.



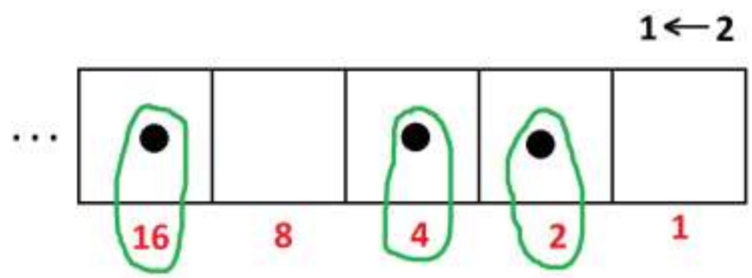
Here's a question to mull on if you like. Remember my solutions to all questions appear at the end of the chapter.

1. The value of a dot one further place to the left is 16. Can you see why? What are the values of dots in the next few boxes even further to the left?

We saw earlier that the code for thirteen in a $1 \leftarrow 2$ machine is 1101. Now we can see that this is absolutely correct: one 8 and one 4 and no 2 s and one 1 does indeed make thirteen.



We also asked what number has code 10110 in a $1 \leftarrow 2$ machine. We now readily see that the answer is $16+4+2=22$.



Can you see that the $1 \leftarrow 2$ code for thirty is 11110?

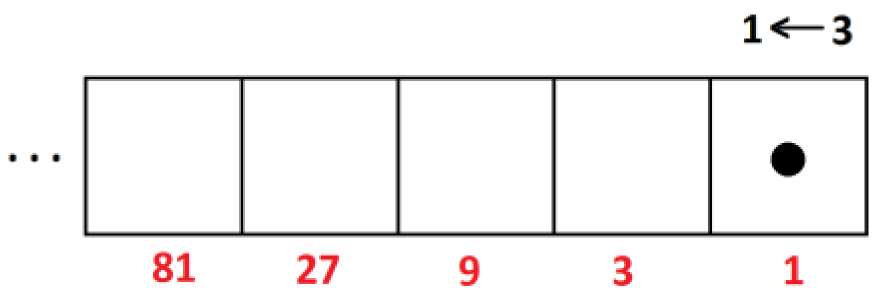
2. What number has $1 \leftarrow 2$ code 100101 ?

3. What is the $1 \leftarrow 2$ code for the number two hundred?

People call the $1 \leftarrow 2$ codes for numbers the *binary* representations of numbers (with the prefix *bi-* meaning "two). They are also called *base two* representations. One only ever uses the two symbols 0 and 1 when writing numbers in binary.

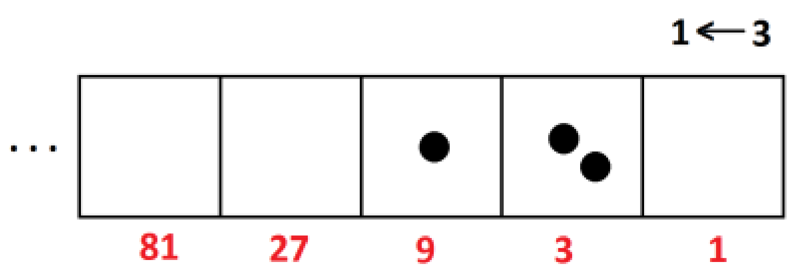
Computers are built on electrical switches that are either on or off. So it is very natural in computer science to encode all arithmetic in a code that uses only two symbols: say 1 for "on" and 0 for "off." Thus base two, binary, is the right base to use in computer science.

4. In a $1 \leftarrow 3$ machine, three dots in any one box are equivalent to one dot one place to the left. (And each dot in the rightmost box is again worth 1.) We get the dot values in this machine by noting that three 1s is 3, and three 3s is 9, and three 9s is 27, and so on.



a) What is the value of a dot in the next box to the left after this?

At one point we said that the $1 \leftarrow 3$ code for fifteen is 120. And we see that this is correct: one 9 and two 3s does indeed make fifteen.



b) Could we say that the $1 \leftarrow 3$ code for fifteen is 0120? That is, is it okay to put zeros in the front of these codes? What about zeros at the ends of codes? Are they optional? Is it okay to leave off the last zero of the code 120 for fifteen and just write instead 12?

c) What number has $1 \leftarrow 3$ machine code 21002?

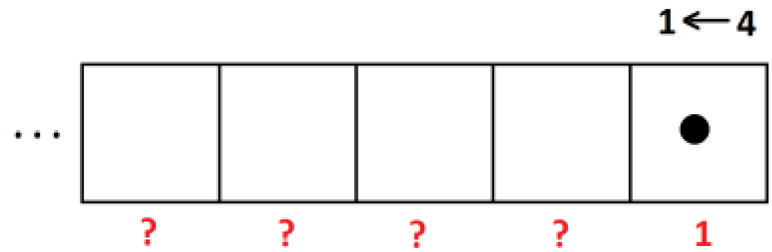
d) What is the $1 \leftarrow 3$ machine code for two hundred?

The $1 \leftarrow 3$ machine codes for numbers are called *ternary* or *base three* representations of numbers. Only the three symbols 0, 1, and 2 are ever needed to represent numbers in this system.

There is talk of building optic computers based on polarized light: either light travels in one plane, or in a perpendicular plane, or there is no light. For these computers, base three arithmetic would be the appropriate notational system to use.

5.

a) In the $1 \leftarrow 4$ system four dots in any one box are equivalent to one dot one place to their left. What is the value of a dot in each box?

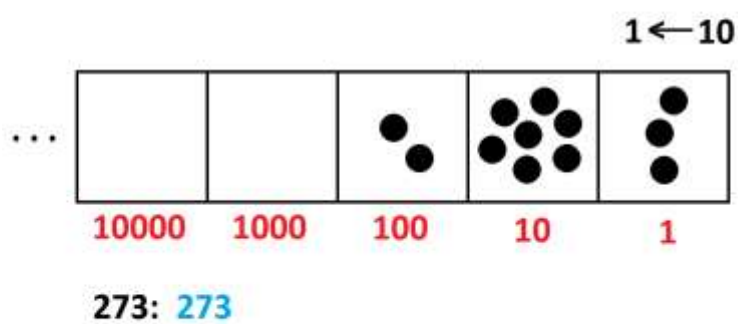


b) What is the $1 \leftarrow 4$ machine code for twenty nine?

c) What number has 132 as its $1 \leftarrow 4$ machine code?

And finally, for a $1 \leftarrow 10$ machine, we see that ten ones makes 10, ten tens makes 100, ten one-hundreds makes 1000, and so on. A $1 \leftarrow 10$ has ones, tens, hundreds, thousands, and so on, as dot values.

We saw that the code for the number 273 in a $1 \leftarrow 10$ machine is 273, and this is absolutely correct: 273 is two hundreds, seven tens, and three ones.



In fact, we even speak the language of a $1 \leftarrow 10$ machine. When we write 273 in words, we write

273 = two hundred seventy three

We literally say, in English at least, two HUNDREDS and seven TENS (that "ty" is short for "ten") and three.

So, through this untrue story of dots and boxes we have discovered *place-value* and *number bases*: base two, base three, base ten, and so on. And society has decided to speak the language of base ten machine.

Why do you think we humans have a predilection for the $1 \leftarrow 10$ *machine? Why do we like the number ten for counting?*

One answer could be because of our human physiology: we are born with ten digits on our hands. Many historians do believe this could well be the reason why we humans have favored base ten.

> **6.** I happen to know that Martians have six fingers on each of two hands. What base do you think they might use in their society?

There are some cultures on this planet that have used base twenty. Why might they have chosen that number do you think?

In fact there are vestiges of base twenty thinking in western European culture of today. For example, in French, the number 87 is spoken and written as *quatre-vingt-sept*, which translates, word for word, as "four twenties seven." In the U.S. the famous Gettysburg address begins: "Four score and seven years ago." That's four-twenties and seven years ago.

All right. The point of today's lesson has been made. We have discovered base-ten place value for writing numbers and seen their context in the whole story of place value. We humans happen to like base-ten in particular because that is the number of fingers most of us seem to have.

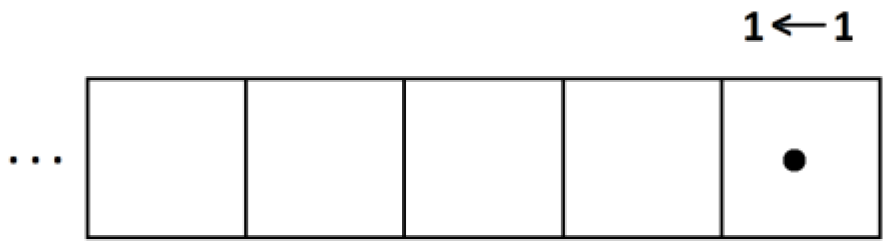
In the next chapter we'll start doing arithmetic with numbers, but in new and fabulous ways!

WILDS EXPLORATIONS

Here are some “big question” investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: CAN MACHINES “GO THE OTHER WAY”?

Jay decides to play with a machine that follows a $1 \leftarrow 1$ rule. He puts one dot into the right-most box. What happens? Do assume there are infinitely many boxes to the left.



Suggi plays with a machine following the rule $2 \leftarrow 1$. She puts one dot into the right-most box. What happens for her?

Do you think these machines are interesting? Is there much to study about them?

EXPLORATION 2: CAN WE PLAY WITH WEIRD MACHINES?

Poindexter decides to play with a machine that follows the rule $2 \leftarrow 3$.

a) Describe what happens when there are three dots in a box.

b) Work out the $2 \leftarrow 3$ machine codes for the numbers 1 up to 30. Any patterns?

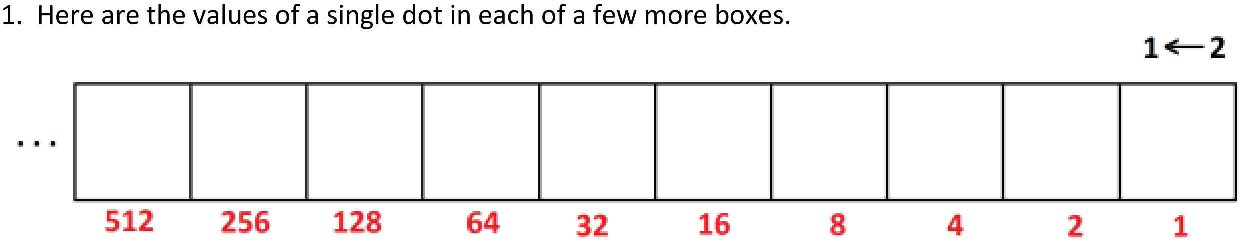
c) The code for ten in this machine turns out to be 2101. Look at your code for twenty. Can you see it as the answer to “ten plus ten”? Does your code for thirty look like the answer to “ten plus ten plus ten”?

Comment: We’ll explore this weird $2 \leftarrow 3$ machine in chapter 9. It is mighty weird!

SOLUTIONS

As promised, here are my solutions to the questions posed.

1. Here are the values of a single dot in each of a few more boxes.



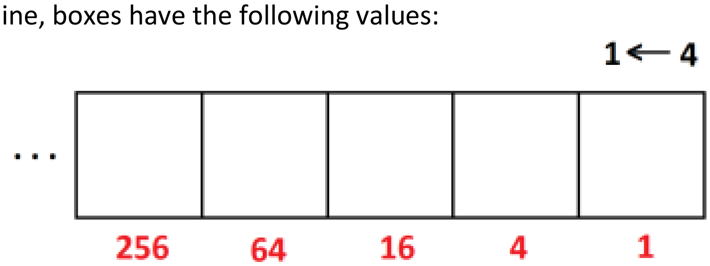
Care to keep going?

2. Thirty-seven.

3. 11001000

4.
a) Each dot in the next box to the left is worth three 81s, that's 243.
b) Yes it is okay to insert a zero at the front of the code. This would say that there are no 27 s, which is absolutely correct. Deleting the end zero at the right, however, is problematic. 120 is the code for fifteen (one 9 and two 3s) but 12 is the code for five (one 3 and two 1s).
c) One hundred and ninety one. (Two 81s, one 27, and two 1s.)
d) 21102

5.
a) For a $1 \leftarrow 4$ machine, boxes have the following values:



b) The number twenty-nine has code 131 in a $1 \leftarrow 4$ machine.
c) Thirty. (This is one more than the code for twenty-nine!)

6. Might Martians use base twelve? This means they will need twelve different symbols for writing numbers.

By the way, have you noticed that we use ten different symbols - 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 - which we call *digits*. (We call our fingers *digits* too!)