EXPLODING DOTS
CHAPTER 9

WEIRD AND WILD MACHINES

All right. It is time to go wild and crazy.

Here is a whole host of quirky and strange machines to ponder on, some yielding baffling mathematical questions still unresolved to this day! We're now well-and-truly in the territory of original thinking and new exploration. Any patterns you observe and explain could indeed be new to the world!

So go wild! Play with the different ideas presented in this chapter. Make your own extensions and variations. Most of all, have fun!

WILD IDEA 1: BASE ONE-AND-A-HALF?

Let's get weird!

> **What do you think of a $1 \leftarrow 1$ machine?**
> What happens if you put in a single dot? Is a $1 \leftarrow 1$ machine interesting? Helpful?
>
> **What do you think of a $2 \leftarrow 1$ machine?**
> What happens if you put in a single dot?
> What do you think of the utility of a $2 \leftarrow 1$ machine?

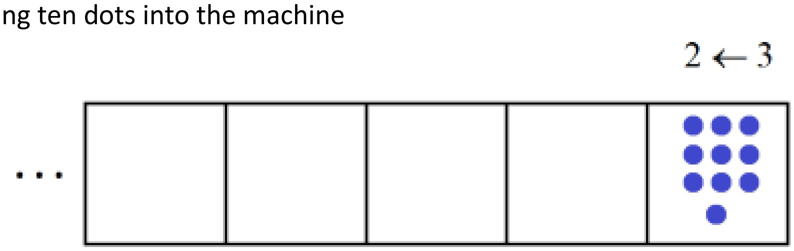
After pondering these machines for a moment you might agree there is not much one can say about them. Both fire "off to infinity" with the placement of a single dot and there is little control to be had over the situation.

How about this then?

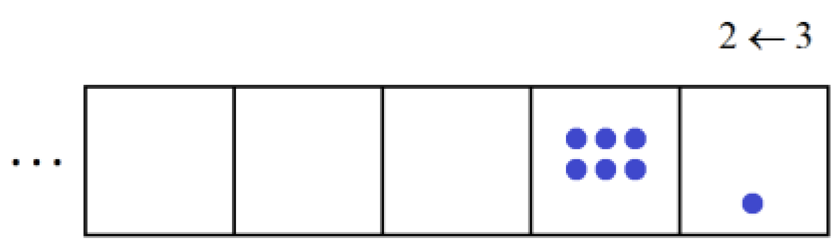
> **What do you think of a $2 \leftarrow 3$ machine?**
> *This machine replaces three dots in one box with two dots one place to their left.*

Ah! Now we're on to something. This machine seems to do interesting things.

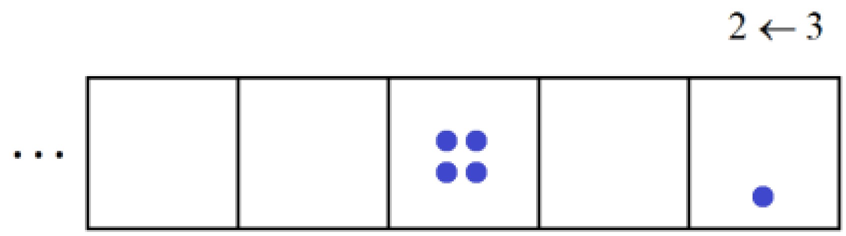
For example, placing ten dots into the machine



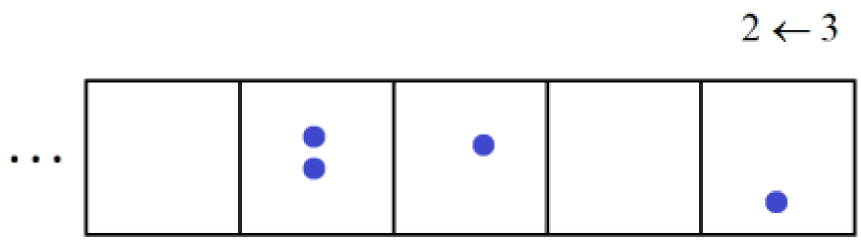
first yields three explosions,



then another two,



followed by one more.



We see the code 2101 appear for the number ten in this $2 \leftarrow 3$ machine.

In fact, here are the $2 \leftarrow 3$ codes for the first fifteen numbers. (Check these!)

1: 1	**6: 210**	**11: 2102**
2: 2	**7: 211**	**12: 2120**
3: 20	**8: 212**	**13: 2121**
4: 21	**9: 2100**	**14: 2122**
5: 22	**10: 2101**	**15: 21010**

Some beginning questions:

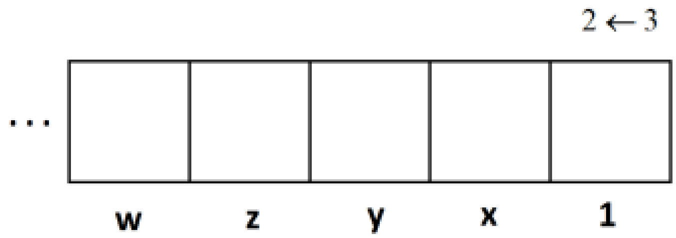
- Does it make sense that only the digits 0, 1, and 2 appear in these codes?
- Does it make sense that the final digits of these codes cycle $1, 2, 0, 1, 2, 0, 1, 2, 0, \ldots$?
- Can one do arithmetic in this weird system? For example, here is what $6+5$ looks like. Is this answer indeed eleven, which has code 2102?

$$\begin{array}{r} 210 \\ +22 \\ \hline 232 \end{array}$$

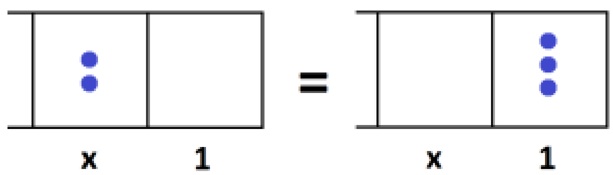
But the real question is: *What are these codes? What are we doing representing numbers this way? Are these codes for place-value in some base?*

Of course, the title of this section gives the answer away, but let's reason our way through the mathematics of this machine.

Dots in the rightmost box, as always, are each worth 1. Let's call the values of dots in the remaining boxes x, y, z, w, ….

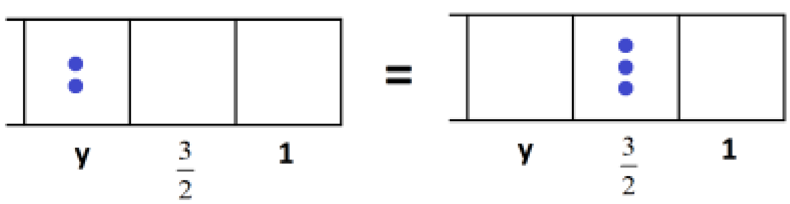


Now three dots in the 1s place are equivalent to two dots in the x place.



This tells us that $2x = 3 \cdot 1$, giving the value of $x = \frac{3}{2}$, one-and-a-half.

In the same was we see that $2y = 3 \cdot \frac{3}{2}$.



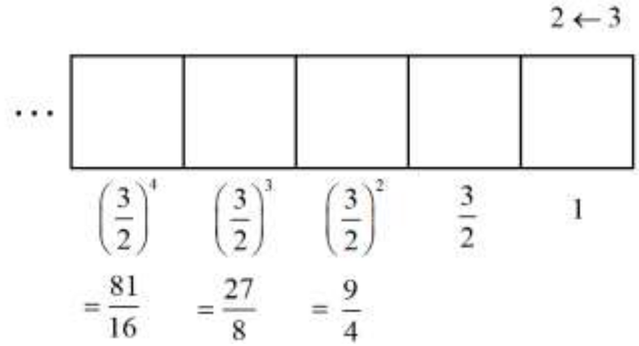
This gives $y = \frac{3}{2} \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^2$, which is $\frac{9}{4}$.

And in the same way,

$$2z = 3\left(\frac{3}{2}\right)^2 \text{ giving } z = \left(\frac{3}{2}\right)^3 \text{, which is } \frac{27}{8},$$

$$2w = 3\left(\frac{3}{2}\right)^3 \text{ giving } w = \left(\frac{3}{2}\right)^4 \text{, which is } \frac{81}{16},$$

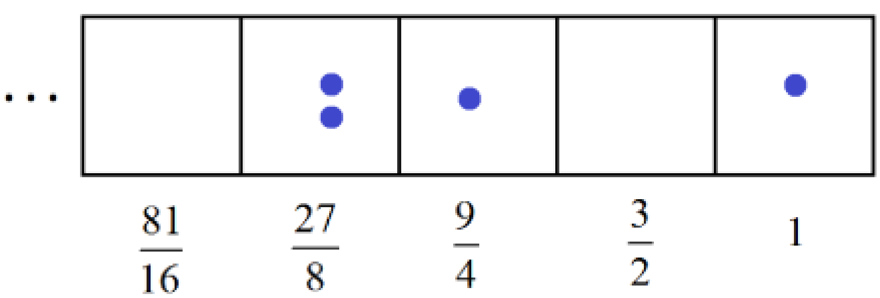
and so on. We are indeed working in something that looks like base one-and-a-half!



Comment: Members of the mathematics community might prefer not to call this base-one-a-half in a technical sense since we are using the digit "2" in our work here. This is larger than the base number. To see the language and the work currently being done along these lines, look up "beta expansions" and "non-integer representations" on the internet. In the meantime, understand that when I refer with "base one-and-a-half" in these notes I really mean "the representation of integers as sums of powers of one-and-a-half using the coefficients 0, 1, and 2." That is, I am referring to the mathematics that arises from this particular $2 \leftarrow 3$ machine.

I personally find this version of base one-and-a-half intuitively alarming! We are saying that each integer can be represented as a combination of the fractions 1, $\frac{3}{2}$, $\frac{9}{4}$, $\frac{27}{8}$, $\frac{81}{16}$, and so on. These are ghastly fractions!

For example, we saw that the number ten has the code 2101.



Is it true that this combination of fractions, $2\times\frac{27}{8}+1\times\frac{9}{4}+0\times\frac{3}{2}+1\times1$, turns out to be the perfect whole number ten? Yes! And to that, I say: whoa!

There are plenty of questions to be asked about numbers in this $2 \leftarrow 3$ machine version of base one-and-a-half, and many represent unsolved research issues of today! For reference, here are the codes to the first forty numbers in a $2 \leftarrow 3$ machine (along with zero at the beginning).

0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2101120
2101	21202	212020	2101121

QUESTION 1: PATTERNS?

Are there any interesting patterns to these representations?
Why must all the representations (after the first) begin with the digit 2?
Do all the representations six and beyond begin with 21?
If you go along the list far enough do the first three digits of the numbers become "stable"?
What can you say about final digits? Last two final digits?

Comment: Dr. Jim Propp of UMass Lowell, who opened my eyes to the $2 \leftarrow 3$ machine suggests these more robust questions.

What sequences can appear at the beginning of infinitely many $2 \leftarrow 3$ machine codes?
What sequences can appear at the end of infinitely many $2 \leftarrow 3$ machine codes?
What sequences can appear somewhere in the middle of infinitely many $2 \leftarrow 3$ machine codes?

QUESTION 2: EVEN NUMBERS?

Look the list of the first forty $2 \leftarrow 3$ codes of numbers. One sees the following "divisibility rule" for three.

A number written in $2 \leftarrow 3$ code is divisible by three precisely when its final digit is zero.

What is a divisibility rule for the number two for numbers written in $2 \leftarrow 3$ code? What common feature does every second code have?

0			
2	2120	21221	212022
21	2122	210110	212211
210	21011	210112	2101100
212	21200	212001	2101102
2101	21202	212020	2101121

QUESTION 3: UNIQUENESS?

Prove that these $2 \leftarrow 3$ machine representations of numbers using the digits 0 , 1, and 2 are unique.

[And as an infinite number of asides: Prove that every whole number can be uniquely written as sums of powers of $\frac{7}{5}$ using the coefficients $0,1,2,3,4$. And that every whole number can be uniquely written as sums of powers of $\frac{13}{8}$ using the coefficients $0,1,2,3,4,5,6,7$. And that every whole number can be uniquely written as sums of powers of $\frac{339}{56}$ using the coefficients $0,1,2,3,..,55$. And so on!]

QUESTION 4: IS IT AN INTEGER?

Not every collection of 0 s, 1s, and 2 s will represent a whole number code in the $2 \leftarrow 3$ machine. For example, looking at the list of the first forty codes we see that 201 is skipped. This combination of powers of one-and-a-half thus is not an integer. (It's the number $5\frac{1}{2}$.)

Here's a question: *Is*

$$2102212020120020122011201102202010221020100202212$$

the code for a whole number in a $2 \leftarrow 3$ *machine?*

Of course, we can just work out the sum of powers this represents and see whether or not the result is a whole number. But that doesn't seem fun!

Is there some quick and efficient means to look as a sequence of 0 s, 1s, and 2 s and determine whether or not it corresponds to a code of a whole number? (Of course, how one defines “quick” and “efficient” is up for debate.)

QUESTION 5: NUMBER OF DIGITS

Looks again at the first forty $2 \leftarrow 3$ codes.

0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2101120
2101	21202	212020	2101121

Notice

0 gives the first one-digit code. (Some might prefer to say 1 here.)
3 gives the first two-digit code.
6 gives the first three-digit code.
9 gives the first four-digit code.

and so on.

This gives the sequence: **3, 6, 9, 15, 24, ….** (Let’s skip the questionable start.)

Are there any patterns to this sequence?

If you are thinking Fibonacci, then, sadly, you will be disappointed with the few numbers of the sequence.

$$36, 54, 81, 123, 186, 279, 420, 630, \ldots$$

A Recursive Formula.

Let a_N represent the N th number in this sequence, regarding 1 as the first one-digit answer. It is known that

$$a_{N+1} = \begin{cases} \dfrac{3a_N}{2} & \text{if } a_N \text{ is even} \\ \dfrac{3(a_N+1)}{2} & \text{if } a_N \text{ is odd.} \end{cases}$$

(If m dots are needed in the rightmost box to get a code N digits long, how many dots do we need to place into the $2 \leftarrow 3$ machine to ensure that m dots appear the second box? This will then give us a code $N+1$ digits long.)

An Explicit Formula?

Is there an explicit formula for a_N ? Is it possible to compute a_{1000} without having to compute a_{999} and a_{998} and so on before it? (This question was posed by Dr. Jim Propp.)

FURTHER: There is a lot of interest about problems involving the power of two, and three, and of three-halves. See Terry Tao's 2011 piece https://terrytao.wordpress.com/2011/08/25/the-collatz-conjecture-littlewood-offord-theory-and-powers-of-2-and-3/ , for instance.

QUESTION 6: COUNTING EXPLOSIONS

The following table shows the total number of explosions that occur in the $2 \leftarrow 3$ machine to obtain the code of each of the first forty numbers.

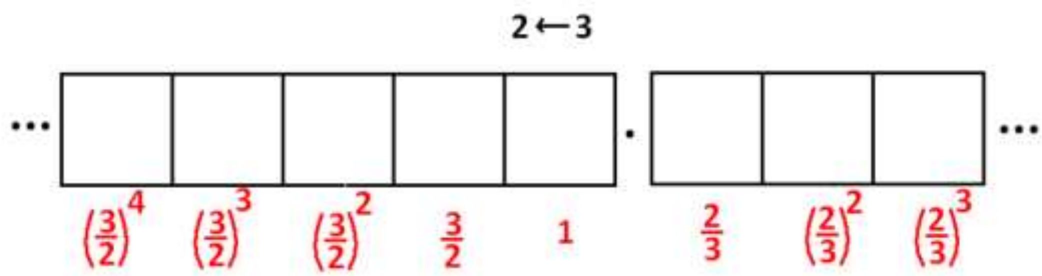
Number of Explosions

0	0						
1	0	2102	6	21220	14	212021	23
2	0	2120	7	21221	14	212022	23
20	1	2121	7	21222	14	212210	25
21	1	2122	7	210110	19	212211	25
22	1	21010	11	210111	19	212212	25
210	3	21011	11	210112	19	2101100	31
211	3	21012	11	212000	22	2101101	31
212	3	21200	13	212001	22	2101102	31
2100	6	21201	13	212002	22	2101120	32
2101	6	21202	13	212020	23	2101121	32

Any patterns?

QUESTION 7: RATIONAL DECIMAL EXPANSIONS

We can go to "decimals" in a $2 \leftarrow 3$ machine.



Here is what $\frac{1}{2}$ looks like as a "decimal" in this base. (Work out the division $1 \div 2$.)

1/2 = 0.01 000001 001 001 01 0000000001 0000001 0001 0000001 001 001 01 …

(Do you have choices to make along the way? Is this representation unique?)

Here are some more "decimal" places.

1/2 = 0.01 000001001 0010100000000001000000100001 0000001 001001 01 00000000000010001001 0000000001001 000001 00001000
0001 0001 0001 000001001 00001000000001000000001 001000000100001 00001001 0001 000001000001001 001001 001 0000100
01 000000000001000001 000001 0010100000010100000 1001 0001001 0001 0001 00100000000001 000000100000000001 00000
01 000000001001 0000000100000100010 01001 000010001010000000000000000100000000001 0001 000000010000000 1 0001 0
01 001000100001 0001001 0000001000 01 001 001001 001000000100100000000 1 00100000000010001001 001001 0001000100
100000100001 00000001…

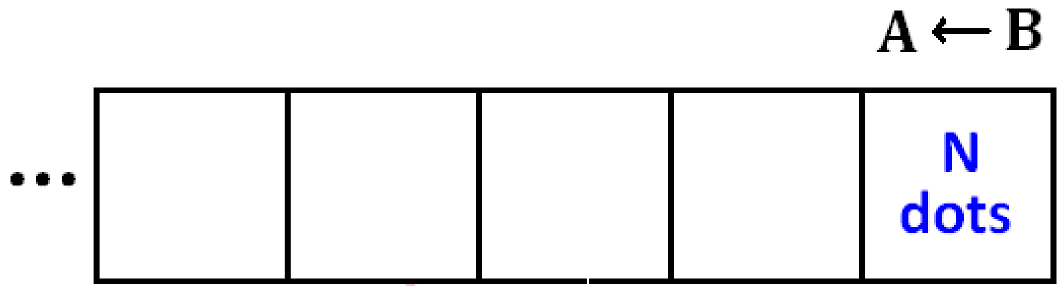
Can $\frac{1}{2}$ have a repeating "decimal" representation in a $2 \leftarrow 3$ machine?

What's a "decimal" representation for $\frac{1}{3}$ in this machine?

Develop a general theory about which fractions have repeating "decimal" representations in the $2 \leftarrow 3$ machine. (I don't personally have one!)

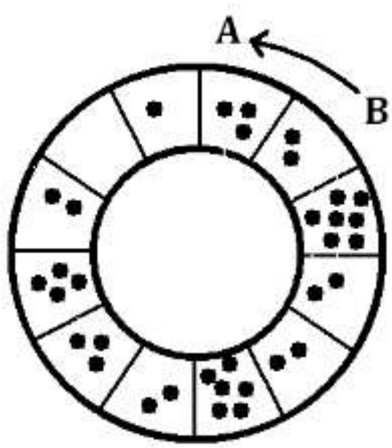
WILD IDEA 2: DOES ORDER MATTER?

Prove that the order in which one conducts explosions in a $1 \leftarrow 10$ machine, without decimals (or for any $A \leftarrow B$ machine, for that matter), does not matter. That is, for a given number of dots placed into the machine, the total number of explosions that occur will always be the same and the final distribution of dots will always be the same, no matter the order in which on chooses to conduct those explosions.



Hint: The total number of dots in the rightmost box is fixed and so the total number of explosions that occur there is fixed too. This means that the total number of dots that ever appear in the second box are fixed too, and so the total number of explosions that occur there is pre-ordained as well. And so on.

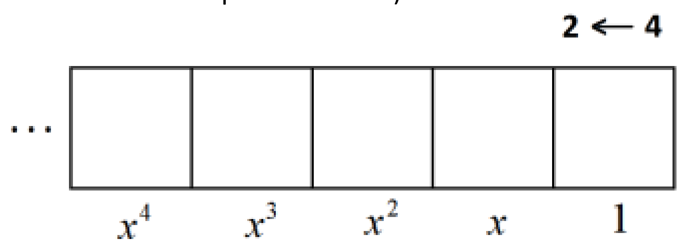
Question: The proof I outlined relies on there being a right boundary to the machine. If machines were circular, then I don't know if the order of explosions matters. It would be worth playing with this! (When can we be certain that dots will eventually land in a stable distribution?)



WILD IDEA 3: BASE TWO? BASE THREE?

BASE TWO AND BASE TWO

a) Verify that a $2 \leftarrow 4$ machine is a base-two machine. (That is, explain why $x = 2$ is the appropriate value for x in the picture below.)

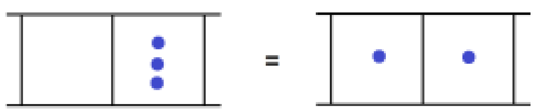


b) Write the numbers 1 through 30 as given by a $2 \leftarrow 4$ machine and as given by a $1 \leftarrow 2$ machine.

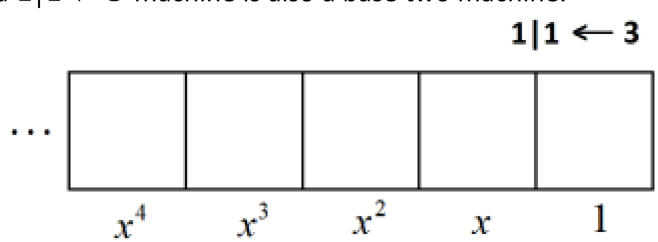
c) Does there seem to be an easy way to convert from one representation of a number to the other?

(Explore representations in $3 \leftarrow 6$ and $5 \leftarrow 10$ machines too?)

Now consider a $1|1 \leftarrow 3$ machine. Here three dots in a box are replaced by two dots: one in the original box and one one place to the left. (Weird!)



d) Verify that a $1|1 \leftarrow 3$ machine is also a base two machine.



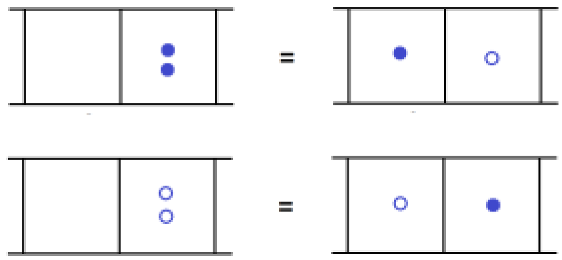
e) Write the numbers 1 through 30 as given by a $1|1 \leftarrow 3$ machine. Is there an easy way to convert the $1|1 \leftarrow 3$ representation of a number to its $1 \leftarrow 2$ representation, and vice versa?

FUN QUESTION: What is the "decimal" representation of the fraction $\frac{1}{3}$ in each of these machines?

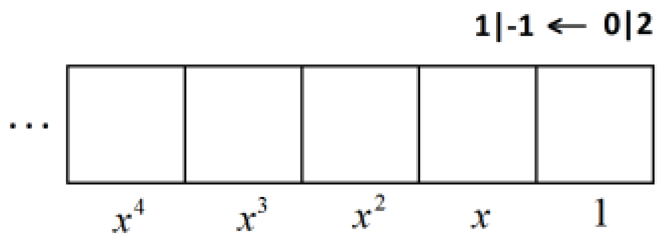
How does long division work for these machines?

A DIFFERENT BASE THREE

Here's a new type of base machine. It is called a $1\ |{-1} \leftarrow 0\ |\ 2$ machine and operates by converting any two dots in one box into an antidot in that box and a proper dot one place to the left. It also converts two antidots in one box to an antidot/dot pair.



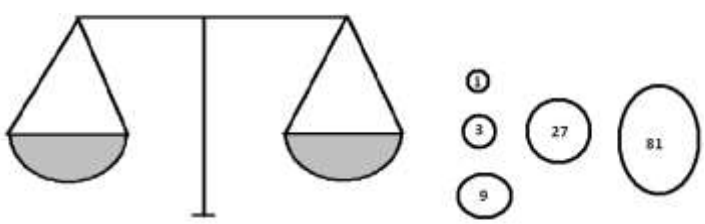
a) Show that the number twenty has representation $1|-1|1|-1$ in this machine.
b) What number has representation $1|1|0|-1$ in this machine?
c) This machine is a base machine:



Explain why x equals 3.

Thus the $1\ |{-1} \leftarrow 0\ |\ 2$ machine shows that every number can be written as a combination of powers of three using the coefficients 1, 0 and -1.

d) A woman has a simple balance scale and five stones of weights 1, 3, 9, 27 and 81 pounds.



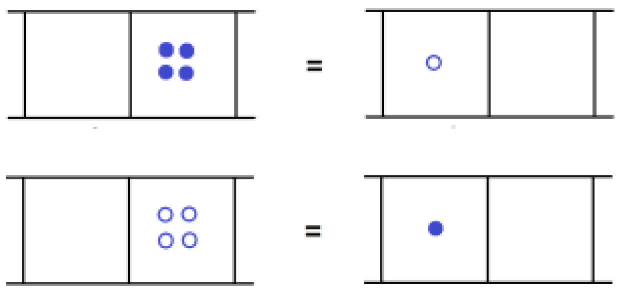
I place a rock of weight 20 pounds on one side of the scale. Explain how the women can place some, or all, of her stones on the scale so as to make it balance.

e) Suppose instead I place a 67 pound rock on the woman's scale. Can she make that stone balance too?

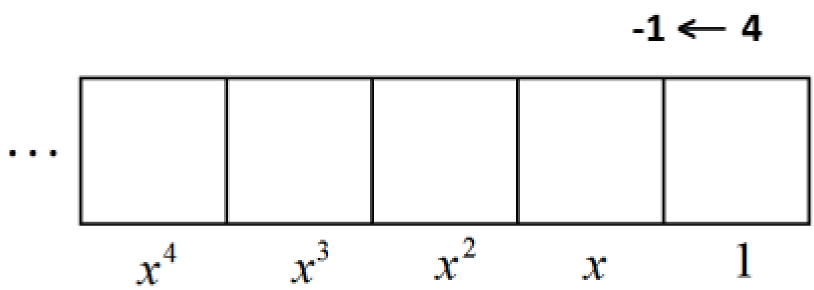
WILD IDEA 4: GOING REALLY WILD!

BASE NEGATIVE FOUR

A $-1 \leftarrow 4$ machine operates by converting any four dots in one box into an antidot one place to the left (and converts four antidots in one box to an actual dot one place to the left).



a) This machine is a base machine:



Explain why x equals -4.

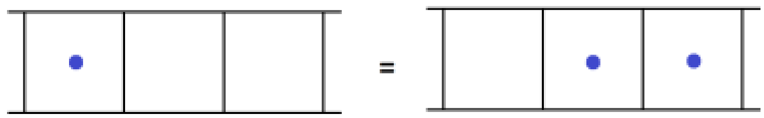
b) What is the representation of the number one hundred in this machine? What is the representation of the number negative one hundred in this machine?

c) Verify that $2 \,|\, -3 \,|\, -1 \,|\, 2$ is a representation of some number in this machine. Which number? Write down another representation for this same number.

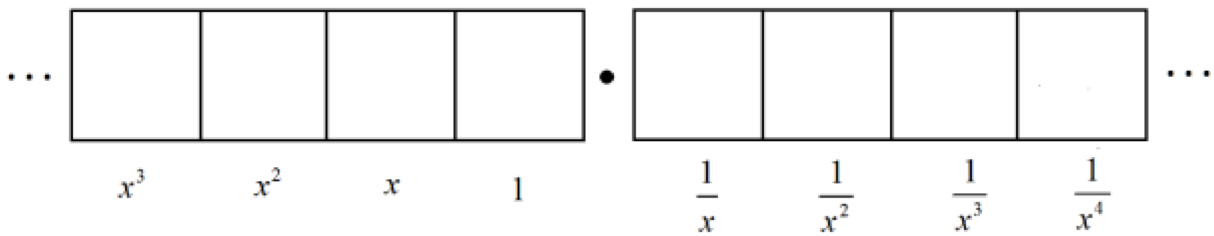
d) Write the fraction $\frac{1}{3}$ as a "decimal" in base -4 by performing long division in a $-1 \leftarrow 4$ machine. Is your answer the only way to represent $\frac{1}{3}$ in this base?

BASE PHI

Consider the very strange machine $1|0|0 \leftrightarrow 0|1|1$. Here two dots in consecutive boxes can be replaced with a single dot one place to the left of the pair and, conversely, any single dot can be replaced with a pair of consecutive dots to its right.



Since this machine can move both to the left and to the right, let's give it its full range of "decimals" as well.



a) Show that, in this machine, the number 1 can be represented as 0.10101010101…. (It can also be represented just as 1!)

b) Show that the number 2 can be represented as 10.01.

c) Show that the number 3 can be represented as 100.01.

d) Explain why each number can be represented in terms of 0s and 1s with no two consecutive 1s. (**TOUGH:** Are such representations unique?)

Let's now address the question: *What base is this machine?*

e) Show that in this machine we need $x^{n+2} = x^{n+1} + x^n$ for all n.

f) Dividing throughout by x^n this tells us that x must be a number satisfying $x^2 = x + 1$. There are two numbers that work. What is the positive number that works?

g) Represent the numbers 4 through 20 in this machine with no consecutive 1s. Any patterns?

RELATED ASIDE?

The Fibonacci numbers are given by:

1, 1, 2, 3, 5, 8, 13, 21, 34, ….

They have the property that each number is the sum of the previous two terms.

In 1939, Edouard Zeckendorf proved (and then published in 1972) that every positive integer can be written as a sum of Fibonacci numbers with no two consecutive Fibonacci numbers appearing in the sum. For example

$$17 = 13 + 3 + 1$$

and

$$46 = 34 + 8 + 3 + 1.$$

(Note that 17 also equals $8+5+3+1$ but this involves consecutive Fibonacci numbers.)

Moreover, Zeckendorf proved that the representations are unique.

Each positive integer can be written as a sum of non-consecutive Fibonacci numbers in precisely one way.

This result has the “feel” of a base machine at its base.

Is there a way to construct a base machine related to the Fibonacci numbers in some way and use it to establish Zeckendorf’s result?

Comment: Of course, one can prove Zeckendorf's result without the aid of a base machine. (To prove that a number N has a Zeckendorf representation adopt a "greedy" approach: subtract the largest Fibonacci number smaller than N from it, and repeat. To prove uniqueness, set two supposed different representations of the same number equal to each other and cancel matching Fibonacci numbers. Use the relation $F(n+2) = F(n+1) + F(n)$ to keep canceling.) It would be lovely, however, to see a visual proof of the result via the mechanics of a machine.

FINAL THOUGHTS

Invent other crazy machines …

Invent $a\,|\,b\,|\,c \leftrightarrow d\,|\,e\,|\,f$ machines for some wild numbers a,b,c,d,e,f .

Invent a base half machine.

Invent a base negative two-thirds machine.

Invent a machine that has one rule for boxes in even positions and a different rule for boxes in odd positions.

Invent a base i machine or some other complex number machine.

How does long division work in your crazy machine?

What is the fraction $\frac{1}{3}$ in your crazy machine?

Do numbers have unique representations in your machines or multiple representations?

Go wild and see what crazy mathematics you can discover!