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# EXPLODING DOTS

***ONE DOCUMENT WITH EVERYTHING IN ONE PLACE***
***for a coherent self-guided learning experience***

**JUST WATCH THE VIDEOS AND IGNORE THIS DOCUMENT**

OR

**WATCH THE VIDEOS AND WORK THROUGH THIS DOCUMENT**

OR

**IGNORE THE VIDEOS AND JUST HAVE FUN WITH THIS DOCUMENT**

# DOCUMENT CONTENT

# INSTRUCTIONS

1. **Start on PAGE 4 and think about the puzzle presented there.**

   Don't fret about trying to explain it – just play with it a bit and then let it sit at the back of your mind.

2. **Start the EXPLODING DOTS story on page 6.**

   Watch the two videos. That might be enough to entice you to keep watching the videos and the material that follow it on that site. If that is the case, ignore these notes and just keep watching the videos on that site.

   Have a fun experience watching videos! Check out the web app too.

3. **If you want, after watching the videos, read through the text in these notes and try some or all of the practice problems – as many as you feel like.**

   You can check your answers in the solutions section of these notes.

**Just enjoy having a mathematical experience with a topic that has swept the globe. Well over 5.9 million students from over 150 countries on this globe have played with the stunning story of EXPLODING DOTS. Enjoy it too!**

*Tanzania, 2018*

# 1. PLACE-VALUE MACHINES

## A MIND-READING TRICK

Here's a classic trick you can perform with a friend.

Write on a board or on a piece of paper the following five groups of numbers for all to see.

| GROUP A | GROUP B | GROUP C | GROUP D | GROUP E |
|---|---|---|---|---|
| 16 20 24 28 | 8 12 24 28 | 4 12 20 28 | 2 10 18 26 | 1 9 17 25 |
| 17 21 25 29 | 9 13 25 29 | 5 13 21 29 | 3 11 19 27 | 3 11 19 27 |
| 18 22 26 30 | 10 14 26 30 | 6 14 22 30 | 6 14 22 30 | 5 13 21 29 |
| 19 23 27 31 | 11 15 27 31 | 7 15 23 31 | 7 15 23 31 | 7 15 23 31 |

Ask your friend to silently think of a number between 1 and 31. (She or he may choose the number 1 itself if they like, or the number 31.)

Now perform the mindreading trick by having the following conversation with your friend.

| | |
|---|---|
| "Suzzy. Is the number you are thinking of in group A?" | "Yes." |
| "Is the number you are thinking of in group B?" | "Yes." |
| "Is it in group C?" | "No." |
| "Is it in group D?" | "No." |
| "Group E?" | "Yes!" |
| "Ahh … your number is 25." | "Wow! Yes it is!" |

Repeat the game a few more with your friend noting each time which groups elicit a "yes" answer as the game is played. The secret number they have in mind is simply the sum of the top-left corner numbers in each group with a yes answer. For example, Suzzy answered YES YES NO NO YES. Groups A, B, and E have top left numbers 16, 8, and 1, respectively, and indeed $16+8+1=25$.

| GROUP A | GROUP B | GROUP C | GROUP D | GROUP E |
|---|---|---|---|---|
| 16 20 24 28 | 8 12 24 28 | 4 12 20 28 | 2 10 18 26 | 1 9 17 25 |
| 17 21 25 29 | 9 13 25 29 | 5 13 21 29 | 3 11 19 27 | 3 11 19 27 |
| 18 22 26 30 | 10 14 26 30 | 6 14 22 30 | 6 14 22 30 | 5 13 21 29 |
| 19 23 27 31 | 11 15 27 31 | 7 15 23 31 | 7 15 23 31 | 7 15 23 31 |
| YES | YES | NO | NO | YES |

*Suzzy's responses for the number 25.*

As practice, check that if Sameer is thinking the number 13, he will answer NO YES YES NO YES and indeed $8+4+1=13$.

Invite your friend to figure out what you are doing. And then, can you and your friends figure out why this trick works? What's special about the top-left numbers of each group?

## Some Things You Might Notice or Question

1. It is curious that group E contains all the odd numbers.
2. Group A contains all the numbers 16 and above.
3. What's up with the number 31? Why must we choose between 1 and 31?
4. There is definitely a pattern to the numbers in the top-left corner of each card.

As we move through this chapter, keep this puzzle in mind. Its secrets will unfold.

## A STORY OF STRANGE MACHINES

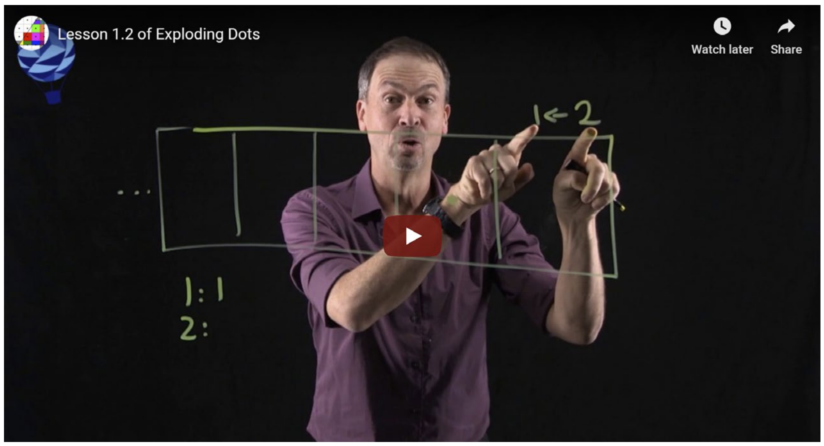
Videos:

Introductory video.

The first machine video (+ all follow-on content you see!)

Web App:

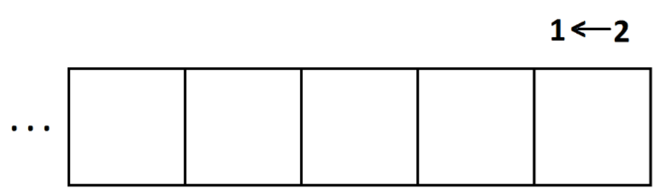
First Experience.



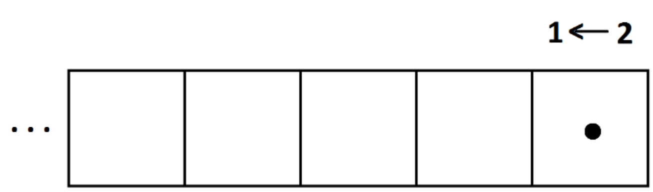

Here's a story that is not true.

When I was a child I, James, invented a machine – not true – and this machine is nothing more than a row of boxes that extends as far to the left as I could ever desire.

I gave this machine of mine a name. I called it a "two-one machine" both written and read in a funny backwards way. (I knew no different as a child.)

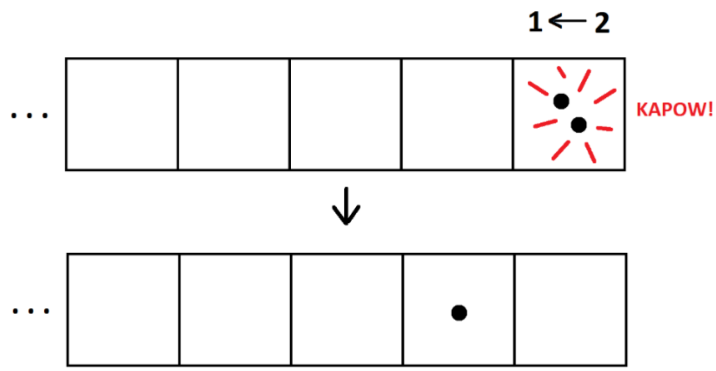



And what do you do with this machine? You put in dots. Dots always go into the rightmost box. Put in one dot, and, well, nothing happens: it stays there as one dot. Ho hum!



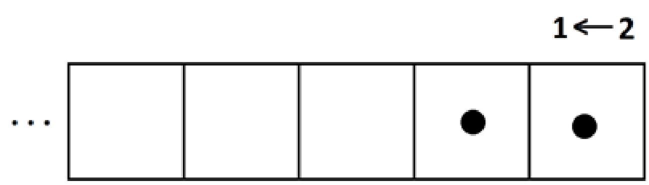
But put in a second dot – always in the rightmost box – and then something exciting happens.

> *Whenever there are two dots in a box they explode and disappear – KAPOW! – to be replaced by one dot, one box to the left.*




(Do you see now why I called this a "1 ⟵ 2 machine" written in this funny way?)

We see that two dots placed into the machine yields one dot followed by zero dots.
Putting in a third dot – always the rightmost box – gives the picture one dot followed by one dot.

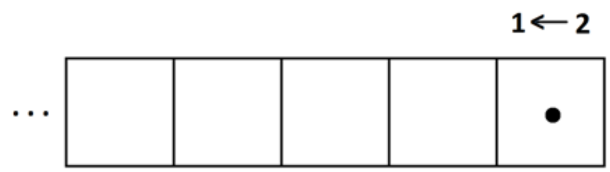



I realized that this machine, in my untrue story, was giving codes for numbers.

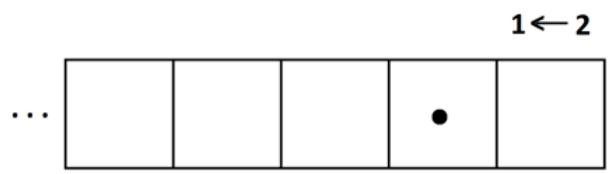
Just one dot placed in the machine, stayed as one dot. Let's say that that the 1 ⟵ 2 machine code for the number one is 1.

Two dots placed into the machine, one after the other, yielded one dot in a box followed by zero dots. Let's say that the 1 ⟵ 2 machine code for the number two is 10.

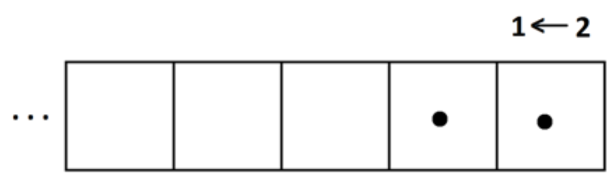
Putting a third dot in the machine gives the code 11 for three.




**1:** **1**



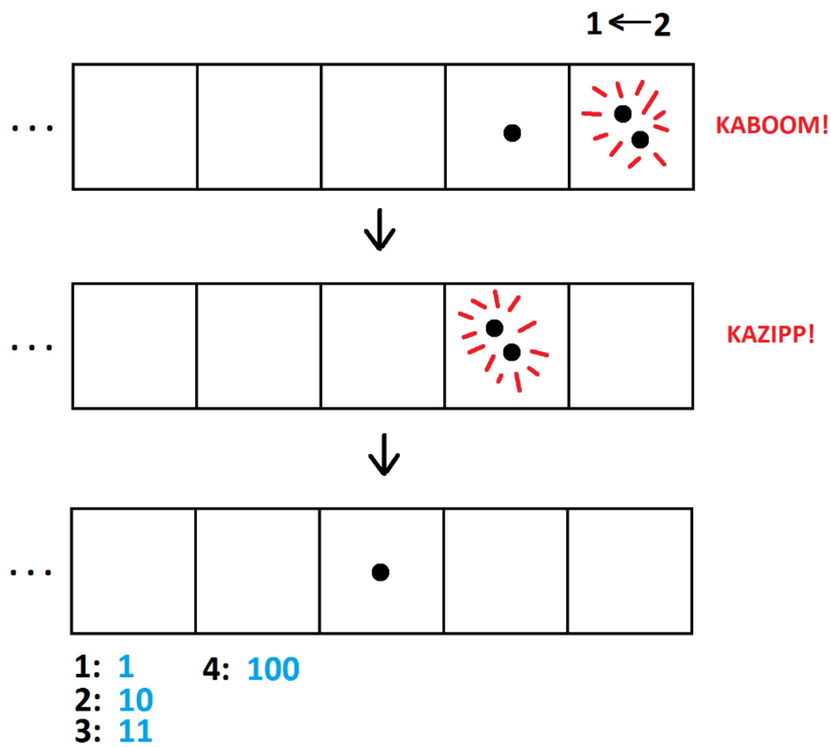

**2:** **10**




**3:** **11**

**Question***: What's the 1 ← 2 machine code for four?*

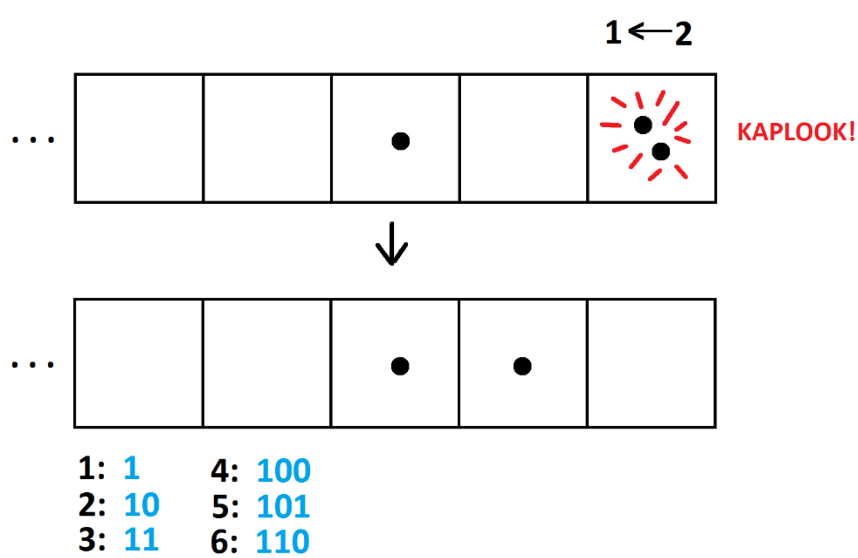
Putting a fourth dot into the machine is particularly exciting: we are in for multiple explosions!




The 1 ← 2 code for four is 100.

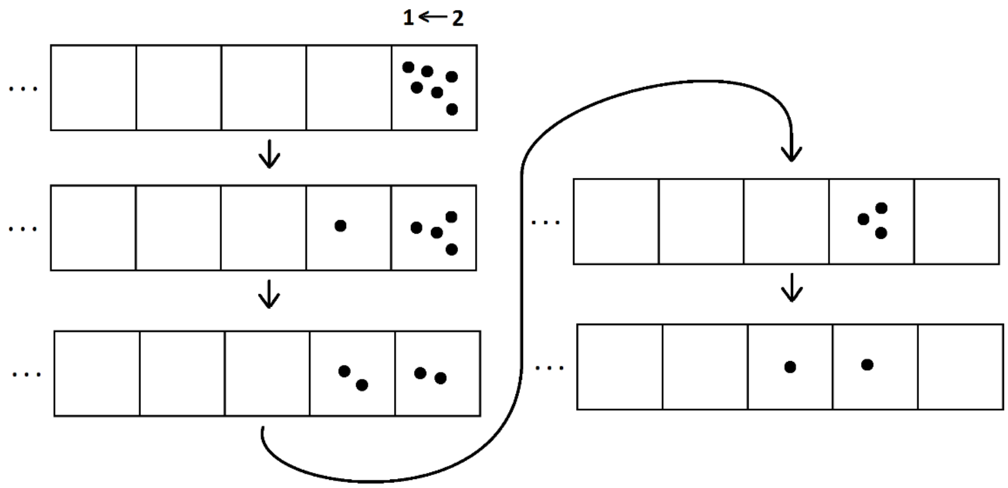
**Question:** *What will be the code for five? Can you see it's 101?*

And the code for six? Adding one more dot to the code for five gives 110 for six.



Actually, we can also get this code for six by clearing the machine and then putting in six dots all at once. Pairs of dots will explode in turn to each produce one dot, one box to their left.

Here is one possible series of explosions. Sound effects omitted!




**Question:** *Do you get the same final code of 110 if you perform explosions in a different order? (Try it!)*

Here are some practice questions you might or might not want to try. Solutions are at the end of this document.

**Practice 1:**

a) What is the 1 ← 2 machine code for the number thirteen? (It turns out to be 1101. Can you get that answer?)

b) What is the code for fifty in this machine? (Whoa!)

**Practice 2:** Could a number ever have code 100211 in a 1 ← 2 machine, assuming we always choose to explode dots if we can?

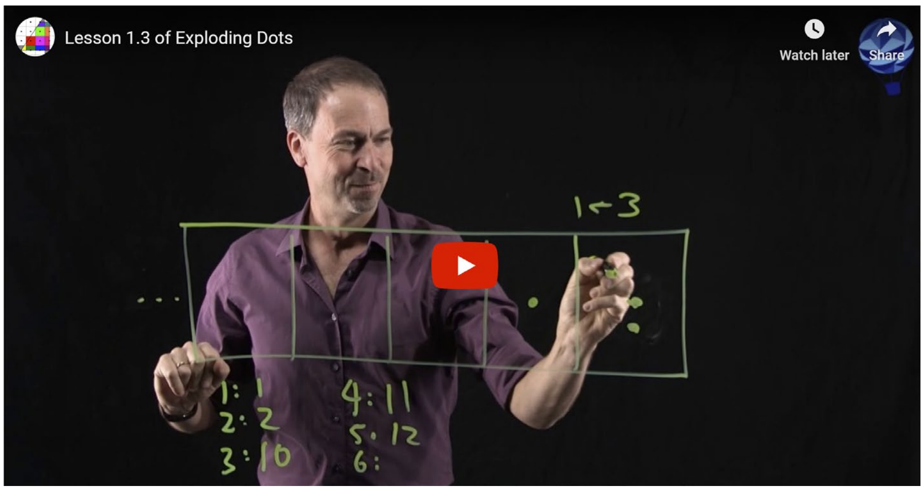
**Practice 3 (Challenge):** Which number has code 10011 in a 1 ← 2 machine?

There are hours of fun to be had playing with codes in a 1 ← 2 machine.

## OTHER MACHINES

Video:
Other machines video (+ all follow-on content you see).

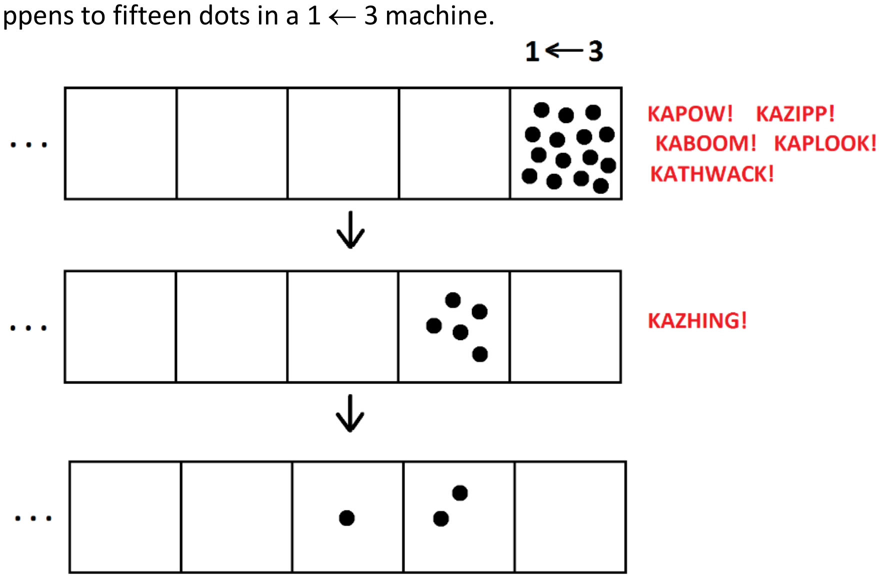
Web App:
First Experience.




But then one day, I had an astounding flash of insight!

Instead of playing with a 1 ← 2 machine, I realized I could play with a 1 ← 3 machine (again written and read backwards, a "three-one "machine). Now whenever there are *three* dots in a box, they explode away to be replaced with one dot, one box to the left.

Here's what happens to fifteen dots in a 1 ← 3 machine.



First there are five explosions in the first box, with each explosion making a dot in the second box to the left. Then three of those dots explode away. This leaves behind two dots and makes one new dot one place to the left. We thus see the code 120 for fifteen in a $1 \leftarrow 3$ machine.

**Question:** *What is the $1 \leftarrow 3$ machine code for the number thirteen?*

There are hours of fun to be had working out the codes for numbers in a $1 \leftarrow 3$ machine.

But then I had another flash if insight. Instead of doing a $1 \leftarrow 3$ machine, I realized I could do a $1 \leftarrow 4$ machine, or a $1 \leftarrow 5$ machine, or any numbered machine I like!

Try some of these practice problems if you like.

**Practice 4:**

a) Show that the code for four in a $1 \leftarrow 3$ machine is 11.
b) Show that the code for thirteen in a $1 \leftarrow 3$ machine is 111.
c) Show that the code for twenty in a $1 \leftarrow 3$ machine is 202.

**Practice 5:** Could a number have code 2041 in a $1 \leftarrow 3$ machine? If so, would the code be "stable"?

**Practice 6:** Which number has code 1022 in a $1 \leftarrow 3$ machine?

We can keep going!

**Practice 7:** What do you think rule is for a $1 \leftarrow 4$ machine?

**Practice 8:** What is the $1 \leftarrow 4$ code for the number thirteen?

**Practice 9:** What is the $1 \leftarrow 5$ code for the number thirteen?

**Practice 10:** What is the $1 \leftarrow 9$ code for the number thirteen?

**Practice 11:** What is the $1 \leftarrow 5$ code for the number twelve?

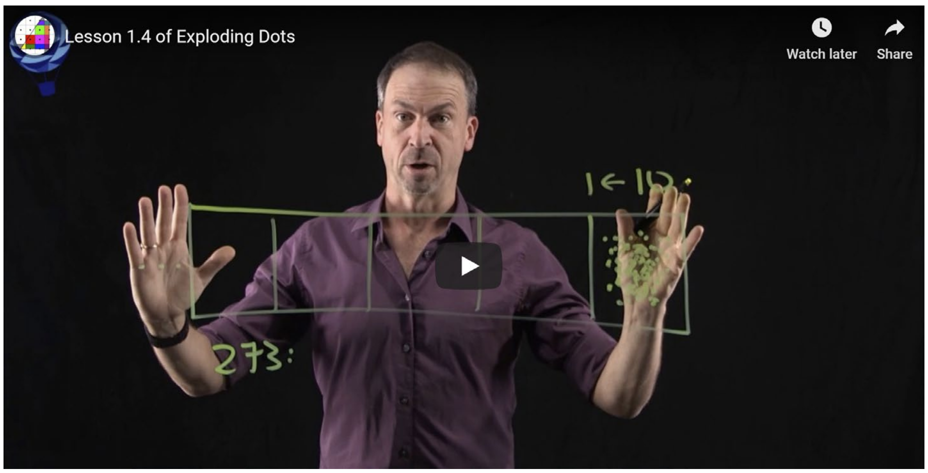
**Practice 12:** What is the $1 \leftarrow 9$ code for the number twenty?

**Practice 13:**

a) What is the $1 \leftarrow 10$ code for the number thirteen?
b) What is the $1 \leftarrow 10$ code for the number thirty-seven?
c) What is the $1 \leftarrow 10$ code for the number 5846?

## THE $1 \leftarrow 10$ MACHINE

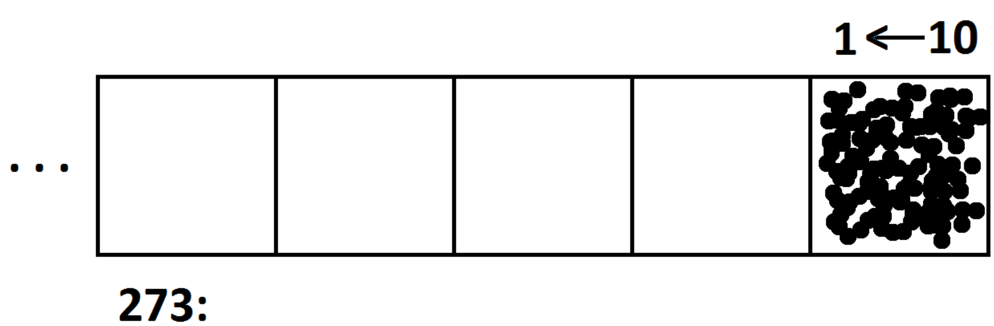
| |
|---|
| Video:<br>The 1 ← 10 machine [video](#) (+ all follow-on content you see).<br><br>Web App:<br>[First Experience](#). |




Okay. Let's now go wild.

Let's go all the way up to a 1 ← 10 machine and put in 273 dots in a 1 ← 10 machine!

What is the secret 1 ← 10 code for the number 273?



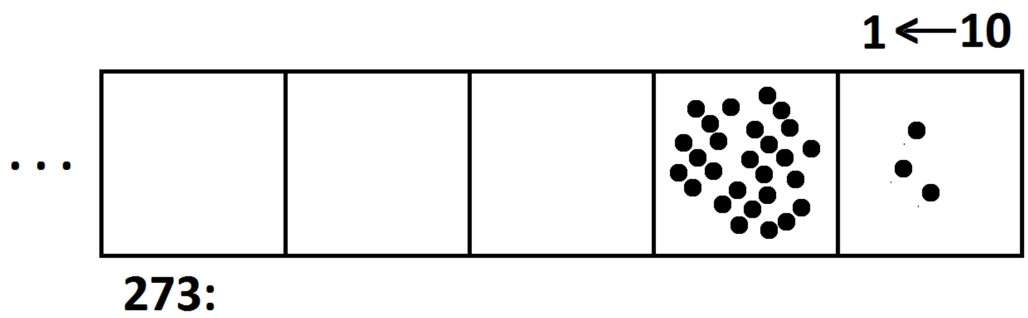

I thought my way through this by asking a series of questions.

*Will there be any explosions? Are there any groups of ten that will explode?* Certainly!

*How many explosions will there be initially?* Twenty-seven.

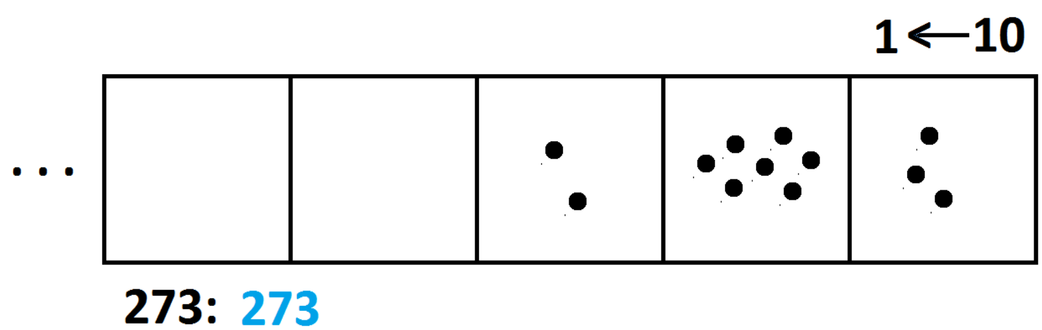
*Any dots left behind?* Yes. Three.

Okay. So, there are twenty-seven explosions, each making one dot one place to the left, leaving three dots behind.




*Any more explosions?* Yes. Two more.

*Any dots left behind?* Seven left behind.




The $1 \leftarrow 10$ code for two hundred seventy-three is…273. Whoa!

Something curious is going on!

What is the natural big question to now ask?

**WILD EXPLORATIONS**

Here are some "big question" investigations you might want to explore, or just think about before moving on. All will become clear as the story unfolds in further chapters, but it could be fun to mull on these ideas now.

**Wild Exploration 1:** Can you indeed figure out what these machines are actually doing?

Why is the code for two hundred and seventy-three in a $1 \leftarrow 10$ machine, "273"? Are all the codes for numbers in a $1 \leftarrow 10$ sure to be identical to how we normally write numbers?

If you can answer that question, can you then also make sense of all the codes for a $1 \leftarrow 2$ machine? What does the code 1101 for the number thirteen mean?

[All this will be explained in the next section.]

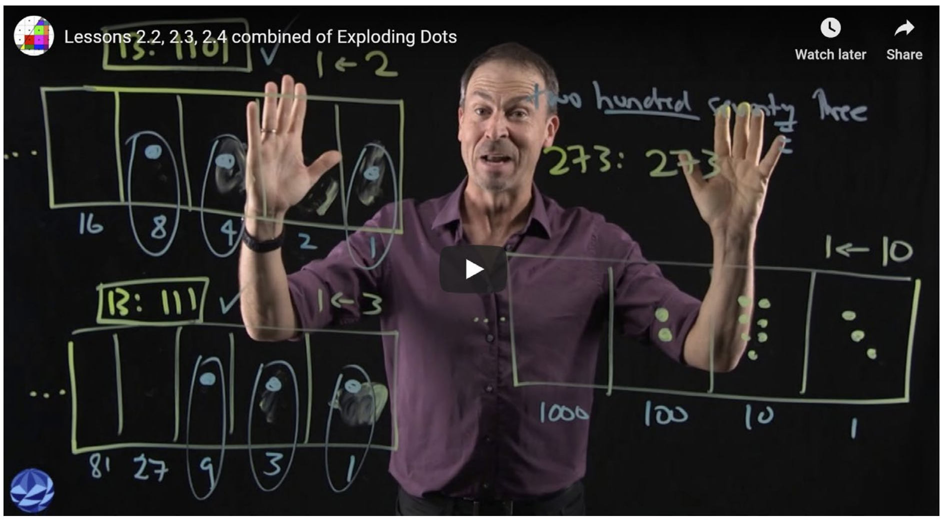
**Wild Exploration 2:** Put nineteen dots into the rightmost box of a $1 \leftarrow 2$ machine and explode pairs of dots in a haphazard manner: explode a few pairs in the right most box, and then some in the second box, and then a few more in the rightmost box, and then some in the second box again, and so on.

Do it again, this time changing the order in which you do explosions. And then again!

Does the same final code of 10011 appear each, and every time?

## EXPLAINING THE MACHINES

| |
|---|
| Videos:<br>Introductory video.<br>Explaining the machines video (+ all follow-on content you see).<br><br>Web App:<br>Second Experience. |

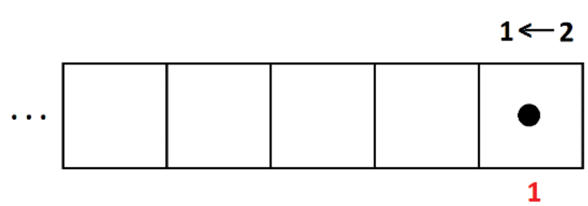



All right. It's time to explain what the machines are really doing. (Have you already figured it all out?)

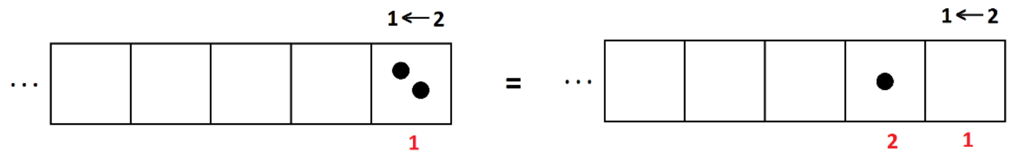
Let's go back to the 1 ← 2 machine and first make sense of that curious device. Recall that it follows the rule:

> *Whenever there are two dots in any one box they "explode," that is, disappear, and are replaced by one dot, one place to their left.*

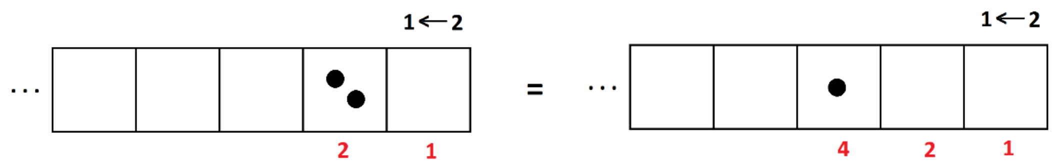
And this machine is set up so that dots in the rightmost box are always worth one.



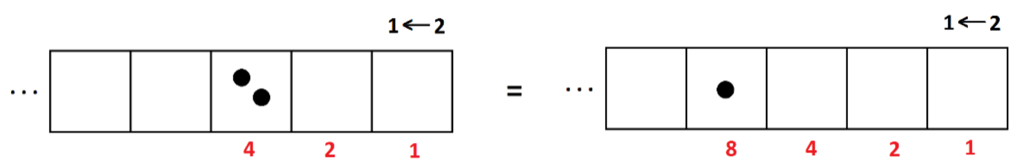
With an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1's, that is, 2.




And two dots in this second box is equivalent to one dot, one place to the left. Such a dot must be worth two 2's, that is, worth 4.

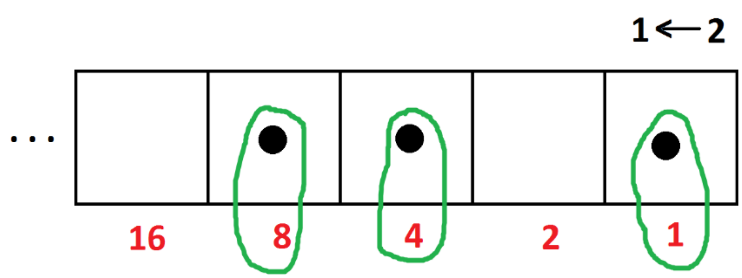



And two 4's makes 8 for the value of a dot the next box over.




And two 8's make 16, and two 16's make 32, and two 32's make 64, and so on.

We saw earlier that the code for thirteen in a 1 ← 2 machine is 1101. Now we can see that this is absolutely correct: one 8 and one 4 and no 2's and one 1 does indeed make thirteen.

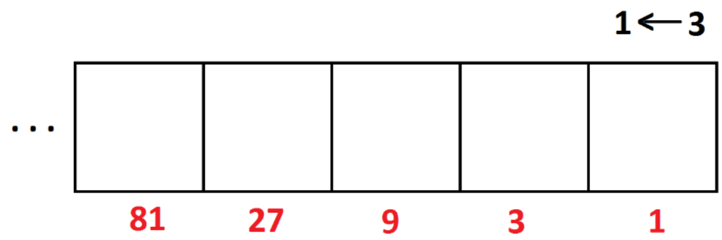



People call the 1 ← 2 codes for numbers the *binary* representations of numbers (with the prefix *bi-* meaning "two"). They are also called *base two* representations. One only ever uses the two symbols 0 and 1 when writing numbers in binary.

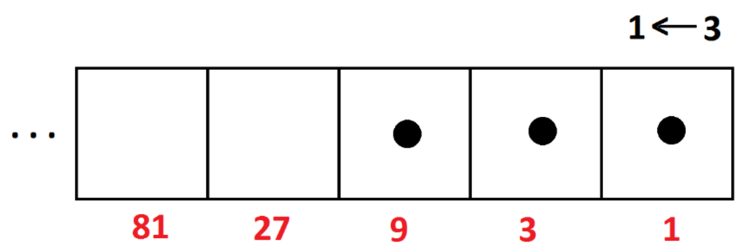
> **Comment:** Computers are built on electrical switches that are either on, or off. So, it is very natural in computer science to encode all arithmetic in a code that uses only two symbols: say 1 for "on" and 0 for "off." Thus, base two, binary, is the right base to use in computer science.

Now to the next machine.

In a 1 ← 3 machine, three dots in any one box are equivalent to one dot, one place to the left. (And each dot in the rightmost box is again worth 1.) We get the dot values in this machine by noting that three 1's is 3, and three 3's is 9, and three 9's is 27, and so on.




At one point, we said that the 1 ← 3 code for thirteen is 111. And we see that this is correct: one 9 and one 3 and one 1 does indeed make thirteen.

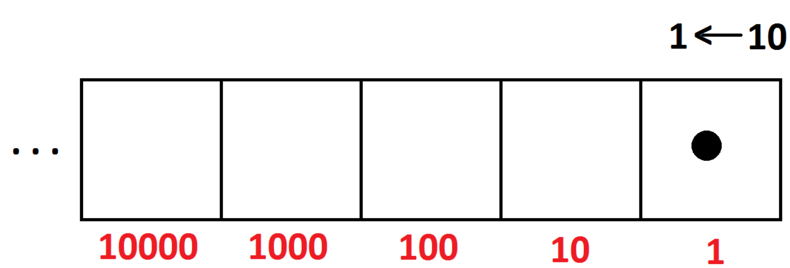



The 1 ← 3 machine codes for numbers are called *ternary* or *base three* representations of numbers. Only the three symbols 0, 1, and 2 are ever needed to represent numbers in this system.

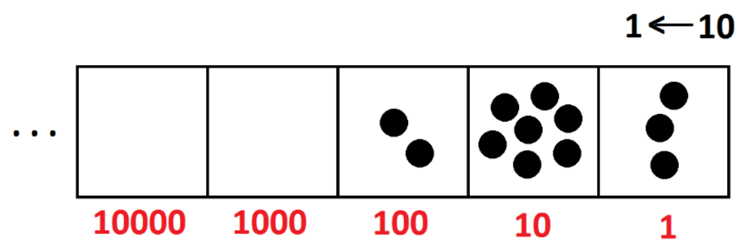
> **Comment:** Scientists are discussing the idea of building optic computers based on polarized light: either light travels in one plane, or in a perpendicular plane, or there is no light. For these computers, base three arithmetic would be the appropriate notational system to use.

### We Speak $1 \leftarrow 10$ Machine

And finally, for a 1 ← 10 machine, we see that ten ones makes 10, ten tens makes 100, ten one-hundreds makes 1000, and so on. A 1 ← 10 has ones, tens, hundreds, thousands, and so on, as dot values.




We saw that the code for the number 273 in a 1 ← 10 machine is 273, and this is absolutely correct: 273 is two hundreds, seven tens, and three ones.




**273: 273**

In fact, we even speak the language of a 1 ← 10 machine. When we write 273 in words, we write

**273 = two hundred seventy three**

We literally say, in English at least, two HUNDREDS and seven TENS (that "ty" is short for "ten") and three.

So, through this untrue story of dots and boxes we have discovered *place-value* and *number bases*: base two, base three, base ten, and so on. And society has decided to speak the language of base ten machine.

> **Question:** *Why do you think we humans have a predilection for the 1 ← 10 machine? Why do we like the number ten for counting?*

One answer could be because of our human physiology: we are born with ten digits on our hands. Many historians do believe this could well be the reason why we humans have favored base ten.

> **Question:** *There are some cultures on this planet that have used base twenty. Why might they have chosen that number do you think?*

In fact, there are vestiges of base twenty thinking in western European culture of today. For example, in French, the number 87 is spoken and written as *quatre-vingt-sept*, which translates, word for word, as "four twenties seven." In the U.S. the famous Gettysburg address begins: "Four score and seven years ago." That's four-twenties and seven years ago.

> **Question:** *I happen to know that Martians have four fingers on each of two hands. What base do you think they might use in their society?* (Probably base eight?)

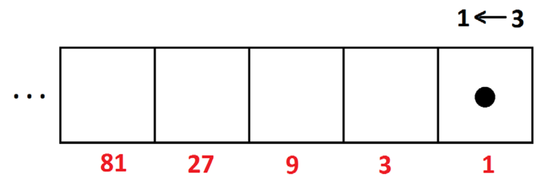
All right. The point of this lesson has been made. We have discovered base-ten place value for writing numbers and seen their context in the whole story of place value. We humans happen to like base ten in particular, because it matches is the number of fingers most of us seem to have.

Here are some practice questions you might or might not want to try.

**Practice 13:** What number has 1 ← 2 machine code 100101?

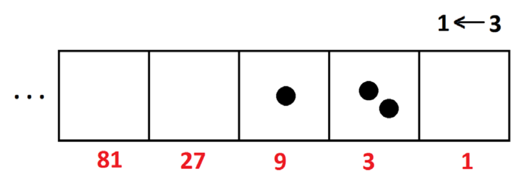
**Practice 14:** What is the 1 ← 2 machine code for the number two hundred?

**Practice 15:** In a 1 ← 3 machine, three dots in any one box are equivalent to one dot one place to the left. (And each dot in the rightmost box is again worth 1.) We get the dot values in this machine by noting that three 1's is 3, and three 3's is 9, and three 9's is 27, and so on.




a) What is the value of a dot in the next box to the left after the ones shown?

b) The 1 ← 3 machine code for fifteen is 120. We see that this is correct as one 9 and two 3's does indeed make fifteen.



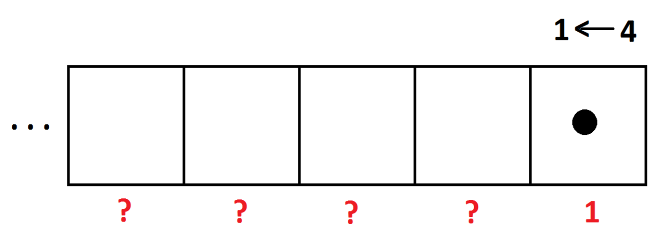

Could we also say that the 1 ← 3 code for fifteen is 0120? That is, is it okay to put zeros in the front of these codes? What about zeros at the ends of codes? Are they optional? Is it okay to leave off the last zero of the code 120 for fifteen and just write instead 12?

c) What number has 1 ← 3 machine code 21002?

d) What is the 1 ← 3 machine code for two hundred?

**Practice 16:**

a) In the 1 ← 4 system four dots in any one box are equivalent to one dot, one place to their left. What is the value of a dot in each box?




b) What is the 1 ← 4 machine code for twenty-nine?

c) What number has 132 as its 1 ← 4 machine code?

**Practice 17:** I happen to know that *Venutians* have six fingers on each of two hands. What base do you think they might use in their society?

## WILD EXPLORATIONS

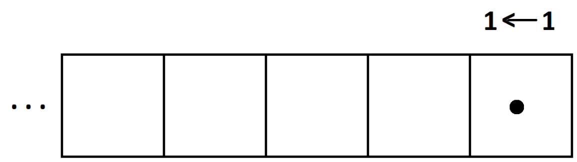
Here are some "big question" investigations you might want to explore, or just think about before moving on. Enjoy!

### Wild Exploration 1: THE MIND-READING TRICK

Can you explain it now by looking at $1 \leftarrow 2$ machine codes?

### Wild Exploration 2: CAN MACHINES GO THE OTHER WAY?

Jay decides to play with a machine that follows a $1 \leftarrow 1$ rule. He puts one dot into the right-most box. What happens? Do assume there are infinitely many boxes to the left.




Suggi plays with a machine following the rule $2 \leftarrow 1$. She puts one dot into the right-most box. What happens for her?

Do you think these machines are interesting? Is there much to study about them?

### Wild Exploration 3: CAN WE PLAY WITH WEIRD MACHINES?

Poindexter decides to play with a machine that follows the rule $2 \leftarrow 3$.

a) Describe what happens when there are three dots in a box.
b) Work out the $2 \leftarrow 3$ machine codes for the numbers 1 up to 30. Any patterns?
c) The code for ten in this machine turns out to be 2101. Look at your code for twenty. Can you see it as the answer to "ten plus ten"? Does your code for thirty look like the answer to "ten plus ten plus ten"?

# 2. BASE TEN ARITHMETIC

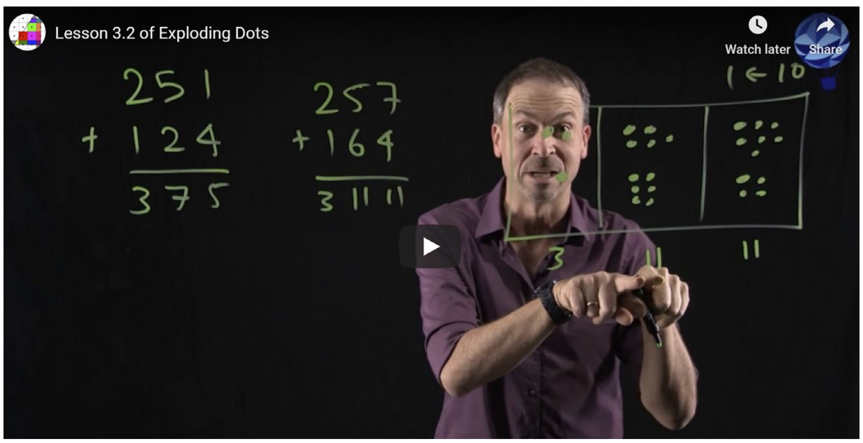
Videos:

Introductory video.

Addition video (+ all follow-on content you see).

Web App:

Third Experience.




Society loves working in base ten. So, let's stay with a $1 \leftarrow 10$ machine for a while and make good sense of all the arithmetic we typically learn in school.

We have just seen how to write numbers. What is the first mathematical thing students learn to do with numbers, once they know how to write them? It's usually addition.

Here's an addition problem.

*Compute* $251 + 124$.

Such a problem is usually set up this way.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline \end{array}$$

This addition problem is fine to compute: $2 + 1$ is $3$, $5 + 2$ is $7$, and $1 + 4$ is $5$. The answer $375$ appears.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline 375 \end{array}$$

But did you notice something curious just then?

Yes. I worked from left to right just as I was taught to read. I guess this is opposite to what most people are taught to do in a mathematics class: go right to left.

But does it matter? Do you get the same answer 375 if you go right to left instead? Yes!

So why then are we taught to work right to left in mathematics classes?

Many people suggest that the problem we just did is “too nice.” We should do a more awkward addition problem, one like $358 + 287$.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

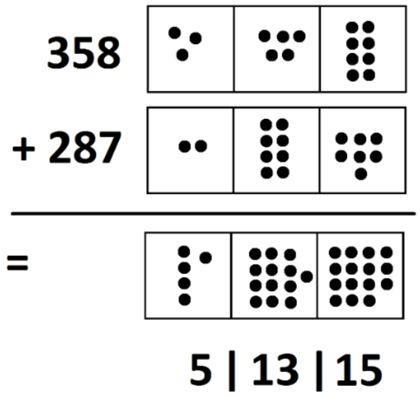
Okay. Let’s do it!
If we go from left to right again we get $3 + 2$ is $5$; $5 + 8$ is $13$; and $8 + 7$ is $15$. The answer “five-hundred thirteen-ty fifteen” appears. (Remember, “ty” is short for *ten*.)

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

$$5\,|\,13\,|\,15$$

And this answer is absolutely, mathematically correct! You can see it is correct in a $1 \leftarrow 10$ machine.

Here are 358 and 287.




Adding 3 hundreds and 2 hundreds really does give 5 hundreds.

Adding 5 tens and 8 tens really does give 13 tens.

Adding 8 ones and 7 ones really does give 15 ones.

"Five-hundred thirteen-ty fifteen" is absolutely correct as an answer – and I even said it correctly. We really do have 5 hundreds, 13 tens, and 15 ones. There is nothing mathematically wrong with this answer. It just sounds weird. Society prefers us not to say numbers this way.

So, the question is now:

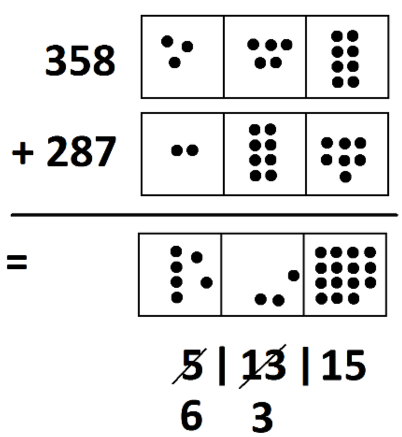
Can we fix up this answer for society's sake – not mathematics' sake – just for society's sake?

The answer is yes! We can do some explosions. (This is a 1 ← 10 machine, after all.)

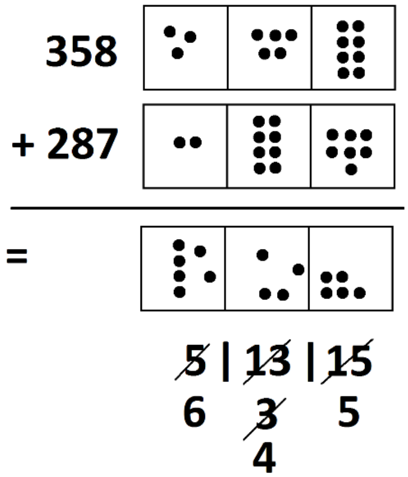
Which do you want to explode first: the 13 or the 15?

Just to be unexpected, let's explode the 15 first. (Most people still want to go right to left!)

Ten dots in the middle box explode to be replaced by one dot, one place to the left.



The answer “six hundred three-ty fifteen” now appears. This is still a lovely, mathematically correct answer. But society at large might not agree. Let’s do another explosion: ten dots in the rightmost box.




Now we see the answer “six hundred four-ty five,” which is one that society understands. (Although, in English, “four-ty” is usually spelled *forty*.)

> Video:
> Traditional Addition video (+ all follow-on content you see).
> Web App:
> Third Experience.

So how does this dots-and-boxes approach to addition compare to the standard algorithm most people know?

Let’s go back to the example $358 + 287$. Most people are surprised (maybe even perturbed) by the straightforward left-to-right answer 5 | 13 | 15.

```
  358
+ 287
-----
5|13|15
```

This is because the traditional algorithm has us work from right to left, looking at $8 + 7$ first.

But, in the algorithm we don't write down the answer 15. Instead, we explode ten dots right away and write on paper a 5 in the answer line together with a small 1 tacked on to the middle column. People call this *carrying the one* and it – correctly – corresponds to adding an extra dot in the tens position.

$$\begin{array}{cccc} & & {\scriptstyle 1} & \\ & 3 & 5 & 8 \\ + & 2 & 8 & 7 \\ \hline & & & 5 \end{array}$$

Now we attend to the middle boxes. Adding gives 14 dots in the tens box ($5 + 8$ gives thirteen dots, plus the extra dot from the previous explosion).

And we perform another explosion.

$$\begin{array}{cccc} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 3 & 5 & 8 \\ + & 2 & 8 & 7 \\ \hline & & 4 & 5 \end{array}$$

On paper, one writes "4" in the tens position of the answer line, with another little "1" placed in the next column over. This matches the idea of the dots-and-boxes picture precisely.

And now we finish the problem by adding the dots in the hundreds position.

$$\begin{array}{cccc} & {\scriptstyle 1} & {\scriptstyle 1} & \\ & 3 & 5 & 8 \\ + & 2 & 8 & 7 \\ \hline & 6 & 4 & 5 \end{array}$$

So, the traditional algorithm works right to left and does explosions (*"carries"*) as one goes along. On paper, it is swift and compact, and this might be why it has been the favored way of doing long addition for centuries.

The *Exploding Dots* approach works left to right, just as we are taught to read in English, and leaves all the explosions to the end. It is easy to understand and kind of fun.

Both approaches, of course, are good and correct. It is just a matter of taste and personal style which one you choose to do. (And feel free to come up with your own new, and correct, approach too!)

Here are some practice problems you might or might not want to try.

**Practice 18:** Write down the answers to the following addition problems working left to right, not worrying about what society thinks! Then, do some explosions to translate each answer into something society understands.

$$\begin{array}{r} 148 \\ +\ 323 \\ \hline = \end{array} \qquad \begin{array}{r} 567 \\ +\ 271 \\ \hline = \end{array} \qquad \begin{array}{r} 377 \\ +\ 188 \\ \hline = \end{array} \qquad \begin{array}{r} 582 \\ +\ 714 \\ \hline = \end{array}$$

$$\begin{array}{r} 310462872 \\ +\ 389107123 \\ \hline = \end{array} \qquad \begin{array}{r} 87263716381 \\ +\ 18778274824 \\ \hline = \end{array}$$

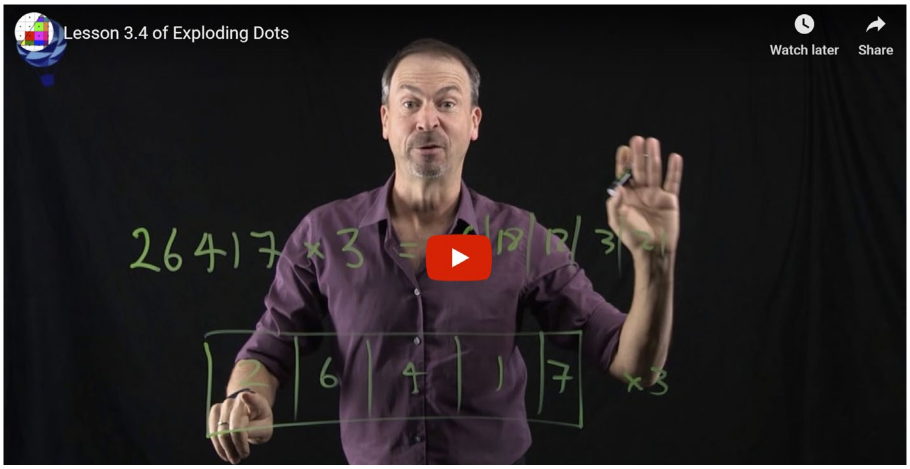
## MULTIPLICATION

Video:

Multiplication video (+ all follow-on content you see).

Web App:

Third Experience.




Okay. Addition. What do students usually learn to do next in school? Subtraction.

But that is too hard. Let's do multiplication instead!

> **ON THE SPOT CHALLENGE:** You've got less than three seconds to write down an absolutely, correct speedy answer to this multiplication problem. Without regard to society's opinions, what's a good answer?

**26417 x 3**

Can you see that 6 | 18 | 12 | 3 | 21, that is, "six ten thousand, eighteen thousand, twelve hundred and three-ty twenty-one," is correct and does the speedy trick?

Here's what's going on.

Let's start with a picture of 26417 in a 1 ← 10 machine. (Is it okay if I just write numbers rather than draw dots?)

| | | | | |
|---|---|---|---|---|
| **2** | **6** | **4** | **1** | **7** |

We're being asked to triple this number.

| | | | | | |
|---|---|---|---|---|---|
| **2** | **6** | **4** | **1** | **7** | **x 3** |

Right now, we have 2 ten-thousands. If we triple this, we'd have 6 ten-thousands.

Right now, we have 6 thousands, and tripling would make this 18 thousands.

Also, 4 hundreds becomes 12 hundreds; 1 ten becomes 3 tens; and 7 ones becomes 21 ones.

| | | | | |
|---|---|---|---|---|
| **6** | **18** | **12** | **3** | **21** |

We see the answer "sixty eighteen thousand, twelve hundred and three-ty twenty-one." Absolutely solid and mathematically correct!

Now, how can we fix up this answer for society? Do some explosions of course!

Which explosion do you want to do first? Umm, I'll explode the 12 first. It gives

$$6 \mid 19 \mid 2 \mid 3 \mid 21$$

**Question:** If you keep going with explosions, can you see the societally acceptable answer 79251 appear?

---

**Practice 19:** If you feel like it, compute each of the following.

a) $26417 \times 4$

b) $26417 \times 5$

c) $26417 \times 9$

d) Compute $26417 \times 10$ too and explain why the answer has to be 264170.
(This answer looks like the original number with the digit zero tacked on to its end.)

e) Care to compute $26417 \times 11$ and $26417 \times 12$ too?
(The answer could be, "No! I do not care to do this!)

## ON MULTIPLYING BY TEN

*Why must the answer to $26417 \times 10$ look like the original number with a zero tacked on to its end?*

I remember being taught this rule in school: to multiply by ten tack on a zero. For example,

$37 \times 10 = 370$

$98989 \times 10 = 989890$

$100000 \times 10 = 1000000$

and so on.

This observation makes perfect sense in the dots-and-boxes thinking.

Here's the number $26417$ again in a $1 \leftarrow 10$ machine.

| | | | | |
|---|---|---|---|---|
| **2** | **6** | **4** | **1** | **7** |

Here's $26417 \times 10$.

| | | | | |
|---|---|---|---|---|
| **20** | **60** | **40** | **10** | **70** |

Now let's perform the explosions, one at a time. (We'll need an extra box to the left.)

We have that $2$ groups of ten explode to give $2$ dots one place to the left, and $6$ groups of ten explode to give $6$ dots one place to the left, and $4$ groups of ten explode to give $4$ dots one place to the left, and so on. The digits we work with stay the same. In fact, the net effect of what we see is all digits shifting one place to the left to leave zero dots in the ones place.

| | | | | | |
|---|---|---|---|---|---|
| | 20 | 60 | 40 | 10 | 70 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 0 | 60 | 40 | 10 | 70 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 6 | 0 | 40 | 10 | 70 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 6 | 4 | 0 | 10 | 70 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 6 | 4 | 1 | 0 | 70 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 6 | 4 | 1 | 7 | 0 |

Indeed, it looks like we just tacked on a zero to the right end of $26417$. But really, we performed a whole lot of explosions to make a zero appear at the end.

These next two practice problems are worth dwelling on. Can you explain them with clarity?

**Practice 20:** What must be the answer to $476 \times 10$? To $476 \times 100$?

**Practice 21:** What must be the answer to $9190 \div 10$? To $3310000 \div 100$?

## (OPTIONAL) TRADITIONAL LONG MULTIPLICATION

Is it possible to do, say, $37 \times 23$, with dots and boxes?

Here we are being asked to multiply three tens by $23$ and seven ones by $23$. If you are good with your multiples of 23, this must give $3 \times 23 = 69$ tens and $7 \times 23 = 161$ ones. The answer is thus $69|161$. With explosions, this becomes $851$.

But this approach is hard! It requires you to know multiples of $23$.

**Thinking Exercise:** Suzzy thought about $37 \times 23$ for a little while, she eventually drew the following diagram.

| | | | |
|---|---|---|---|
| | 6 | 14 | 0 |
| + | | 9 | 21 |
| = | 6 | 23 | 21 |

She then said that $37 \times 23 = 6|23|21 = 8|3|21 = 851$.

a) Can you work out what Suzzy was thinking? If not, perhaps look at this next example she did.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | 3 | 1 | 2 | 5 |
| | | x | | | 8 | 3 | 2 |
| = | | | | 6 | 2 | 4 | 10 |
| | | | 9 | 3 | 6 | 15 | 0 |
| | | 24 | 8 | 16 | 40 | 0 | 0 |
| | | 24 | 17 | 25 | 48 | 19 | 10 |
| = | 2 | 6 | 0 | 0 | 0 | 0 | 0 |

Does this look anything like the long multiplication you were taught to do in grade school?
Can you make sense of if?

b) What diagram do you think Suzzy might draw for $236 \times 34$ (and what answer will she get from it)?

**WILD EXPLORATIONS**

Here are some "big question" investigations you might want to explore, or just think about before moving on. Have fun!

**Wild Exploration 1: THERE IS NOTHING SPECIAL ABOUT BASE TEN FOR ADDITION.**

Here is an addition problem in a $1 \leftarrow 5$ machine. (That is, it is a problem in base five.) This is not a $1 \leftarrow 10$ machine addition.

$$\begin{array}{r} 20413 \\ +\ 13244 \\ \hline \end{array}$$

a) What is the $1 \leftarrow 5$ machine answer?
b) What number has code 20413 in a $1 \leftarrow 5$ machine?
   What number has code 13244 in a $1 \leftarrow 5$ machine?
   What is the sum of those two numbers and what is the code for that sum in a $1 \leftarrow 5$ machine?

[Here are the answers so that you can check your clever thinking.
The sum, as a $1 \leftarrow 5$ machine problem, is

$$20413 + 13244 = 3\,|\,3\,|\,6\,|\,5\,|\,7 = 3\,|\,4\,|\,1\,|\,5\,|\,7 = 3\,|\,4\,|\,2\,|\,0\,|\,7 = 3\,|\,4\,|\,2\,|\,1\,|\,2 = 34212.$$

In a $1 \leftarrow 5$ machine, 20413 is two 625's, four 25's, one 5, and three 1's, and so is the number 1358 in base ten; 13244 is the number 1074 in base ten; and 34212 is the number 2432 in base ten. We have just worked out $1358 + 1074 = 2432$.]

**Wild Exploration 2: THERE IS NOTHING SPECIAL ABOUT BASE TEN FOR MULTIPLICTION.**

Let's work with a $1 \leftarrow 3$ machine.

a) Find $111 \times 3$ as a base three problem. Also, what are $1202 \times 3$ and $2002 \times 3$?
   Can you explain what you notice?

**Comment:** For base three, we could write "10" here instead of "3".

Let's now work with a $1 \leftarrow 4$ machine.

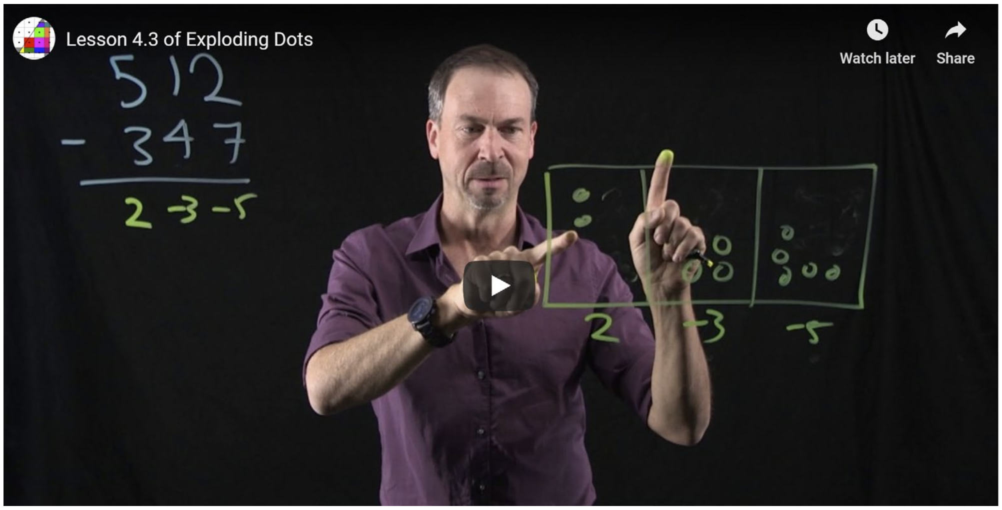
b) What is $133 \times 4$ as a base four problem? What is $2011 \times 4$? What is $22 \times 4$?
   Can you explain what you notice?

**Comment:** For base four, we could write "10" here instead of "4".

c) In general, if we are working with a $1 \leftarrow \boldsymbol{b}$ machine, can you explain why multiplying a number in base $\boldsymbol{b}$ by $\boldsymbol{b}$ returns the original number with a zero tacked on to its right?

## SUBTRACTION

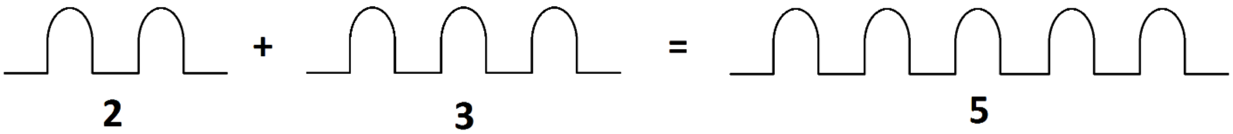
Videos:
Introductory video.
Antidots video.
Subtraction video (+ all follow-on content you see).
Web App:
Fourth Experience.




So far, we've made sense of addition and multiplication. But we skipped over subtraction. Why? Because I don't believe in subtraction! To me, subtraction is just the addition of the opposite.

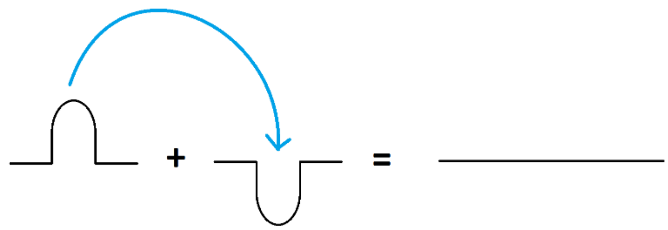
My disbelief in subtraction comes from another story that isn't true. Briefly, it goes as follows.

*As a young child, I used to regularly play in a sandbox. And there I discovered the positive counting numbers as piles of sand: one pile, two piles, and so on. And I also discovered the addition of positive numbers simply by lining up piles. For example, I saw that two plus three equals five simply by lining up piles like this.*

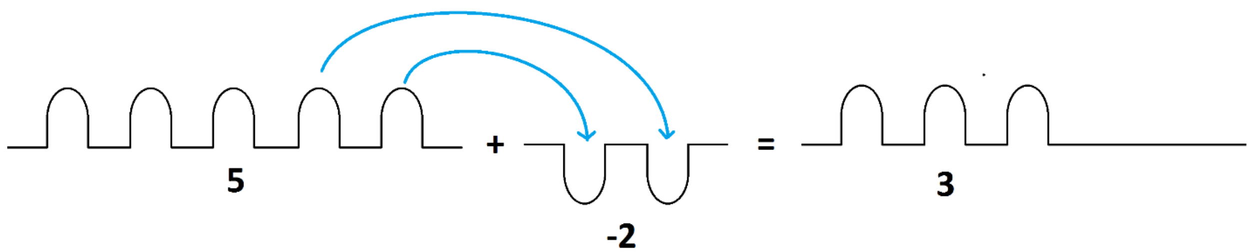



*I had hours of fun counting and lining up piles to explore addition.*

*But then one day I had an astounding flash of insight! Instead of making piles of sand, I realized I could also make holes. And I saw right away that* ***a hole is the opposite of a pile: place a pile and a hole together and they cancel each other out.*** *Whoa!*




*Later in school I was taught to call a hole "*$-1$*", and two holes "*$-2$*," and so on, and was told to do this thing called "subtraction." But I never really believed in subtraction. My colleagues would read* $5-2$*, say, as, "five take away two," but I was thinking of five piles and the addition of two holes. A picture shows that the answer is three piles.*




*Yes. This gives the same answer as my peers, of course: the two holes "took away" two of the piles. But I had an advantage. For example, my colleagues would say that* $7-10$ *has no answer. I saw that it did.*

$7-10=$ seven piles and ten holes

$=$ three holes

$=-3$

*Subtraction is just the addition of the opposite.*

*(By the way, I will happily write* $7-10$ *as "*$7+-10$*." This makes the thinking more obvious.)*

We'll go through the story and arithmetic of negative numbers in a later chapter, but maybe this little introduction is enough for us to keep playing with the *Exploding Dots* fun.

But there's a problem. In our 1 ← 10 machine we've been working with dots, which I've been drawing as solid dots. (And a dot is like a "pile" for a positive number.)

 **= dot**

Now we need the notion of the opposite of dot. Umm…

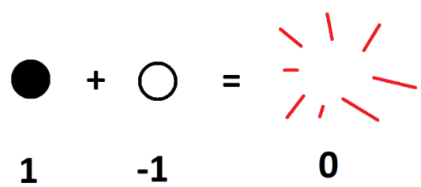
*What should I call the opposite of a dot?*

*What do I draw for the opposite of a dot?*

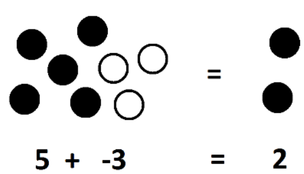
Let's draw an open circle for the opposite of a dot and call it an "antidot" (or "tod").

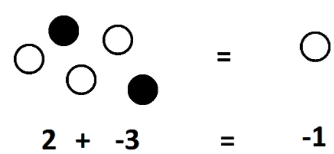
 **= antidot**

Like matter and antimatter (or a pile and hole), which each annihilate one another when brought together, a dot and an antidot should also annihilate – POOF! – when brought together, to leave nothing behind.




We can conduct basic arithmetic with dots and antidots, just like we did with piles and holes.



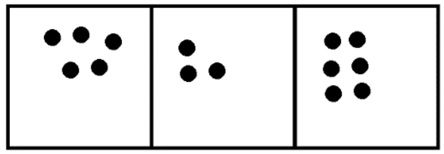

Now we're ready to do subtraction in a $1 \leftarrow 10$ machine.

Consider this subtraction problem.

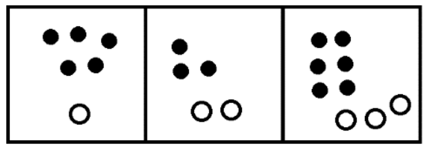
$$\begin{array}{r} 536 \\ -\ 123 \\ \hline \end{array}$$

To me, this is 536 plus the opposite of 123.

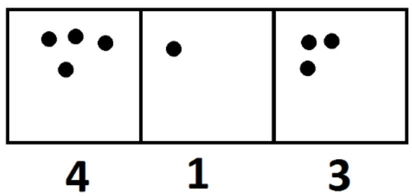
The first number, 536, looks like this in a 1 ← 10 machine: five dots, three dots, six dots.



To this we are adding the opposite of 123. That is, we're adding one anti-hundred, two anti-tens, and three anti-ones.



And now there are a lot of annihilations: POOF!;  POOF POOF!;  POOF POOF POOF!




We see the answer 413 appear.

And notice, we get this answer as though we just work left to right and say

5 take away 1 is 4,
3 take away 2 is 1,
and
6 take away 3 is 3.

Yes! Left to right again!

$$\begin{array}{r} 536 \\ -\ 123 \\ \hline 413 \end{array}$$

All right. That example was too nice. How about 512 – 347?

$$\begin{array}{r} 5\ 1\ 2 \\ -3\ 4\ 7 \\ \hline \end{array}$$

Going from left to right, we get

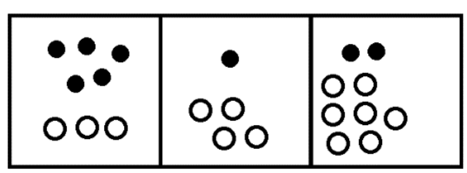
5 take away 3 is 2,
1 take away 4 is –3,
and
2 take away 7 is –5.

The answer is two-hundred negative-three-ty negative-five.

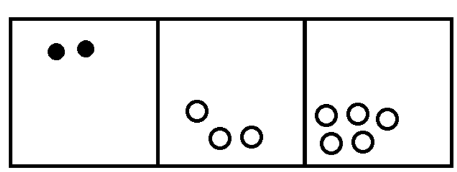
$$\begin{array}{r} 5\ 1\ 2 \\ -3\ 4\ 7 \\ \hline 2|{-3}|{-5} \end{array}$$

And this answer is absolutely, mathematically correct! The picture shows it is.

Here's five hundreds, one ten and two ones together with three anti-hundreds, four anti-tens, and seven anti-ones.



And after lots of annihilations we are left with two actual hundreds, three anti-tens, and five anti-ones.



2 -3 -5

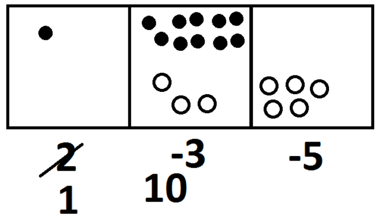
The answer really is two-hundred negative-three-ty negative-five!

But, of course, saying the answer to our subtraction problem this way seems mighty weird to society.

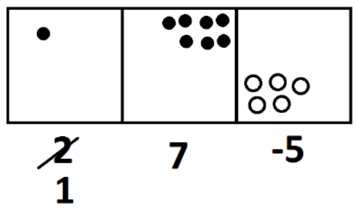
*Can we fix up this mathematically correct answer for society's sake?*

After some time you might think to do something wild: **unexplode** dots! Any dot in a box to the left must have come from ten dots in the box just to its right, so we can "unexplode" it to make ten dots.

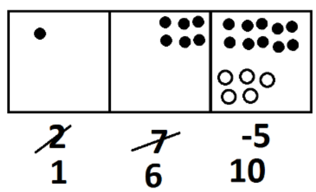
Okay. Let's unexplode one of the two dots we have in the leftmost box. Doing so gives this picture.



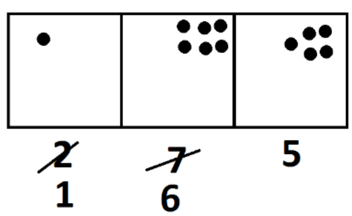

After annihilations, we see we now have the answer one-hundred seventy negative-five. Beautiful!




Let's unexplode again.




And with some more annihilations we see an answer society can understand: one-hundred sixty-five.



## (OPTIONAL) THE TRADITIONAL SUBTRACTION ALGORITHM

How does this dots-and-boxes approach compare with the standard algorithm?

Consider again 512 – 347.

$$\begin{array}{rrrr} & 5 & 1 & 2 \\ - & 3 & 4 & 7 \\ \hline \end{array}$$

The standard algorithm has you start at the right to first look at "2 take away 7," which you can't do. (Well you can do it, it is –5, but you are not to write that for this algorithm.)

So, what do you do?

You "borrow one." That is, you take a dot from the tens column and unexplode it to make ten ones. That leaves zero dots in the tens column. We should write ten ones to go with the two in the ones column.

$$\begin{array}{rrrr} & & 0 & 10 \\ & 5 & \cancel{1} & 2 \\ - & 3 & 4 & 7 \\ \hline \end{array}$$

But we are a bit clever here and just write 12 rather than 10 + 2. (That is, we put a 1 in front of the 2 to make it look like twelve.)

$$\begin{array}{rrrr} & & 0 & \\ & 5 & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline \end{array}$$

Then we say, "twelve take seven is five," and write that answer.

$$\begin{array}{rrrr} & & 0 & \\ & 5 & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline & & & 5 \end{array}$$

The rightmost column is complete. Shift now to the middle column.

We see "zero take away four," which can't be done. So, perform another unexplosion, that is, another "borrow," to see 10 – 4 in that column. We write the answer 6.

We then move to the last remaining column where we have 4 – 3, which is 1.

$$\begin{array}{rrr} 4 & {}^{1}0 & \\ \cancel{5} & \cancel{1} & {}^{1}2 \\ -3 & 4 & 7 \\ \hline & 6 & 5 \end{array} \qquad\qquad \begin{array}{rrr} 4 & {}^{1}0 & \\ \cancel{5} & \cancel{1} & {}^{1}2 \\ -3 & 4 & 7 \\ \hline 1 & 6 & 5 \end{array}$$

Phew!

Again. All correct approaches to mathematics are correct, and it is just a matter of style as to which approach you like best for subtraction. The traditional algorithm has you work from right to left and do all the unexplosions as you go along. The dots-and-boxes approach has you "just do it!" and conduct all the unexplosions at the end. Both methods are fine and correct.

Here's a practice schedule to try if you like.

**Practice 22:**
Compute each of the following two ways: the dots-and-boxes way (and fixing the answer for society to read) and then with the traditional algorithm. The answers should be the same.

$$\begin{array}{r} 6328 \\ -\ 4469 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 78390231 \\ -\ 32495846 \\ \hline \end{array}$$

**Thinking questions along the way:**
As you fix up your answers for society, does it seem easier to unexplode from left to right, or from right to left?

Do you think you could become just as speedy the dots-and-boxes way as you currently are with the traditional approach?

**WILD EXPLORATIONS**

Here are some “big question” investigations you might want to explore, or just think about before moving on. Have fun!

**Wild Exploration 1: IS THERE ANOTHER WAY TO INTERPRET DOTS-AND-BOXES ANSWERS?**

When Sunil saw,

$$\begin{array}{r} 5\ 1\ 2 \\ -3\ 4\ 7 \\ \hline 2\,|{-3}\,|{-5} \end{array}$$

he wrote on his paper the following lines:

$$\begin{array}{r} 200 \\ -30 \\ -5 \end{array}$$

He then said that the answer has to be 165.

a) Can you explain what he is seeing and thinking?

b) What would Sunil likely write on the page for $7109 - 3384$?

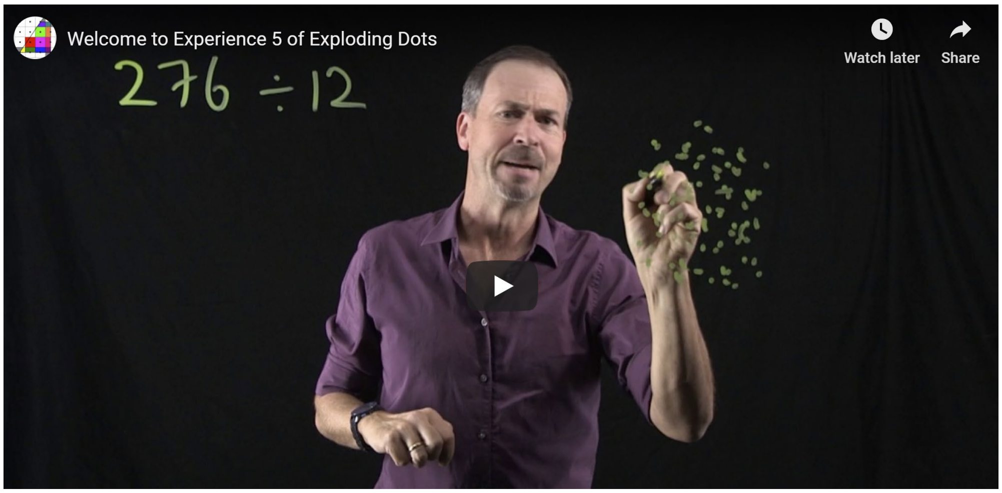
**Wild Exploration 2: WHAT ABOUT NEGATIVE ANSWERS?**

How might you handle and interpret this subtraction problem?

$$\begin{array}{r} 148 \\ -\ 677 \\ \hline \end{array}$$

## DIVISION

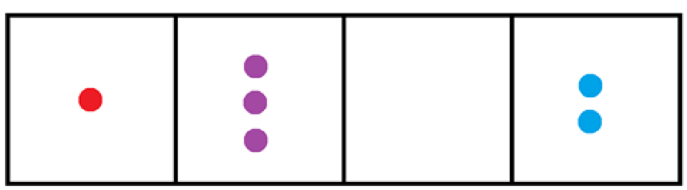
Videos:
Introductory video.
Division video (+ all follow-on content you see).
Web App:
Fifth Experience.



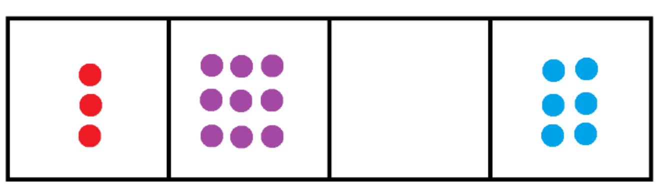

Addition, subtraction, multiplication. Now it is time for division.

Division is linked with multiplication. In fact, many people think of division as the reverse of multiplication. So, let's revisit multiplication for a moment to then see if we can follow it backwards to get to division. We'll start with a straightforward multiplication problem, say, $1302 \times 3$ (with answer $3906$).

Here's what $1302$ looks like in a $1 \leftarrow 10$ machine. (I've colored the dots for fun.)

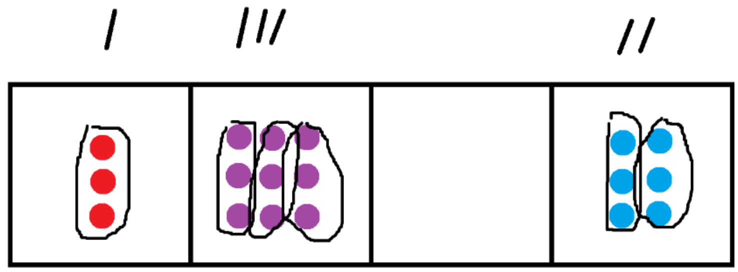
**1302 =** 

To triple this quantity, we just need to replace each dot in the picture with three dots. We see the answer $3906$.

**1302 x 3 =** 

Now suppose I gave you the second picture first and said "Please use it to divide $3906$ by three." Could you use the picture to see triples of dots that must have come from single dots? Yes! Look at the picture this way.

3906 =



We see the picture as the result of tripling one dot at the thousands level, tripling three dots at the hundreds level, and tripling two dots at the ones level. That is, we see $3906$ as the number $1302$ tripled.

We have just deduced, from the picture, that $3906 \div 3 = 1302$ !

So, to divide a number by three, all we need to do is to look for groups of three in the picture of the number. Each group of three corresponds to a dot that must have been tripled. We can just read off the answer to the division problem then by looking at the groups we find!

Of course, we can do the same for any single-digit division problem.

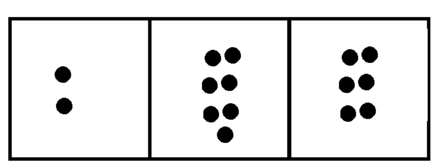
> **Question;** *Draw a dots and boxes picture of the number* $426$ *and use it to explain why* $426 \div 2$ *equals* $213$.
>
> **Question:** *Try doing* $402 \div 3$ *with just a dots-and-boxes picture. Do you see that unexplosions unlock this problem to reveal the answer* $134$ *?*

Division by single-digit numbers is all well and good. What about division by multi-digit numbers? People usually call that *long division*.

Let's consider the problem $276 \div 12$.

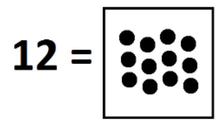
Here is a picture of $276$ in a $1 \leftarrow 10$ machine.

276 =

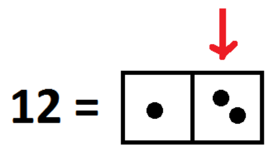


And we are looking for groups of twelve in this picture of $276$. Could this picture be the result of a different picture whose single dots were each replaced by a group of twelve?

Here's what twelve looks like.

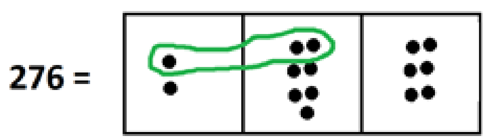



Actually, this is not quite right as there would be an explosion in our $1 \leftarrow 10$ machine. Twelve will look like one dot next to two dots. (But we need to always keep in mind that this really is a picture with all twelve dots residing in the rightmost box. An explosion caused a dot to spill to the left.)

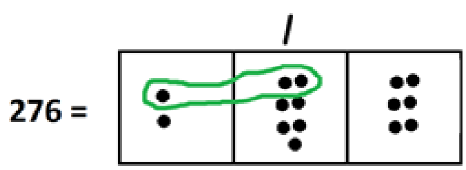



Okay. So we're looking for groups of $12$ in our picture of $276$. Do we see any one-dot-next-to-two-dots in the diagram?

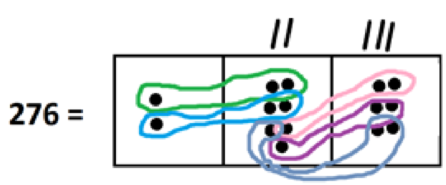
Yes. Here's one.




Within each loop of $12$ we find, the $12$ dots actually reside in the right part of the loop. So we have found one group of $12$ at the tens level.




And there are more groups of twelve.




This shows that our picture of $276$ is actually a picture of $23$ each of whose dots was replaced by $12$. We see $276 = 23 \times 12$ and so $276 \div 12 = 23$.

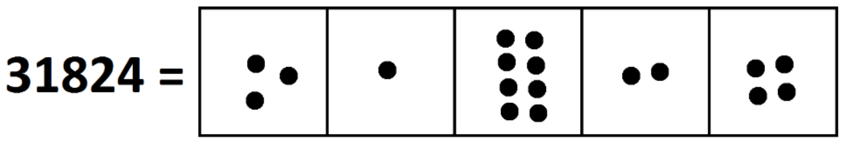
Alternatively, we can say in our picture of $276$ we've identified two groups of $12$ at the tens level and three $12$s at the ones level, and so there are $23$ groups of $12$ in $276$.

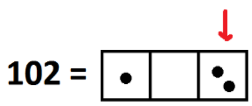
No matter how you wish to interpret matters, the picture reveals all!

**Question:** Use a picture to show that 2783 ÷ 23 equals $121$.

Let's do another example. Let's compute $31824 \div 102$.

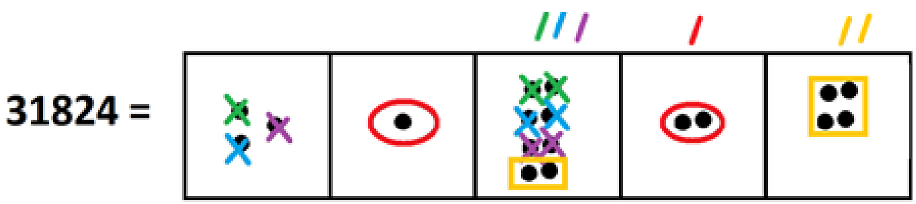
Here's the picture.




Now we are looking for groups of one dot–no dots–two dots in our picture of $31824$. (And, remember, all $102$ dots are physically sitting in the rightmost position of each set we identify.)

We can spot a number of these groups. (I now find drawing loops messy so I am drawing Xs and circles and boxes instead. Is that okay? Do you also see how I circled a double group in one hit at the very end?)

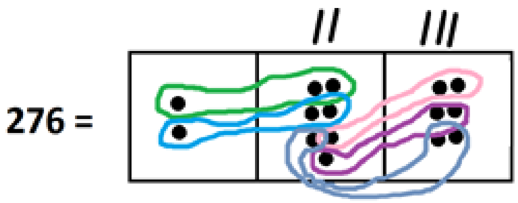



The answer $312$ to $31824 \div 102$ is now apparent. (The colors show that $31824$ is a picture of $312$ with each dot multiplied by $102$. Or, if you prefer, we've found $312$ groups of $102$ in our picture of $31824$.)

## REMAINDERS

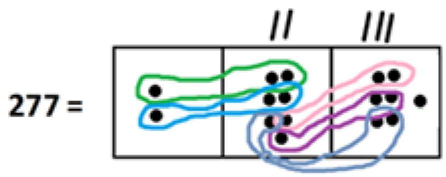
Video:
Remainders video (+ all follow-on content you see).
Web App:
Fifth Experience.

We saw that 276 ÷ 12 equals 23.



Suppose we tried to compute 277 ÷ 12 instead. What picture would we get? How should we interpret the picture?

Well, we'd see the same picture as before except for the appearance of one extra dot, which we fail to include in a group of twelve.



This shows that 277 ÷ 12 equals 23 with a remainder of 1.

You might write this as

$$277 \div 12 = 23 \ R \ 1$$

or with some equivalent notation for remainders. (People use different notations for remainders in different countries.)

Or you might be a bit more mathematically precise and say that 277 ÷ 12 equals 23 with one more dot still to be divided by twelve.

$$277 \div 12 = 23 + \frac{1}{12}.$$

(Some people might prefer to think of that single dot as one-twelfth of a group of twelve. All interpretations are good!)

Here are some practice problems you might like to try ... or not!

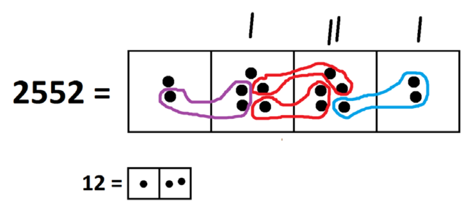
**Practice 23:** Compute 2783 ÷ 23 by the dots-and-boxes approach by hand.
**Practice 24:** Compute 3900 ÷ 12.
**Practice 25:** Compute 46632 ÷ 201.
**Practice 26:** Show that 31533 ÷ 101 equals 312 with a remainder of 21.
**Practice 27:** Compute 2789 ÷ 11.
**Practice 28:** Compute 4366 ÷ 14.
**Practice 29:** Compute 5481 ÷ 131.
**Practice 30:** Compute 61230 ÷ 5.

**WILD EXPLORATIONS**

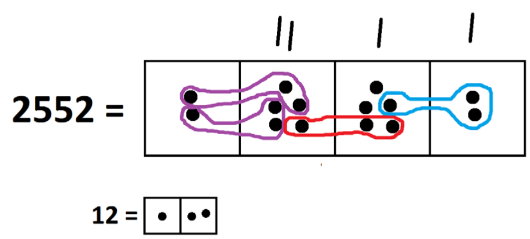
Here are some "big question" investigations you might want to explore, or just think about before moving on. Have fun!

**Wild Exploration 1: LEFT TO RIGHT? RIGHT TO LEFT? ANY ORDER?**

When asked to compute 2552÷12, Kaleb drew this picture, which he got from identifying groups of twelve working right to left.




He said the answer to 2552 ÷ 12 is 121 with a remainder of 1100.
Mabel, on the other hand, identified groups of twelve from left to right in her diagram for the problem.




She concluded that 2552 ÷ 12 is 211 with a remainder of 20. Both Kaleb and Mabel are mathematically correct, but their teacher pointed out that most people would expect an answer with smaller remainders: both $1100$ and $20$ would likely be considered strange remainders for a problem about division by twelve. She also showed Kaleb and Mabel the answer to the problem that is printed in the textbook.

$$2552 \div 12 = 212\ R\ 8$$

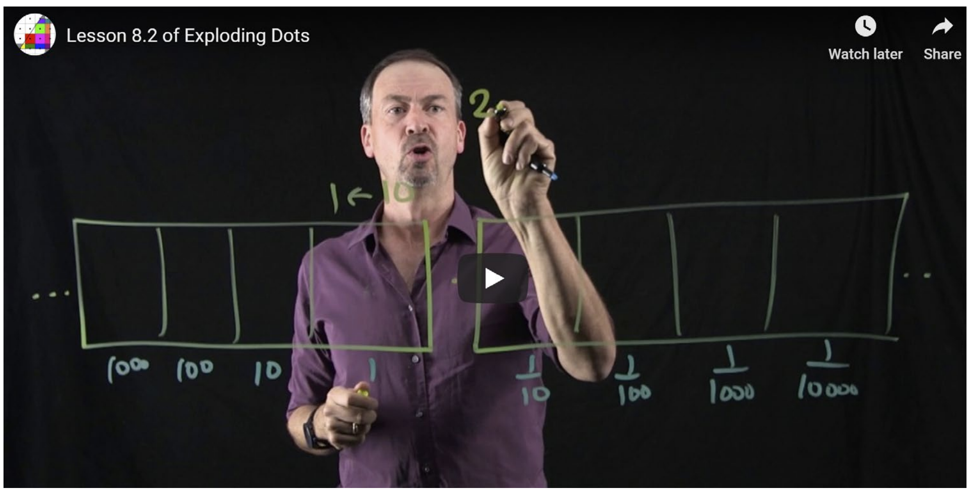
How could Kaleb and Mabel each continue work on their diagrams to have this textbook answer appear?

**Wild Exploration 2: HOW DOES THE DOTS-AND-BOXES DIVISON COMPARE WITH THE TRADTIONAL ALGORITHM?**

You can see the comparison here if you are interested.

# 3. DECIMALS

Videos:
Introductory video.
Decimals video(+ all follow-on content you see).
Web App:
Eighth Experience.




All our machines, so far, have had boxes going to the left as far as we please. But that seems awfully lopsided! Why can’t we have boxes going infinitely far to the right as well?

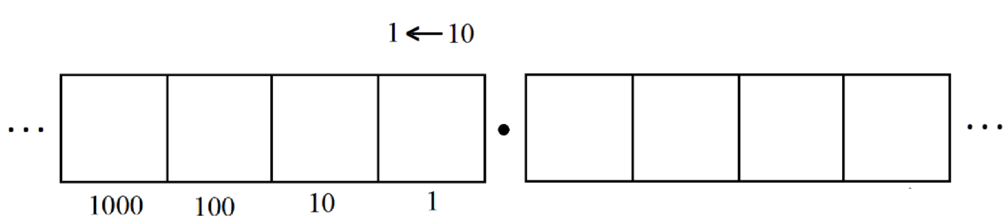
Mathematicians like symmetry and so let’s follow suit and now make all our machines symmetrical. Let’s have boxes going to the left and to the right.

But the challenge now is to figure out what those boxes to the right mean.

## DISCOVERING DECIMALS

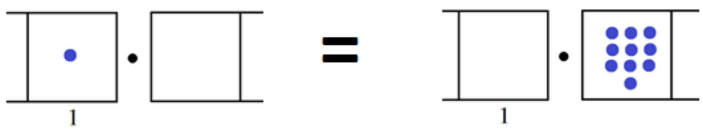
Still working with a $1 \leftarrow 10$ machine let’s see what boxes to the right could mean in that machine.

To keep the left and right boxes visibly distinct, we’ll separate them with a point. (Society calls this point, for a $1 \leftarrow 10$ machine at least, a *decimal point*.)

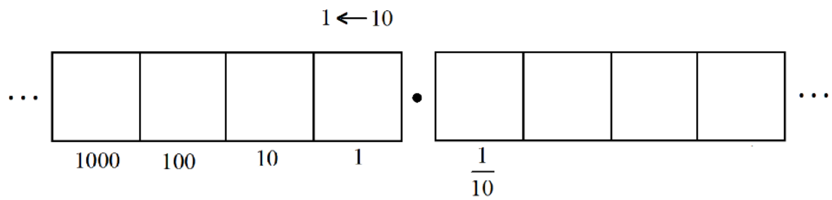


What does it mean to have dots in the right boxes? What are the values of dots in those boxes?

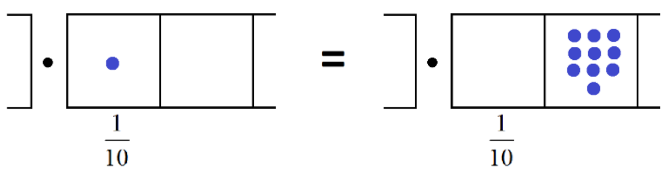
Since this is a $1 \leftarrow 10$ machine, we do know that ten dots in any one box explode to make one dot one place to the left. So, ten dots in the box just to the right of the decimal point are equivalent to one dot in the $1$s box. Each dot in that box must be worth one-tenth.



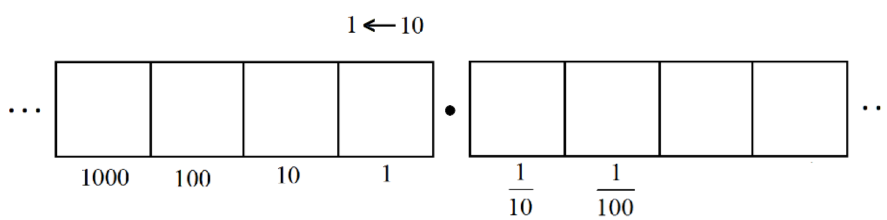

We have our first place-value to the right of the decimal point.



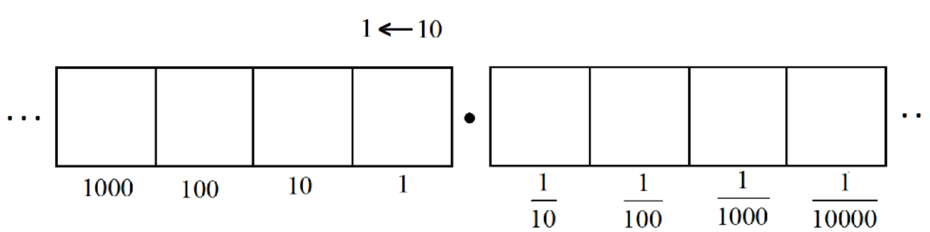

In the same way, ten dots in the next box over are worth one-tenth. And so each dot in that next box must be worth one-hundredth.




We now have two place values to the right of the decimal point.




And ten one-thousandths make a hundredth, and ten ten-thousandths make a thousandth, and so on.

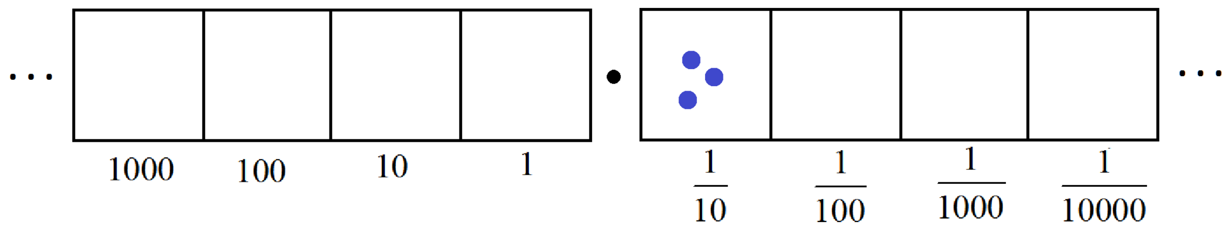


We see that the boxes to the left of the decimal point represent place values as given by the powers of ten, and the boxes to the right of the decimal point represent place values given by the reciprocals of the powers of ten.

We have just discovered decimals!

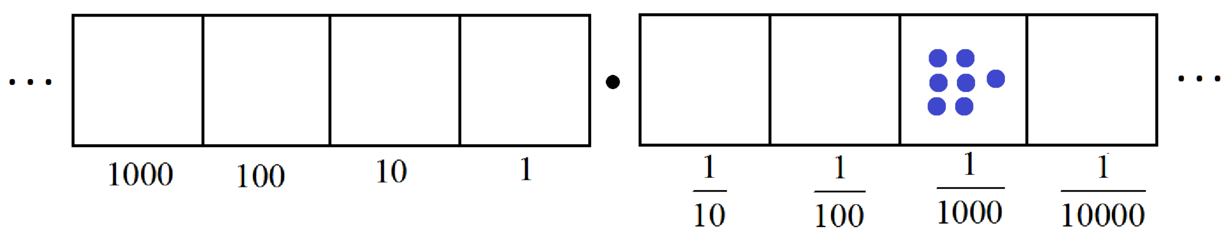
> **Question:** What does the prefix *deci* mean in English? Would we talk about "decimals" in a $1 \leftarrow 2$ machine, for instance? (And why isn't December the tenth month of the year? What happened in the history of the western calendar?)

When people write $0.3$, for example, in base ten, they mean the value of placing three dots in the first box after the decimal point.




We see that $0.3$ equals three tenths: $0.3 = \frac{3}{10}$.

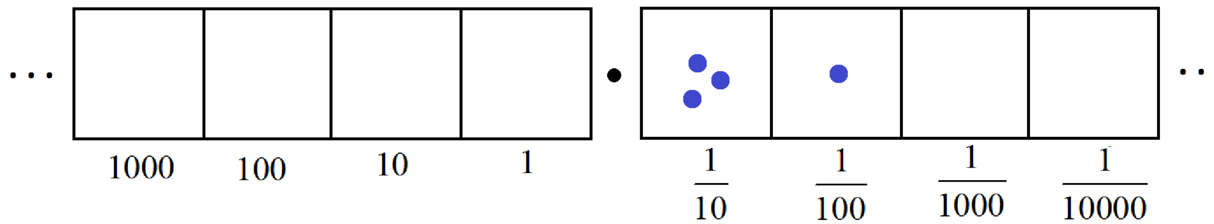
Seven dots in the third box after the decimal point is seven thousandths: $0.007 = \frac{7}{1000}$.



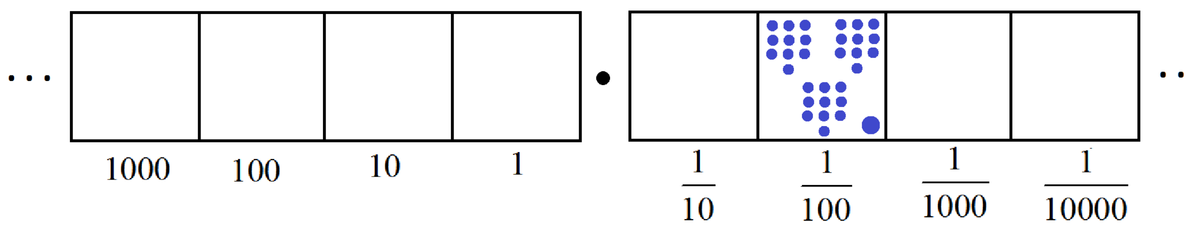

**Comment**: Some people might leave off the beginning zero and just write $.007 = \frac{7}{1000}$. It's just a matter of personal taste.

| |
|---|
| **Practice 31:** Some people read $0.6$ out loud as "point six" and others read it out loud as "six tenths." Which is more helpful for understanding what the number really is? |

There is a possible source of confusion with a decimal such as $0.31$. This is technically three tenths and one hundredth: $0.31 = \frac{3}{10} + \frac{1}{100}$.



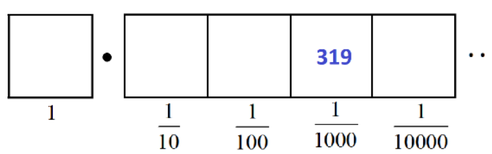

But some people read $0.31$ out loud as "thirty-one hundredths," which looks like this.



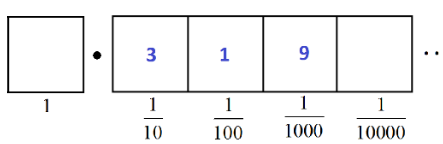

Are these the same thing?

Well, yes! With three explosions we see that thirty-one hundredths becomes three tenths and one hundredth.

**Practice 32:** A teacher asked his students to each draw a $1 \leftarrow 10$ machine picture of the fraction $\frac{319}{1000}$. JinJin drew:




Subra drew:




The teacher marked both students as correct. Are each of these solutions indeed valid? Explain your thinking. (By the way, the teacher doesn't mind if students just write numbers instead of drawing dots.)

**Practice 33:** Multiple choice!

a) The decimal $0.23$ equals: (A) $\frac{23}{10}$ (B) $\frac{23}{100}$ (C) $\frac{23}{1000}$ (D) $\frac{23}{10000}$?

b) The decimal $0.0409$ equals: (A) $\frac{409}{100}$ (B) $\frac{409}{1000}$ (C) $\frac{409}{10000}$ (D) $\frac{409}{100000}$?

**Warning!**
The remainder of this section assumes some familiarity with the arithmetic of fractions. If you are not quite ready for this yet, fret not! Just skip this rest of this section and we'll talk about fractions another day.

## SIMPLE FRACTIONS AS DECIMALS

Some decimals give fractions that simplify further. For example,

$$0.5=\frac{5}{10}=\frac{1}{2}$$

and

$$0.04=\frac{4}{100}=\frac{1}{25}.$$

Conversely, if a fraction can be rewritten to have a denominator that is a power of ten, then we can easily write it as a decimal. For example,

$$\frac{3}{5}=\frac{6}{10} \text{ and so } \frac{3}{5}=0.6$$

and

$$\frac{13}{20}=\frac{13\times 5}{20\times 5}=\frac{65}{100}=0.65.$$

Here are some more problems. Do whichever ones seem interesting to you.

**Practice 34:** What fractions (in simplest terms) do the following decimals represent?

$0.05$ $\quad 0.2$ $\quad 0.8$ $\quad 0.004$

**Practice 35:** Write each of the following fractions as a decimal.

$\frac{2}{5}$ $\quad \frac{1}{25}$ $\quad \frac{1}{20}$ $\quad \frac{1}{200}$ $\quad \frac{2}{2500}$

**Practice 36:** MULTIPLE CHOICE!

a) The decimal $0.050$ equals (A) $\frac{50}{100}$ (B) $\frac{1}{20}$ (C) $\frac{1}{200}$ (D) None of these?

b) The decimal $0.000208$ equals (A) $\frac{52}{250}$ (B) $\frac{52}{2500}$ (C) $\frac{52}{25000}$ (D) $\frac{52}{250000}$?

**Practice 37:** Write each of the following fractions as decimals.

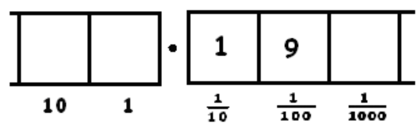
$$\frac{7}{20} \qquad \frac{16}{25} \qquad \frac{301}{500} \qquad \frac{17}{50} \qquad \frac{3}{4}$$

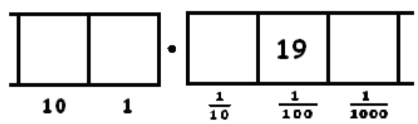
**Practice 38: CHALLENGE**

a) What fraction does the decimal $2.3$ represent?
b) What fraction does $17.04$ represent?
c) What fraction does $1003.1003$ represent?

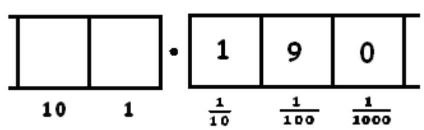
**Practice 39:** Let's explore the question: Do $0.19$ and $0.190$ represent the same number or different numbers?

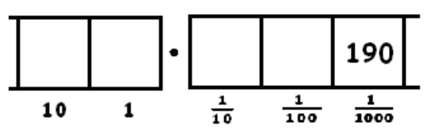
Here are two dots and boxes pictures for the decimal 0.19:

0.19 =



0.19 =



Here are two dots and boxes picture for the decimal 0.190

0.190 =



0.190 =



a) Explain how one "unexplosion" establishes that the first picture of $0.19$ is equivalent to the second picture of $0.19$ .
b) Explain how several unexplosions establishes that the first picture of $0.190$ is equivalent to the second picture of $0.190$ .
c) Explain how explosions and unexplosions in fact establish that all four pictures are equivalent to each other.
d) In conclusion then: Does $0.190$ represent the same number as $0.19$ ?

Following on from question 39 ... To a mathematician, the expressions $0.19$ and $0.190$ represent exactly the same numeric quantity. But you may have noticed in science class that scientists will often write down what seems likes unnecessary zeros when recording measurements. This is because scientists want to impart more information to the reader than just a numeric value.

For example, suppose a botanist measures the length of a stalk. By writing the measurement as $0.190$ meters in her paper, the scientist is saying to the reader that she measured the length of the stalk to the nearest one thousandth of a meter and that she got $1$ tenth, $9$ hundredths, and $0$ thousandths of a meter. Thus we are being told that the true length of the stalk is somewhere in the range of $0.1895$ and $0.1905$ meters.

If she wrote in her paper, instead, just $0.19$ meters, then we would have to assume she measured the length of the stalk only to the nearest hundredth of a meter and so its true length lies somewhere between $0.185$ and $0.195$ meters.

## MIXED NUMBERS AS DECIMALS

How does $12\frac{3}{4}$, for example, appear as a decimal?

Well, $12\frac{3}{4} = 12 + \frac{3}{4}$ and we can certainly write the fractional part as a decimal. (The non-fractional part is already in the $1 \leftarrow 10$ machine format!) We have

$$12\frac{3}{4} = 12 + \frac{75}{100}$$

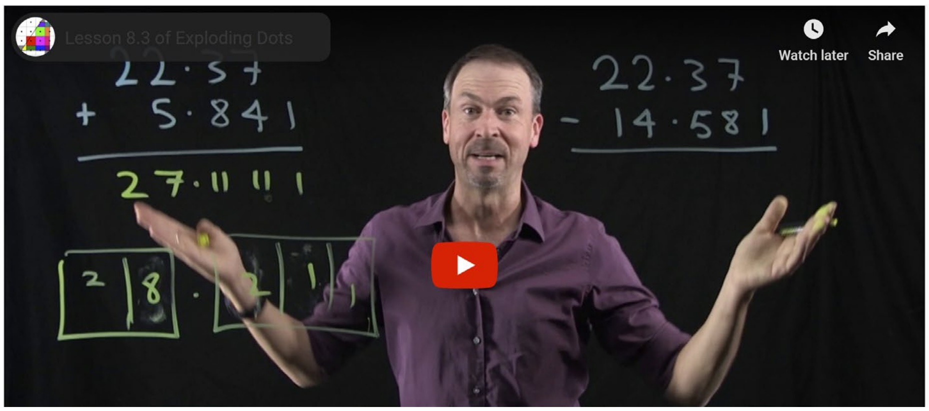
and so we see

$$12\frac{3}{4} = 12.75\,.$$

| |
|---|
| **Practice 40:** Write each of the following numbers in decimal notation, if you like. |
| $5\frac{3}{10}$ $\quad 7\frac{1}{5}$ $\quad 13\frac{1}{2}$ $\quad 106\frac{3}{20}$ $\quad \frac{78}{25}$ $\quad \frac{9}{4}$ $\quad \frac{131}{40}$ |

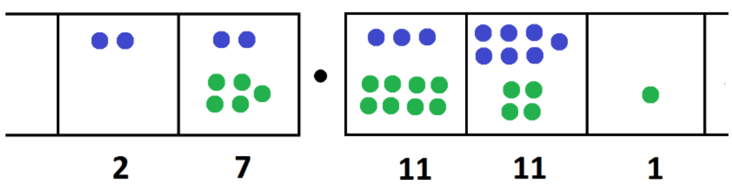
## ADDING AND SUBTRACTING DECIMALS

| Video: |
|---|
| Adding and Subtracting Decimals video (+ all follow-on content you see). |
| Web App: |
| Eighth Experience. |




Performing addition and subtraction with decimals in a $1 \leftarrow 10$ machine is no different from performing addition and subtraction in a $1 \leftarrow 10$ machine without decimals.

For example, here is a picture of $22.37 + 5.841$.

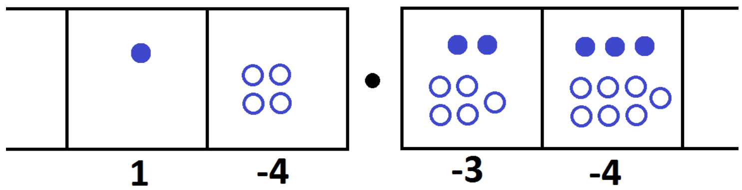



We see the answer $2 \mid 7\ .\ 11 \mid 11 \mid 1$. Just add left to right and don't worry about explosions!

$$
\begin{array}{r}
22.3\ \ 7\ \ \ \ \\
+\ \ \ 5.8\ \ 4\ \ 1 \\
\hline
2|7.11|11|1
\end{array}
$$

With explosions we obtain an answer that society understand. We get $28.211$. (Check this!)

And here is $10.23 - 4.57$.




We see the answer, after some annihilations, as $1\,|\,{-4}\;.\;{-3}\,|\,{-4}$ . (Do you see this?)

$$\begin{array}{rr} & 1\ 0\ .\ 2\ 3 \\ - & 4\ .\ 5\ 7 \\ \hline & 1|{-4}\ .{-3}|{-4} \end{array}$$

Now with some unexplosions, we can fix up this answer for society to read. We get $10.23 - 4.57 = 5.66$. (Check that you follow this.)

$$\begin{array}{rr} & 1\ 0\ .\ 2\ 3 \\ - & 4\ .\ 5\ 7 \\ \hline & \cancel{1}|{-4}\ .{-3}|{-4} \\ & \cancel{6}\ .{-3}|{-4} \\ & 5\ .\ \cancel{7}|{-4} \\ & 5\ .\ 6\,|\,6 \end{array}$$

Here are some practice questions you might, or might not, want to do.

**Practice 41:**

a) What is $0.05 + 0.006$ ?
b) What is $0.05 - 0.006$ ?

**Practice 42:**

Agatha says that computing $0.0348 + 0.0057$ is essentially the same work as computing $348 + 57$ in whole numbers. Do you agree?

Percy says that computing $0.0852 + 0.037$ is essentially the same work as computing $852 + 37$ in whole numbers. Do you agree?

## MULTIPLYING AND DIVIDING DECIMALS

| |
|---|
| Video: |
| Multiplying and Dividing Decimals video (+ all follow-on content you see). |
| Web App: |
| Eighth Experience. |

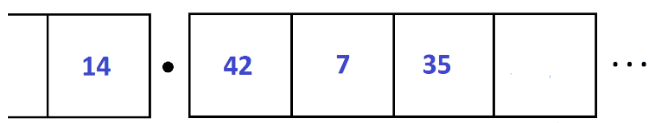
Even though the addition and subtraction of decimals is straightforward in a $1 \leftarrow 10$ machine, performing multiplication and division on them can be awkward. The trouble is that the $1 \leftarrow 10$ machine is based on whole dots, not parts of dots, and so it is tricky to make easy sense of pictures. But there are ways to play with dots-and-boxes nonetheless.

### MULTIPLICATION

We can certainly do some basic multiplication. For example, a picture of dots and boxes shows that

$$2.615 \times 7$$

equals $14.\ 42 \mid 7 \mid 35$.

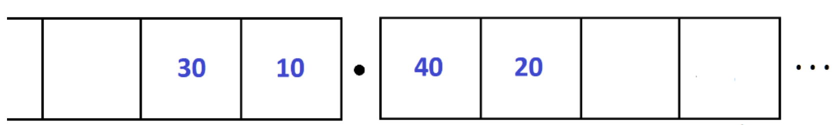



With explosions, this becomes $18.305$. (Check this!)

We can also explain a rule that students are taught to memorize:

*If you multiply a decimal by $10$, just shift the decimal point one place to the right.*

For example, $31.42 \times 10$ gives the answer $30 \mid 10\ .\ 40 \mid 20$. With explosions, this becomes $314.20$ and the answer looks as though we just shifted the decimal point. (Most people leave off the final zero.)




By the way, if you multiply a number a little larger than $31$ by ten, you should get an answer a little larger than $310$. Memorizing the direction of travel of a decimal point is unnecessary.

| |
|---|
| **Practice 43:** Explain why multiplying a decimal by $100$ has the same effect as shifting the decimal point two places? (Do you need to memorize the direction of the shift?) |

We can also multiply decimals by decimals, but now the work is getting less fun.

My advice, is to convert decimals as fractions, and do the same multiplication and division as one does in a $1 \leftarrow 10$ machine. For example, to compute $2.15 \times 0.3$, think of this as

$$\frac{215}{100} \times \frac{3}{10}.$$

This would be $\frac{215 \times 3}{1000} = \frac{6|3|15}{100} = \frac{645}{1000} = 0.646$.

In the same way:

$$0.2 \times 0.4 = \frac{2}{10} \times \frac{4}{10} = \frac{8}{100} = 0.08$$

and

$$0.05 \times 0.006 = \frac{5}{100} \times \frac{6}{1000} = \frac{30}{100000} = \frac{3}{10000} = 0.0003$$

**Practice 44:** Compute $0.04 \times 0.5$ and compute $1000 \times 0.0385$.

## DIVISION

Consider $0.08 \div 0.005$.

Since a fraction is a number that is an answer to a division problem we can think of $0.08 \div 0.005$ as the "fraction"

$$\frac{0.08}{0.005}.$$

(Since we've learnt not to care about societal conventions, why fuss about having whole numbers for the numerators and denominators of fractions?)

To make this fraction look friendlier, let's multiply the top and bottom by factors of ten.

$$\frac{0.08 \times 10 \times 10 \times 10}{0.005 \times 10 \times 10 \times 10} = \frac{80}{5}$$

The division problem $\frac{80}{5}$ is much friendlier. It has the answer $16$. (Use a $1 \leftarrow 10$ machine now to compute it if you like!)

**Example:** Examine $\frac{8.5}{100}$.

Let's rewrite this as $\frac{8.5\times10}{100\times10}=\frac{85}{1000}$ and see the answer $0.085$.

**Example:** Examine $1.51\div0.07$.

I see this as the fraction $\frac{1.51}{0.07}$, which is

$$\frac{1.51\times10\times10}{0.07\times10\times10}=\frac{151}{7}.$$

Ordinary division gives me $21\frac{4}{7}$. If the person asked me to give the answer as a decimal, then I would have to convert the fraction $\frac{4}{7}$ into a decimal too. And that is possible. The next section explains how!

**Practice 45:**

a) Explain why the answer to $0.9\div10$ must be $0.09$.
b) Explain why the answer to $2.34\div1000$ must be $0.00234$.
c) Explain why the answer to $40.04\times0.01$ must be $.4004$.

**Practice 46:** Compute $\frac{0.75}{25}$.

**Practice 47:** Let's put it all together. Here are some unpleasant computations. Do you want to try evaluating them?

a) $0.3\times(5.37-2.07)$

b) $\dfrac{0.1+(1.01-0.1)}{0.11+0.09}$

c) $\dfrac{(0.002+0.2\times2.02)(2.2-0.22)}{2.22-0.22}$

## CONVERTING FRACTIONS INTO DECIMALS

Video:

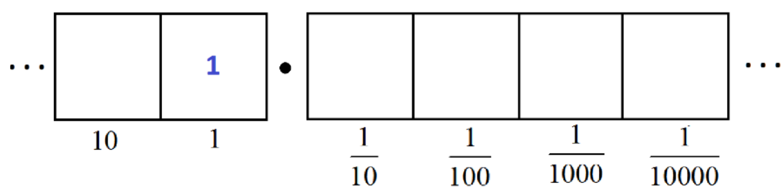
Converting Fractions to Decimals [video](#) (+ all follow-on content you see).

Web App:

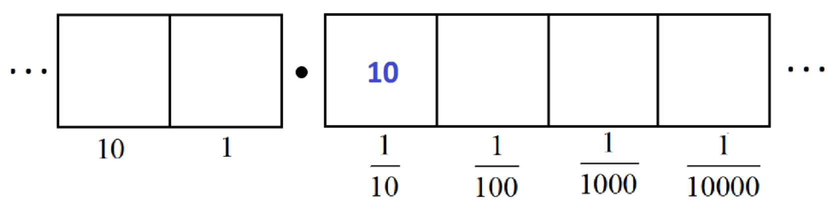
[Eighth Experience](#).

A fraction is a number that is an answer to a division problem. For example, the fraction $\frac{1}{8}$ is the result of dividing $1$ by $8$. And we can compute $1 \div 8$ in a $1 \leftarrow 10$ machine by making use of decimals. The method is exactly the same as for division without decimals.

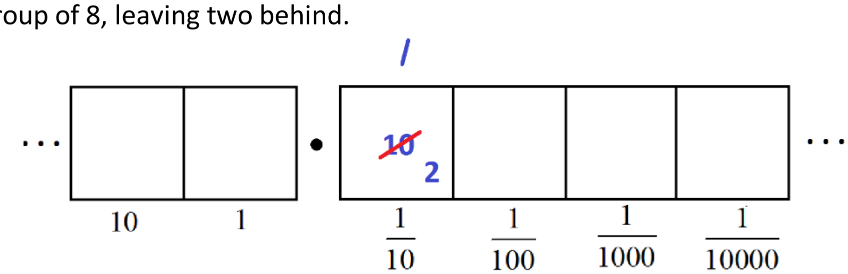
For $1 \div 8$ we seek groups of eight in the following picture.



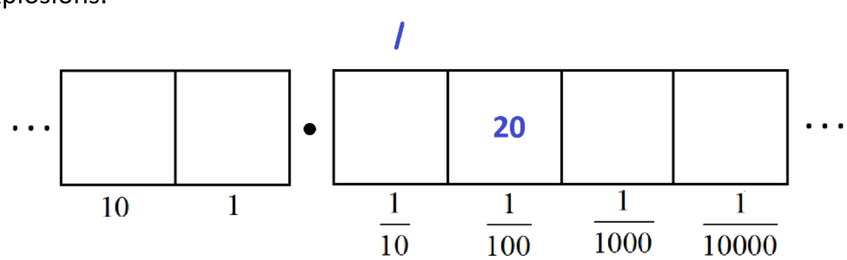

None are to be found right away, so let's unexplode.



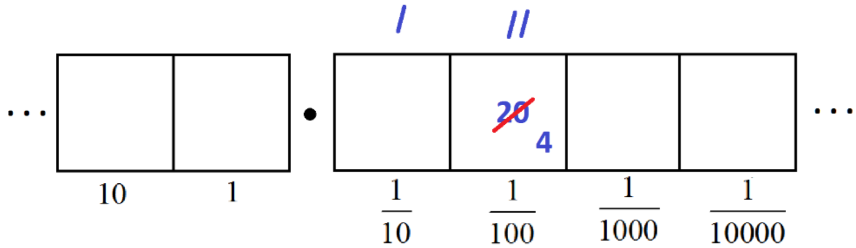

We have one group of 8, leaving two behind.



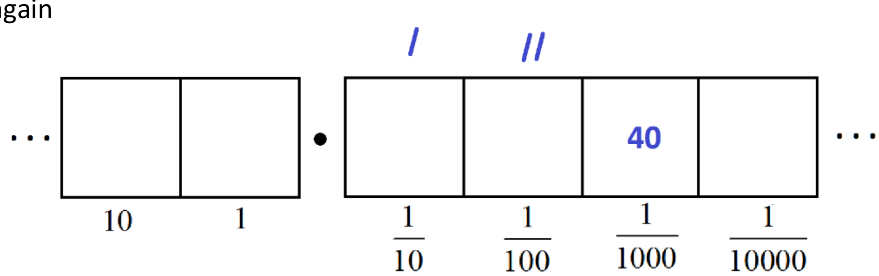

Two more unexplosions.



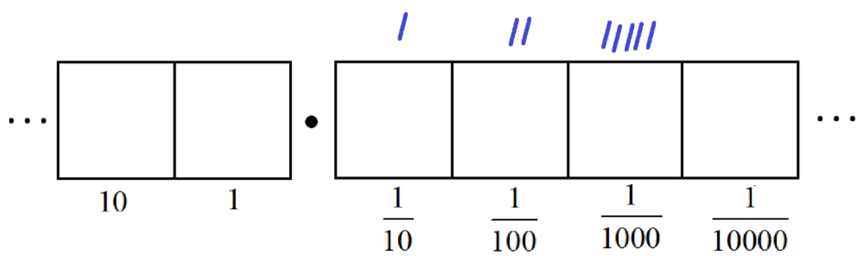
This gives two more groups of 8 leaving four behind.




Unexploding again




reveals five more groups of 8 leaving no remainders.




We see that, as a decimal, $\frac{1}{8}$ turns out to be $0.125$. And as a check we have

$$0.125 = \frac{125}{1000} = \frac{25}{200} = \frac{5}{40} = \frac{1}{8}.$$

Here are some practice problems for you to pick and choose from.

**Practice 48:** Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{1}{4}$, as a decimal, is $0.25$.

**Practice 49:** Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{1}{2}$, as a decimal, is $0.5$.

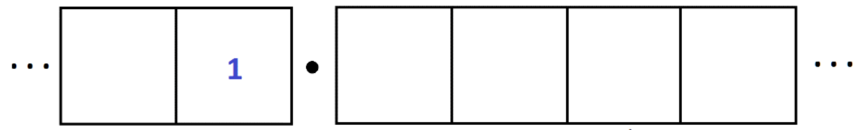
**Practice 50:** Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{3}{5}$, as a decimal, is $0.6$.

**Practice 51:** Perform the division in a $1 \leftarrow 10$ machine to show that $\frac{3}{16}$, as a decimal, is $0.1875$.

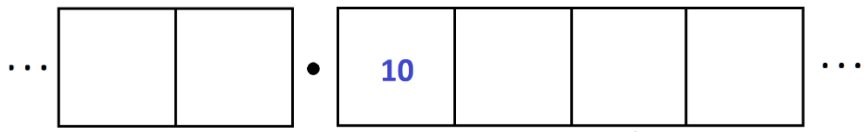
**Practice 52:** In simplest terms, what fraction is represented by each of these decimals?

$0.75 \quad 0.625 \quad 0.16 \quad 0.85 \quad 0.0625$

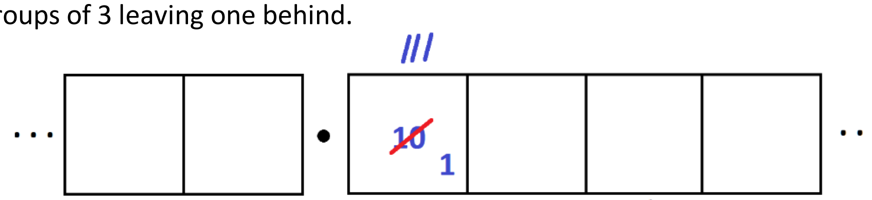
Not all fractions lead to simple decimal representations. For example, consider the fraction $\frac{1}{3}$. To compute it, we seek groups of three in the following picture.



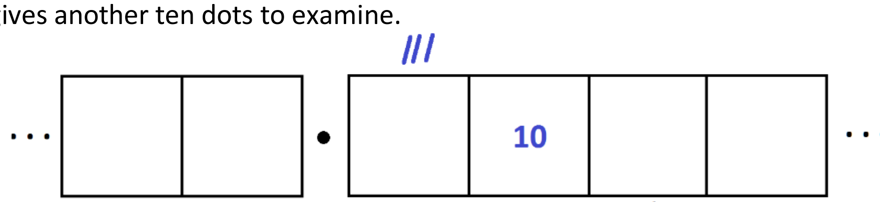

Let's unexplode.



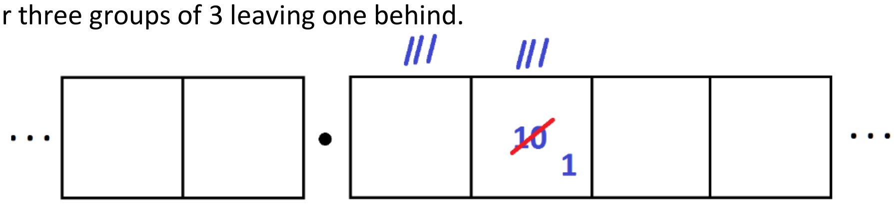

We see three groups of 3 leaving one behind.



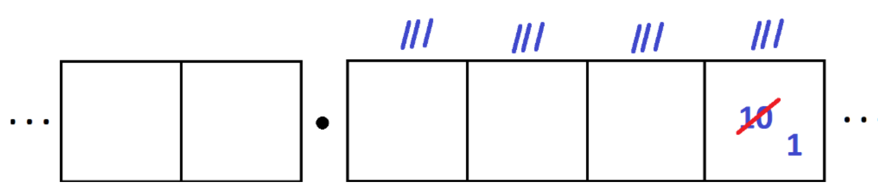

Unexploding gives another ten dots to examine.




We find another three groups of 3 leaving one behind.




And so on. We are caught in an infinitely repeating cycle.



This puts us in a philosophically interesting position. As human beings we cannot conduct this, or any, activity for an infinite amount of time. But it seems very tempting to write

$$\frac{1}{3} = 0.33333\cdots$$

with the ellipsis representing the instruction “keep going with this pattern forever.” In our minds we can almost imagine what this means. But as a practical human being it is beyond our abilities: one cannot actually write down those infinitely many 3s represented by the ellipses.

Nonetheless, many people choose not to contemplate what an infinite statement like this means and just carry on to say that some decimals are infinitely long and not be worried by it. In which case, the fraction $\frac{1}{3}$ is one of those fractions whose decimal expansion goes on forever.

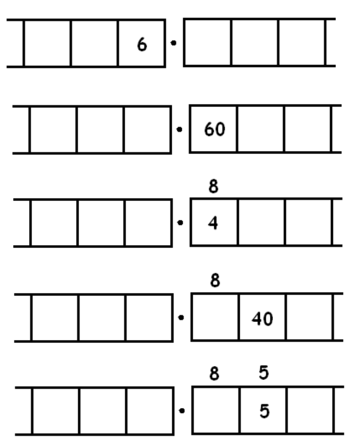
**Notation:** Many people make use of a *vinculum* (a horizontal bar) to represent infinitely long repeating decimals. For example, $0.\overline{3}$ means “repeat the 3 forever”

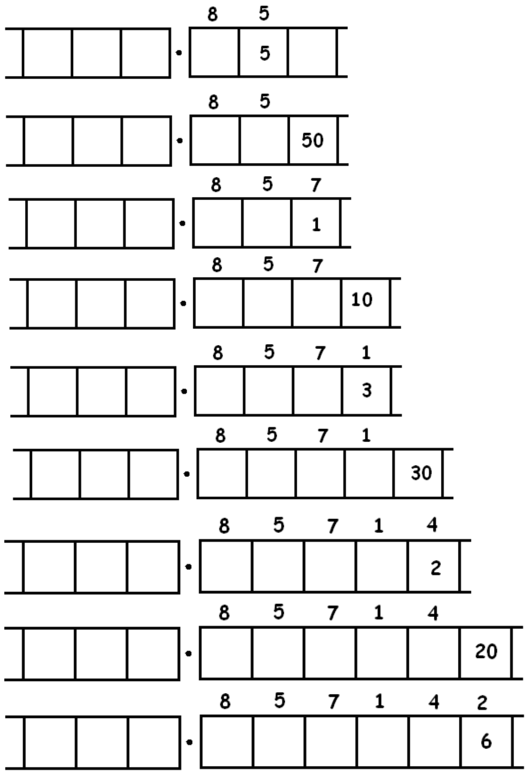
$$0.\overline{3} = 0.3333\cdots$$

and $0.38\overline{142}$ means “repeat the group 142 forever” after the beginning 38 hiccup:

$$0.38\overline{142} = 0.38142142142142\cdots.$$

As another (complicated) example, here is the work that converts the fraction $\frac{6}{7}$ to an infinitely long repeating decimal. Make sure to understand the steps one line to the next.





Do you see, with this 6 in the final right-most box that we have returned to the very beginning of the problem? This means that we shall simply repeat the work we have done and obtain the same sequence 857142 of answers, and then again, and then again. We have

$$\frac{6}{7} = 0.857142857142857142857142\cdots.$$

Some more practice questions if you like.

**Practice 53:** Compute $\frac{4}{7}$ as an infinitely long repeating decimal.

**Practice 54:** Compute $\frac{1}{11}$ as an infinitely long repeating decimal.

**Practice 55:** Which of the following fractions give infinitely long decimal expansions?

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10}$$

**Practice 56:** Use a $1 \leftarrow 10$ machine to compute $133 \div 6$, writing the answer as a decimal.

**Practice 57:** Use a $1 \leftarrow 10$ machine to compute $255 \div 11$, writing the answer as a decimal.

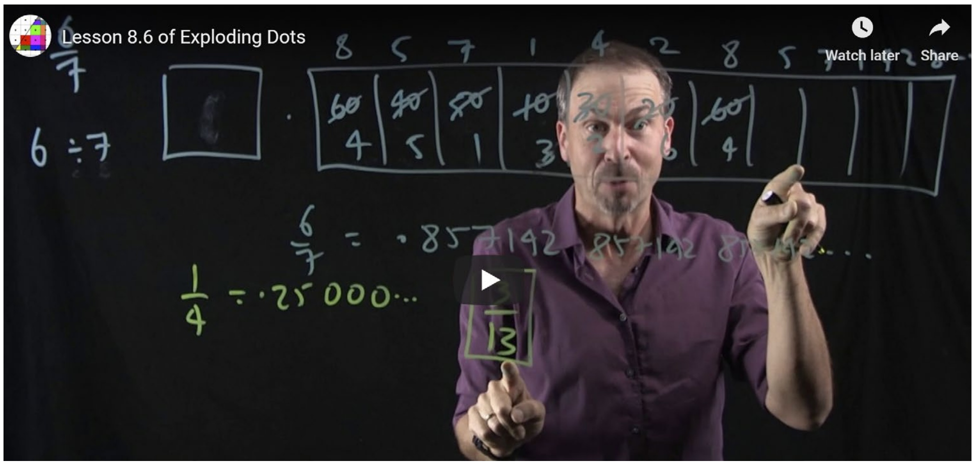
## IRRATIONAL NUMBERS

Video:

Irrational Numbers video (+ all follow-on content you see).

Web App:

Eighth Experience.




We have seen that fractions can possess finitely long decimal expansions. For example,

$$\frac{1}{8} = 0.125 \text{ and } \frac{1}{2} = 0.5 .$$

And we have seen that fractions can possess infinitely long decimal expansion. For example,

$$\frac{1}{3} = 0.3333.... \text{ and } \frac{6}{7} = 0.857142057142857142... .$$

All the examples of fractions with infinitely long decimal expansions we've seen so far fall into a repeating pattern. This is curious.

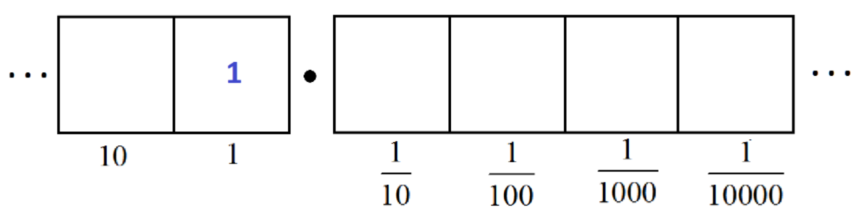
We can even say that our finite examples eventually fall into a repeating pattern too, a repeating pattern of zeros after an initial start.

$$\frac{1}{8} = 0.12500000.... = 0.125\overline{0}$$
$$\frac{1}{2} = 0.50000.... = 0.5\overline{0}$$
$$\frac{1}{3} = 0.\overline{3}$$
$$\frac{6}{7} = 0.\overline{857142}$$

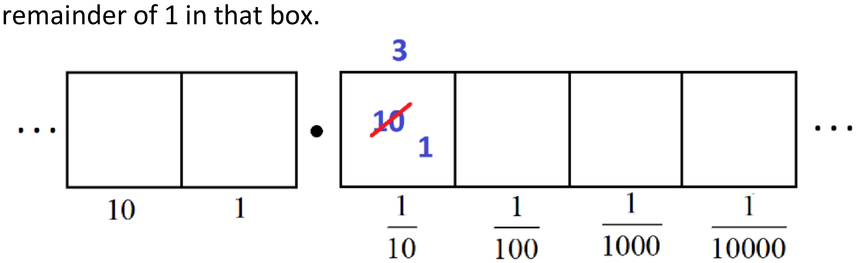
This begs the question: *Does every fraction have a decimal representation that eventually repeats?*

The answer to this question is YES and our method of division explains why.

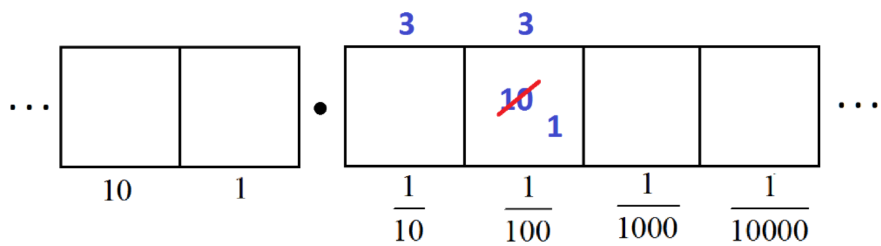
Let's go through the division process again, slowly, first with a familiar example. Let's compute the decimal expansion of $\frac{1}{3}$ again in a $1 \leftarrow 10$ machine. We think of $\frac{1}{3}$ as the answer to the division problem $1 \div 3$, and so we need to find groups of three within a diagram of one dot.




We unexplode the single dot to make ten dots in the tenths position. There we find three groups of three leaving a remainder of 1 in that box.

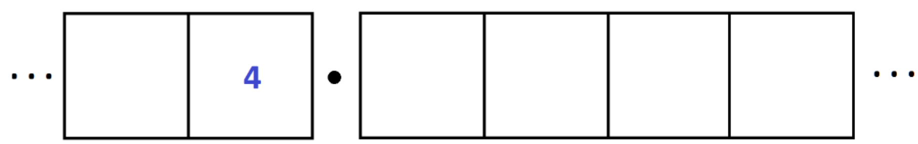



Now we can unexploded that single dot in the tenths box and write ten dots in the hundredths box. There we find three more groups of three, again leaving a single dot behind.

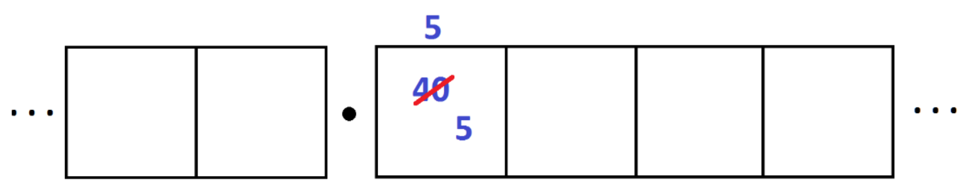



And so on. We are caught in a cycle of having the same remainder of one dot from cell to cell, meaning that the same pattern repeats. Thus we conclude $\frac{1}{3} = 0.333\ldots$. The key point is that the same remainder of a single dot kept appearing.

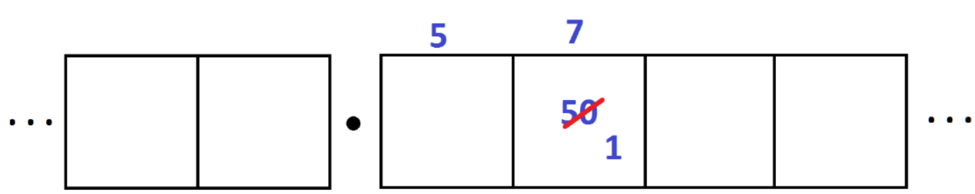
Here's a more complicated example. Let's compute the decimal expansion of $\frac{4}{7}$ in the $1 \leftarrow 10$ machine. That is, let's compute $4 \div 7$.



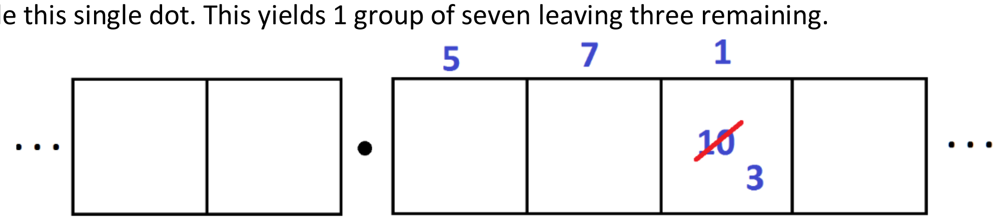

We start by unexploding the four dots to give 40 dots in the tenths cell. There we find 5 groups of seven, leaving five dots over.



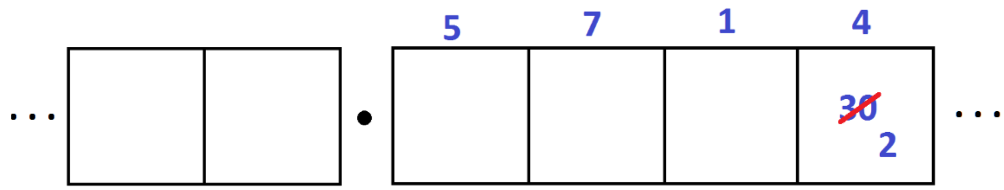

Now unexplode those five dots to make 50 dots in the hundredths position. There we find 7 groups of seven, leaving one dot over.



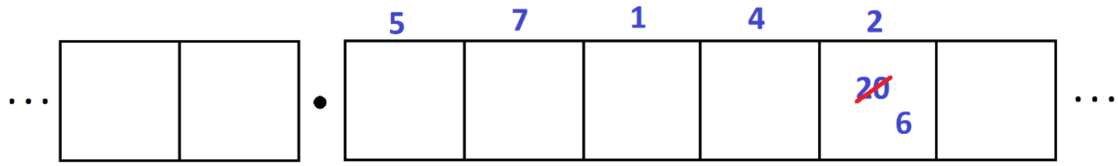

Unexplode this single dot. This yields 1 group of seven leaving three remaining.



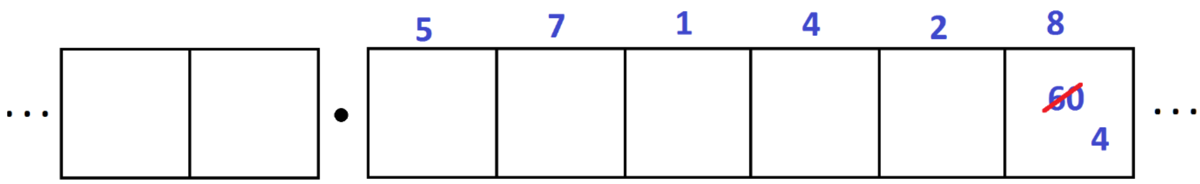

Unexplode these three dots. This gives 4 groups of seven with two remaining.




Unexplode the two dots. This gives 2 groups of seven with six remaining.




Unexplode the six dots. This gives 8 groups of seven with four remaining.




But this is the predicament we started with: four dots in a box!

So now we are going to repeat the pattern and produce a cycle in the decimal representation. We have

$$\frac{4}{7} = 0.571428 \quad 571428 \quad 571428 \quad \cdots .$$

Stepping back from the specifics of this problem, it is clear now that one must be forced into a repeating pattern. In dividing a quantity by seven, there are only seven possible numbers for a remainder number of dots in a cell - 0, 1, 2, 3, 4, 5, or 6 – and there is no option but to eventually repeat a remainder and so enter a cycle.

In the same way, the decimal expansion of $\frac{18}{37}$ must also cycle. In doing the division, there are only thirty-seven possible remainders for dots in a cell (0, 1, 2, ..., 36). As we complete the division computation, we must eventually repeat a remainder and again fall into a cycle.

We have just established a very interesting fact.

**ALL FRACTIONS HAVE A REPEATING DECIMAL REPRESENTATION.**

(A repeating pattern of zeros is possible. In fact, as a check, conduct the division procedure for the fraction $\frac{1}{8}$. Make sure to understand where the cycle of repeated remainders commences.)

This now opens up a curious idea:

**A quantity given by a decimal expansion that does not repeat cannot be a fraction!**

For example, the quantity

$$0.10\ 1100\ 111000\ 11110000\ 1111100000\cdots$$

is designed not to repeat (though there is a pattern to this decimal expansion) and so represents a number that is not a fraction.

A number that is equivalent to the ratio of two whole numbers (that is, a fraction) is called a *rational number*. A number that cannot be represented this way is called an *irrational number.*

It looks like we have just proved that irrational numbers exist! Not all numbers are fractions.

In fact, we can now invent all sorts of numbers that can't be fractions! For example

$$0.102030405060708090100110120130140150\cdots$$

and

$$0.3030030003000030000030000003\cdots$$

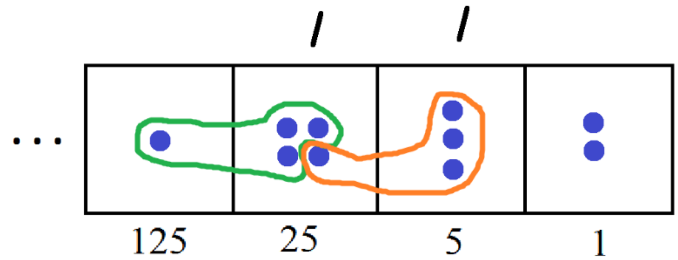
are irrational numbers.

Of course, we have all heard that numbers like $\sqrt{2}$ and $\pi$ are irrational numbers. It is not at all obvious why they are and how you would go about proving that they are. (In fact, it took mathematicians about 2000 years to finally establish that $\pi$ is an irrational number. Swiss mathematician Johann Lambert finally proved it so in 1761.) But if you are willing to believe that these numbers are irrational, then you can say for sure that their decimal expansions possess no repeating patterns!

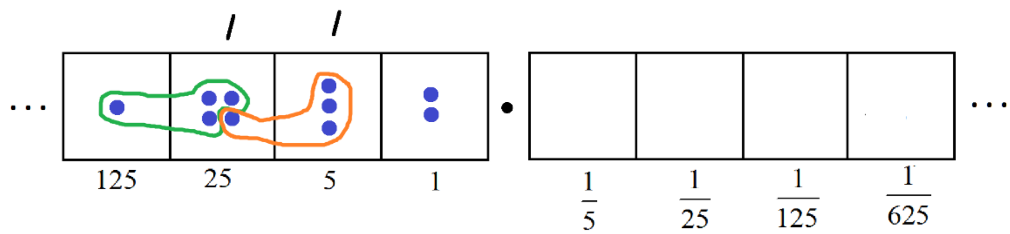
## DECIMALS IN OTHER BASES

Who said we need to stay with a $1 \leftarrow 10$ machine?

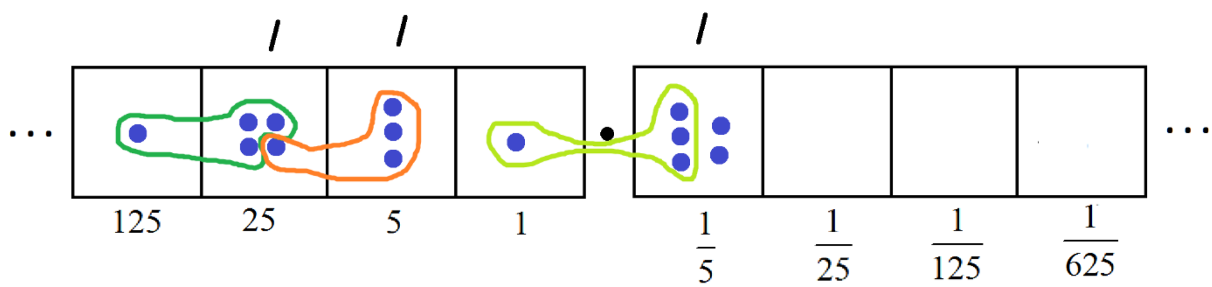
The following picture shows that $1432 \div 13 = 110\ R\ 2$ in a $1 \leftarrow 5$ machine.



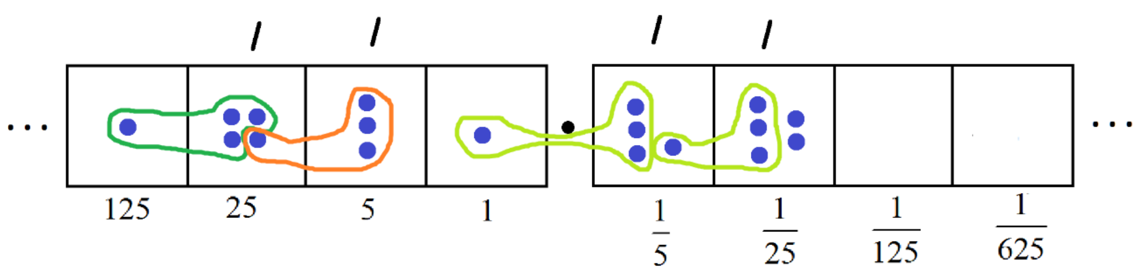

If we work with reciprocals of powers of five we can keep unexploding dots and continue the division process.



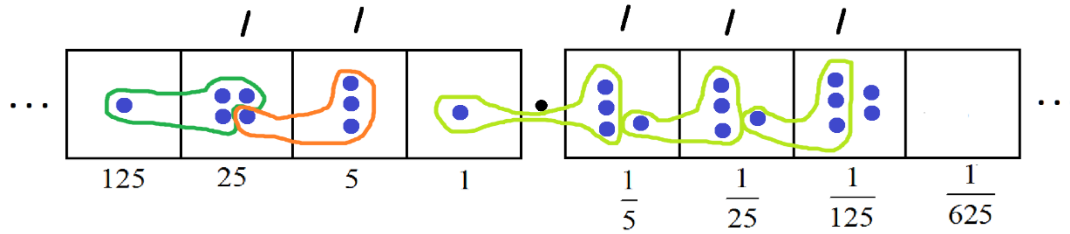

Here goes!




(This reads as $1432 \div 13 = 110.1\ R\ 0.2$ if you like.)




(This reads as $1432 \div 13 = 110.11 R\ 0.02$ if you like.)




And so on. We get, as a statement of base $5$ arithmetic, the following:

$$1432 \div 13 = 110.1111\cdots.$$

To translate this to ordinary arithmetic we have that

$1432$ in base five is $1 \times 125 + 4 \times 25 + 3 \times 5 + 1 \times 1 = 242$,
$13$ in base five is $1 \times 5 + 3 \times 1 = 8$,
$110.111...$ in base five is $30 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$ ,

so we are claiming, in ordinary arithmetic, that

$$242 \div 8 = 30 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots .$$

Whoa!

Here are some (challenging) practice questions if you are up for them.

**Practice 58:** Compute $8 \div 3$ in a base $10$ machine and show that it yields the answer $2.666\cdots$.

**Practice 59:** Compute $1 \div 11$ as a problem in base $3$ and show that it yields the answer $0.0202020202\cdots$.

(In base three, "$11$" is the number four, and so this question establishes that the fraction $\frac{1}{4}$ written in base three is $0.0202020202\cdots$.)

**Practice 60:** Show that the fraction $\frac{2}{5}$, written here in base ten, has the "decimal" representation $0.121212\cdots$ in base four. (That is, compute $2 \div 5$ in a $1 \leftarrow 4$ machine.)

**Practice 61: CHALLENGE:** What fraction has decimal expansion $0.3333\cdots$ in base $7$? Is it possible to answer this question by calling this number $x$ and multiplying both sides by $10$? (Does "$10$" represent ten?)

**Practice 62:** Use an $1 \leftarrow x$ machine and $x$-mals to show that $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots$.

## WILD EXPLORATIONS

Here again are some "big question" investigations you might want to explore, or just think about before moving on. Have fun!

### Wild Exploration 1: WHICH FRACTIONS GIVE FINITE DECIMAL EXPANSIONS?

We've seen that $\frac{1}{2} = 0.5$ and $\frac{1}{4} = 0.25$ and $\frac{1}{8} = 0.125$ each have finite decimal expansions. (We're ignoring infinite repeating zeros now.)

Of course, all finite decimal expansions give fractions with finite decimal expansions! For example, $0.37$ is the fraction $\frac{37}{100}$, showing that $\frac{37}{100}$ has a finite decimal expansion.

What must be true about the integers $a$ and $b$ (or true just about $a$ or just about $b$) for the fraction $\frac{a}{b}$ to have a finite decimal expansion?

### Wild Exploration 2: BACKWARDS: Are repeating decimals fractions?

We have seen that $\frac{1}{3} = 0.\overline{3}$ and $\frac{4}{7} = 0.\overline{571428}$, for example, and that every fraction gives a decimal expansion that (eventually) repeats, perhaps with repeating zeros.

Is the converse true? Does every infinitely repeating decimal fraction correspond to a number that is a fraction?

> Is $0.\overline{17}$ a fraction? If so, which fraction is it?
> Is $0.\overline{450}$ a fraction? If so, which fraction is it?
> Is $0.322222... = 0.3\overline{2}$ a fraction? Is $0.17\overline{023}$ a fraction?

Indeed, does every repeating decimal have a value that is a fraction?

# 4. ALL BASES, ALL AT ONCE

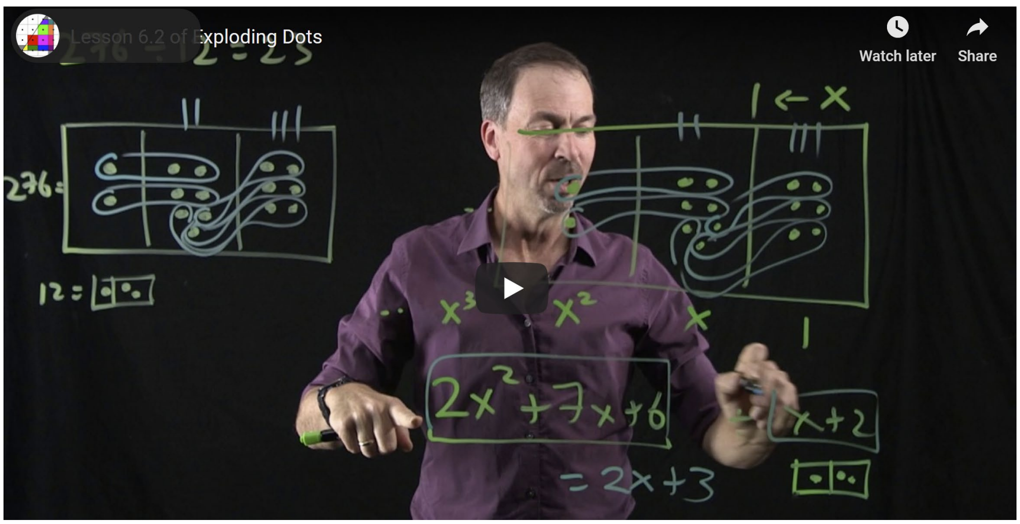
Videos:

Introductory video.

Division in any base video (+ all follow-on content you see).

Web App:

Sixth Experience.




Okay. Up to now we have been dealing with grade-school or primary-school arithmetic. Let's now head on to advanced high school algebra.

Whoa!

But here's the thing: there is nothing to it. We've already done all the work.

The only thing we have to realize is that there is nothing special about a $1 \leftarrow 10$ machine. We could do all of grade school arithmetic in a $1 \leftarrow 2$ machine if we wanted to, or a $1 \leftarrow 5$ machine, or even a $1 \leftarrow 37$ machine. The math doesn't care in which machine we do it. It is only us humans with a predilection for the number ten that draws us to the $1 \leftarrow 10$ machine.

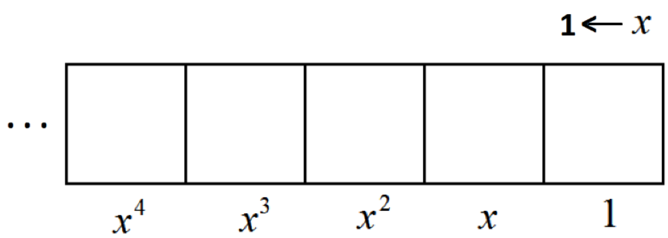
Let's now go through much of what we've done. But let's now do it in all possible machines, all at once! Sounds crazy. But it is surprisingly straightforward.

So, what I am going to do is draw a machine on the board, but I am not going to tell you which machine it is. It could be a $1 \leftarrow 10$ machine again, I am just not going to say. Maybe it will be a $1 \leftarrow 2$ machine, or a $1 \leftarrow 4$ machine or a $1 \leftarrow 13$ machine. You just won't know as I am not telling. It's the mood I am in!

Now, in high school algebra there seems to be a favorite letter of the alphabet to use for a quantity whose value you do not know. It's the letter $x$. (Always $x$ ! It's $x$ and $x$ and $x$ and ….)

So, let's work with an $1 \leftarrow x$ machine with the letter $x$ representing some number whose actual value we do not know.

In a $1 \leftarrow 10$ machine the place values of the boxes are the powers of ten: 1, 10, 100, 1000, … .
In a $1 \leftarrow 2$ machine the place values of the boxes are the powers of two: 1, 2, 4, 8, 16, … .
And so on. Thus, in an $1 \leftarrow x$ machine, the place values of the boxes will be the powers of $x$.

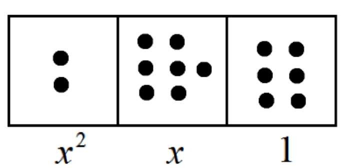



As a check, if I do tell you that $x$ actually is 10 in my mind, then the powers $1,\ x,\ x^2,\ x^3$, … match the numbers 1, 10, 100, 1000, … which is correct for a $1 \leftarrow 10$ machine. If, instead, I tell you that $x$ is really 2 in my mind, then the powers $1,\ x,\ x^2,\ x^3$, … match the numbers 1, 2, 4, 8, … which is correct for a $1 \leftarrow 2$ machine.This $1 \leftarrow x$ machine really is representing all machines, all at once!

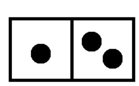
Okay. Out of the blue! Here's an advanced high school algebra problem.

**Compute:** $\left(2x^2 + 7x + 6\right) \div \left(x + 2\right)$.

Here's what $2x^2 + 7x + 6$ looks like in an $1 \leftarrow x$ machine. It's two $x^2$'s, seven $x$'s, and six ones.

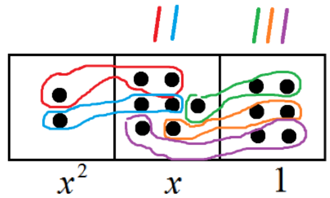
$2x^2 + 7x + 6 =$ 

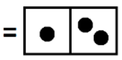

And here's what $x + 2$ looks like.

$x + 2 =$ 

The division problem $(2x^2 + 7x + 6) \div (x + 2)$ is asking us to find copies of $x + 2$ in the picture of $2x^2 + 7x + 6$.

$2x^2 + 7x + 6 =$




$x + 2 =$ 

I see two copies of $x + 2$ at the $x$ level and three copies at the 1 level. The answer is $2x + 3$.

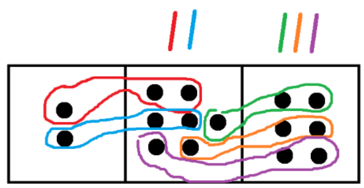
> **Stare at the picture for $(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$. Does it look familiar?**
> Look back at the picture of $276 \div 12 = 23$ we did back in section 1.2.

We have identical pictures!

**In a $1 \leftarrow 10$ machine.**

$276 \div 12$

$= 23$



**In a $1 \leftarrow x$ machine.**

$(2x^2 + 7x + 6) \div (x + 2)$

$= 2x + 3$

**SAME PICTURE!**

We've just done a high school algebra problem as though it is a grade school arithmetic problem!

What's going on?

Suppose I told you that $x$ really was 10 in my head all along. Then $2x^2 + 7x + 6$ is the number $2 \times 100 + 7 \times 10 + 6$, which is 276. And $x + 2$ is the number 10 + 2, that is, 12. And so, we computed 276 ÷ 12. We got the answer $2x + 3$, which is $2 \times 10 + 3 = 23$, if I am indeed now telling you that $x$ is 10.

So, we did just repeat a grade-school arithmetic problem!

> **Aside:** By the way, if I tell you that $x$ was instead 2, then
>
> $2x^2 + 7x + 6 = 2 \times 4 + 7 \times 2 + 6$, which is 28,
>
> $x + 2 = 2 + 2$, which is 4,
>
> and
>
> $2x + 3 = 2 \times 2 + 3$, which is 7.
>
> We just computed 28 ÷ 4 = 7, which is correct!
>
> Doing division in an $1 \leftarrow x$ machine is really doing an infinite number of division problems all in one hit. Whoa!

**Question:** Try computing $(2x^3 + 5x^2 + 5x + 6) \div (x + 2)$ in an $1 \leftarrow x$ machine to get the answer $2x^2 + x + 3$ (And if I tell you $x$ is 10 in my mind, can you see that this matches 2256 ÷ 12 = 213?)

In high school, numbers expressed in an $1 \leftarrow x$ machine are usually called *polynomials*. They are just like numbers expressed in base 10, except now they are "numbers" expressed in base $x$. (And if someone tells you $x$ is actually 10, then they really are base 10 numbers!)

Keeping this in mind makes so much of high school algebra so straightforward: it is a repeat of grade school base 10 arithmetic.

**Practice 63:**

a) Compute $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1)$.
b) Compute $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1)$.

If I tell you that $x$ is actually 10 in both these problems what two division problems in ordinary arithmetic have you just computed?

**Practice 64:** Here's a polynomial division problem written in fraction notation. Can you do it? (Is there something tricky to watch out for?)

$$\frac{x^4 + 2x^3 + 4x^2 + 6x + 3}{x^2 + 3}$$

**Practice 65:** Show that $(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1)$ equals $x^3 + 3x^2 + 3x + 1$.

a) What is this saying for $x = 10$?
b) What is this saying for $x = 2$?
c) What is this saying for $x$ equal to each of 3, 4, 5, 6, 7, 8, 9, and 11?
d) What is this saying for $x = 0$?
e) What is this saying for $x = -1$?

## A PROBLEM!

| |
|---|
| Video:<br>A problem video (+ all follow-on content you see).<br><br>Web App:<br>Sixth Experience. |

Okay. Now that we are feeling really good about doing advanced algebra, I have a confession to make.

I've been fooling you!

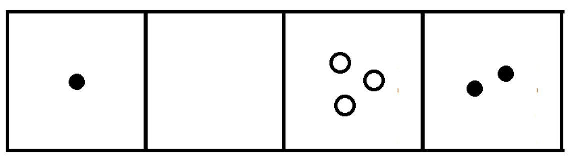
I've been choosing examples that are designed to be nice and to work out just beautifully. The truth is, this fabulous method of ours doesn't usually work so nicely.

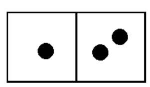
Consider, for example,

$$\frac{x^3 - 3x + 2}{x + 2}$$

Do you see what I've been avoiding all this time? Yep. Negative numbers.

Here's what I see in an $1 \leftarrow x$ machine.

$x^3 - 3x + 2 =$ 

$x+2 =$ 

We are looking for one dot next to two dots in the picture of $x^3 - 3x + 2$. And I don't see any! So, what can we do, besides weep a little?

Do you have any ideas?

We need some amazing flash of insight for something clever to do. Or maybe polynomial problems with negative numbers just can't be solved with this dots and boxes method.

So, what do you think? Any flashes of insight?

## RESOLUTION

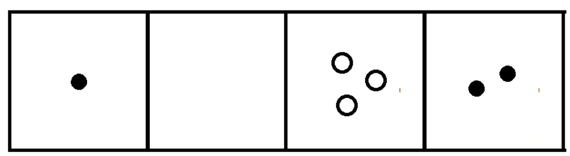
| |
|---|
| Video:<br>Resolution video (+ all follow-on content you see).<br><br>Web App:<br>Sixth Experience. |

We are stuck on

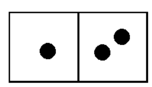
$$\frac{x^3 - 3x + 2}{x + 2}$$

with the $1 \leftarrow x$ machine picture

$\boldsymbol{x^3 - 3x + 2 \;=}$



$\boldsymbol{x+2 \;=}$



We are looking for copies of $x + 2$, one dot next to two dots, anywhere in the picture of $x^3 - 3x + 2$.

We don't see any.

And we can't unexplode dots to help us out as we don't know the value of $x$. (We don't know how many dots to draw when we unexplode.)

The situation seems hopeless at present.

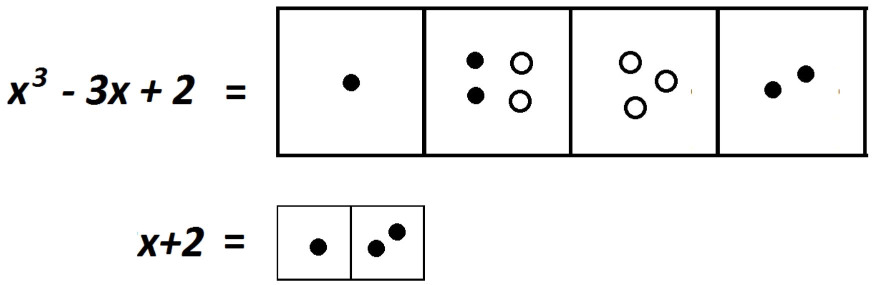
But I have a piece of advice for you, a general life lesson in fact. It's this.

**IF THERE IS SOMETHING IN LIFE YOU WANT, MAKE IT HAPPEN!**
**(And deal with the consequences.)**

Right now, is there anything in life we want?

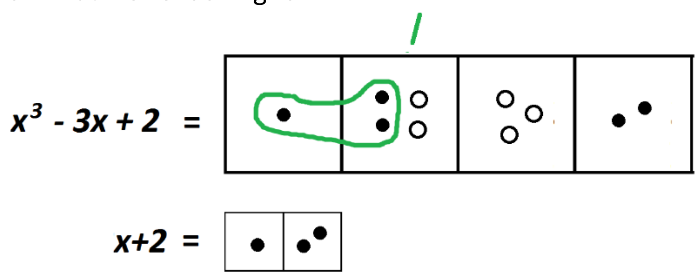
Look at that single dot way out to the left. Wouldn't it be nice if we had two dots in the box next to it, to make a copy of $x + 2$?

So, let's just put two dots into that empty box! That's what I want, so let's make it happen!
But there are consequences: that box is meant to be empty. And in order to keep it empty, we can put in two antidots as well!




Brilliant!

We now have one copy of what we're looking for.

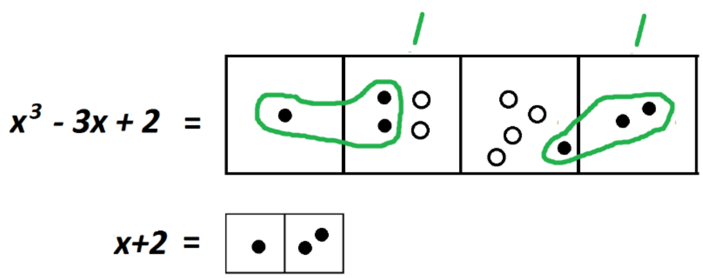



But there is still the question: Is this brilliant idea actually helpful?

Hmm.

Well. Is there anything else in life you want right now? Can you create another copy of $x + 2$ anywhere?

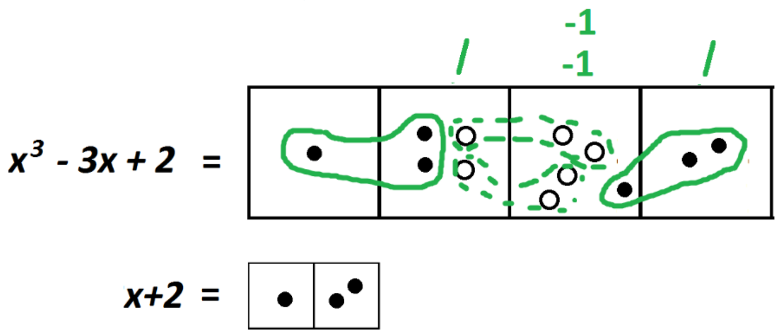
I'd personally like a dot to the left of the pair dots in the rightmost box. I am going to make it happen! I am going to insert a dot and antidot pair. Doing so finds me another copy of $x + 2$.




This is all well and good, but are we now stuck? Maybe this brilliant idea really just isn't helpful.

Stare at this picture for a while. Do you notice anything?

Look closely and we start to see copies of the exact opposite of what we're looking for! Instead of one dot next to two dots, there are copies of one antidot next to two antidots.




Whoa!

And how do we read the answer? We see that $(x^3 - 3x + 2) \div (x + 2)$ is $x^2 - 2x + 1$.
Fabulous!

So actually, I was lying about fooling you. We can actually do all polynomial division problems with this dots and boxes method, even ones with negative numbers!

**Question:** Show that $\frac{x^{10}-1}{x^2-1}$ equals $x^8 + x^6 + x^4 + x^2 + 1$.

**Practice 66:** Compute $\frac{x^3-3x^2+3x-1}{x-1}$.

**Practice 67:** Try $\frac{4x^3-14x^2+14x-3}{2x-3}$.

**Practice 68:** If you can do this problem, $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1}$, you can probably do any problem!

**Practice 69:** Can you deduce what the answer to $(2x^2 + 7x + 7) \div (x + 2)$ is going to be before doing it?

**Practice 70:** Compute $\frac{x^4}{x^2-3}$.

**Practice 71:** Try this crazy one: $\frac{5x^5-2x^4+x^3-x^2+7}{x^3-4x+1}$.

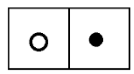
If you do this with paper and pencil, you will find yourself trying to draw 84 dots at some point. Is it swift and easy just to write the number "84"? In fact, how about just writing numbers and not bother drawing any dots at all?

**Aside**: Is there a general way to conduct the dots and boxes approach with ease on paper? Rather than draw boxes and dots, can one work with tables of numbers that keep track of coefficients? (The word *synthetic* is often used for algorithms one creates that are a step or two removed from that actual process at hand.)

## SOMETHING BOLD FOR THE BRAVE!

Here is a picture of the very simple polynomial 1 and the polynomial $1 - x$.

**1 = ⋯** 

**1-x =** 

Can you compute $\frac{1}{1-x}$? Can you interpret the answer?

**WILD EXPLORATIONS**

Here again are some “big question” investigations you might want to explore, or just think about before moving on. Have fun!

**Wild Exploration 1: CAN WE EXPLAIN AN ARITHMETIC TRICK?**

Here’s an unusual way to divide by nine.

To compute $21203 \div 9$, take the digits in “21203” from left to right computing the partial sums along the way as follows

| | |
|---|---|
| **2** | **= 2** |
| **2+1** | **= 3** |
| **2+1+2** | **= 5** |
| **2+1+2+0** | **= 5** |
| **2+1+2+0+3** | **= 8** |

and then read off the answer

$$21203 \div 9 = 2355 \ R \ 8 .$$

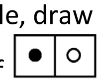
In the same way,

$$1033 \div 9 = 1 | \ 1+0 \ | 1+0+3 | \quad R \quad 1+0+3+3 \quad = 114 \ R \ 7$$

and

$$2222 \div 9 = 246 \ R \ 8$$

Can you explain why this trick works?

Here’s the approach I might take: for the first example, draw a picture of 21203 in a $1 \leftarrow 10$ machine, but think of nine as $10 - 1$. That is, look for copies of  in the picture.

**Wild Exploration 2: Exploding Number Theory**

Use an $1 \leftarrow x$ machine to compute each of the following

a) $\frac{x^2-1}{x-1}$ b) $\frac{x^3-1}{x-1}$ c) $\frac{x^6-1}{x-1}$ d) $\frac{x^{10}-1}{x-1}$

Can you now see that $\frac{x^{number}-1}{x-1}$ will always have a nice answer without a remainder?

Another way of saying this is that

$$x^{number} - 1 = (x-1) \times (\text{something}).$$

For example, you might have seen from part c) that $x^6 - 1 = (x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$.

This means we can say, for example, that $17^6 - 1$ is sure to be a multiple of 16! How? Just choose $x = 17$ in this formula to get

$$17^6 - 1 = (17-1) \times (\text{something}) = 16 \times (\text{something}).$$

a) Explain why $999^{100} - 1$ must be a multiple of 998.

b) Can you explain why $2^{100} - 1$ must be a multiple of 3, and a multiple of 15, and a multiple of 31 and a multiple of 1023? (Hint: $2^{100} = (2^2)^{50} = 4^{50}$, and so on.)

c) Is $x^{number} - 1$ always a multiple of $x + 1$? Sometimes, at least?

d) The number $2^{100} + 1$ is not prime. It is a multiple of 17. Can you see how to prove this?

More content:
Remainders.
Web App:
Sixth Experience.

# 5. WILD THINKING

Want to go truly wild?

Would you like to see and play with base one-and-a-half?

Would you like to play with base negative two?

Check out these extra videos and materials for some seriously wild exploration!

Experience 9: Weird and Wild Machines

Experience 11: Grape Codes and more.

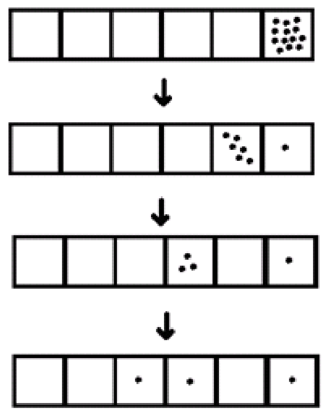
Experience 7: Infinite Sums

Experience 10: Unusual Numbers

The Web App.

# PRACTICE PROBLEM SOLUTIONS

**Practice 1:**
a) Here's how the code 1101 appears from thirteen dots.



b) The number fifty has code 110010.

**Practice 2:** Assuming we want to make the agreement that we'll always choose to explode dots if we can, then the code 100211 is not complete: the two dots in the third-to-last box can explode to give a final code of 101011.

**Practice 3:** This is the code for the number nineteen. (In the next section, we'll discover a swift way to see this.)

**Practice 4:** a) Do it! b) Do this one too! c) You're on a roll. Do this third one as well!

**Practice 5:** Again, if we agree to do all the explosions we can, then this code is not complete: three of the dots in the second-to-last box can explode to give 2111.

**Practice 6:** The number thirty-five has this code.

**Practice 7:** "Four dots in any one box explode and are replaced by one dot one place to the left." The number thirteen has code 31 in a $1 \leftarrow 4$ machine.

**Practice 8:** 23

**Practice 9:** 14

**Practice 10:** 22

**Practice 11:** 22 (Same code as the previous answer – but, of course, the interpretation of the code is different.)

**Practice 12:** a) 13 b) 37 c) 5846 (These are the codes we use for numbers in everyday life!)

**Practice 13:** Thirty-seven. It's a 32 and a 4 and a 1.

**Practice 14:** 11001000

**Practice 15:**

a) Each dot in the next box to the left is worth three 81's, that's 243.

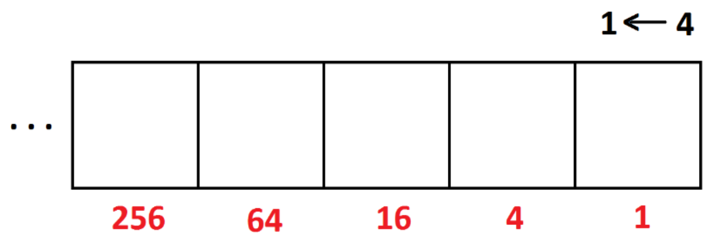
b) Yes it is okay to insert a zero at the front of the code. This would say that there are no 27's, which is absolutely correct. Deleting the end zero at the right, however, is problematic. 120 is the code for fifteen (one 9 and two 3's) but 12 is the code for five (one 3 and two 1's).

c) One hundred and ninety one. (Two 81's, one 27, and two 1's.)

d) 21102

**Practice 16:**

a) For a $1 \leftarrow 4$ machine, boxes have the following values:




b) The number twenty-nine has code 131 in a $1 \leftarrow 4$ machine.

c) Thirty. (This is one more than the code for twenty-nine!)

**Practice 17:** Might Venutians use base twelve? This means they will need twelve different symbols for writing numbers.

By the way, have you noticed that we use ten different symbols – 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 – which we call *digits*. (We call our fingers *digits* too!)

**Practice 18:**

$$148 + 323 = 4 \,|\, 6 \,|\, 11 = 471$$

$$567 + 271 = 7 \,|\, 13 \,|\, 8 = 838$$

$$377 + 188 = 4 \,|\, 15 \,|\, 15 = 5 \,|\, 5 \,|\, 15 = 565$$

$$582 + 714 = 12 \,|\, 9 \,|\, 6 = 1 \,|\, 2 \,|\, 9 \,|\, 6 = 1296$$

$$310462872 + 389107123 = 6 \,|\, 9 \,|\, 9 \,|\, 5 \,|\, 6 \,|\, 9 \,|\, 9 \,|\, 9 \,|\, 5 = 699569995$$

$$87263716381 + 18778274824 = 9 \,|\, 15 \,|\, 9 \,|\, 13 \,|\, 11 \,|\, 9 \,|\, 8 \,|\, 10 \,|\, 11 \,|\, 10 \,|\, 5$$
$$= \ldots = 106041991205$$

**Practice 19:** We have

$$26417 \times 4 = 8\,|\,24\,|\,16\,|\,4\,|\,28 = 10\,|\,4\,|\,16\,|\,4\,|\,28 = 1\,|\,0\,|\,4\,|\,16\,|\,4\,|\,28 = 1\,|\,0\,|\,5\,|\,6\,|\,4\,|\,28 = 105668$$
$$26417 \times 5 = 10\,|\,30\,|\,20\,|\,5\,|\,35 = 10\,|\,30\,|\,20\,|\,8\,|\,5 = 10\,|\,32\,|\,0\,|\,8\,|\,5 = 13\,|\,2\,|\,0\,|\,8\,|\,5 = 132085$$
$$26417 \times 9 = 18\,|\,54\,|\,36\,|\,9\,|\,63 = 18\,|\,54\,|\,36\,|\,15\,|\,3 = \ldots = 237753$$

$$26417 \times 10 = 20\,|\,60\,|\,40\,|\,10\,|\,70 = \cdots = 264170$$

and

$$26417 \times 11 = 22\,|\,66\,|\,44\,|\,11\,|\,77 = \ldots = 290587$$
$$26417 \times 12 = 24\,|\,72\,|\,48\,|\,12\,|\,84 = \ldots = 317004$$

For a full discussion as to why 26417 × 10 is 264170 read on!

**Practice 20:** 4760 and 47600.

**Practice 21:** $9190$ comes from $919 \times 10$, and so $9190 \div 10$ must be $919$.
Also, $3310000$ can be seen as coming from $33100 \times 100$ and so $3310000 \div 100$ must be $33100$.

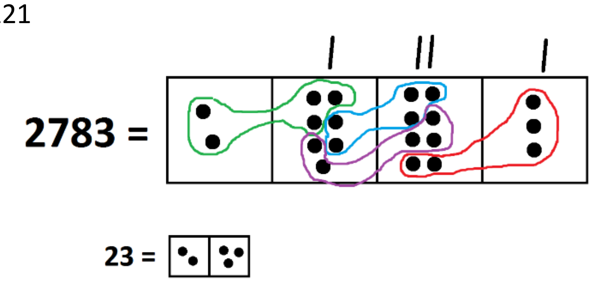
**Practice 22:**

$$6328 - 4469 = 2\,|\,-1\,|\,-4\,|\,-1 = 1\,|\,9\,|\,-4\,|\,-1 = 1\,|\,8\,|\,6\,|\,-1 = 1\,|\,8\,|\,5\,|\,9 = 1859$$

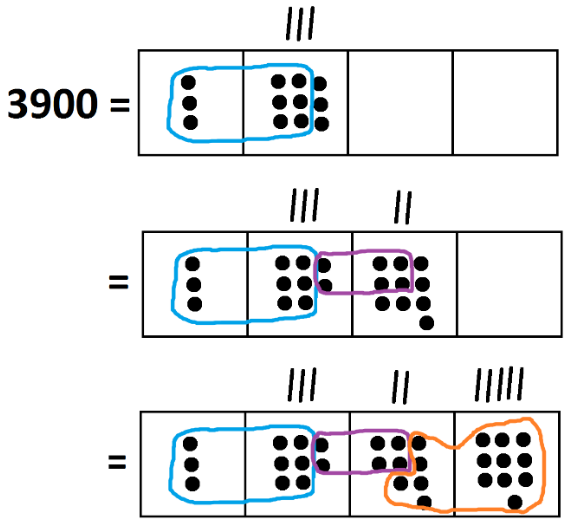
$$\begin{aligned}78390231 - 32495846 &= 4\,|\,6\,|\,-1\,|\,0\,|\,-5\,|\,-6\,|\,-1\,|\,-5\\ &= 4\,|\,5\,|\,9\,|\,0\,|\,-5\,|\,-6\,|\,-1\,|\,-5\\ &= 4\,|\,5\,|\,8\,|\,10\,|\,-5\,|\,-6\,|\,-1\,|\,-5\\ &= 4\,|\,5\,|\,8\,|\,9\,|\,5\,|\,-6\,|\,-1\,|\,-5\\ &= 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,4\,|\,-1\,|\,-5\\ &= 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,3\,|\,9\,|\,-5\\ &= 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,3\,|\,8\,|\,5 \qquad = 45894385\end{aligned}$$

I personally find it much easier to do the unexplosions from left to right.

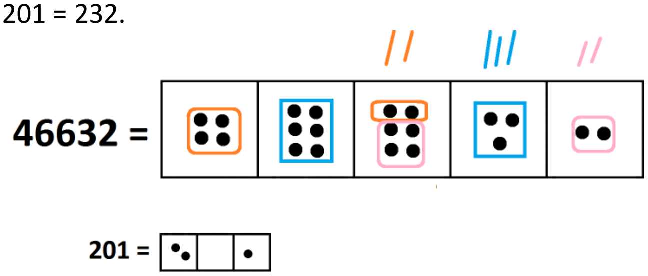
**Practice 23:** 2783 ÷ 23 = 121



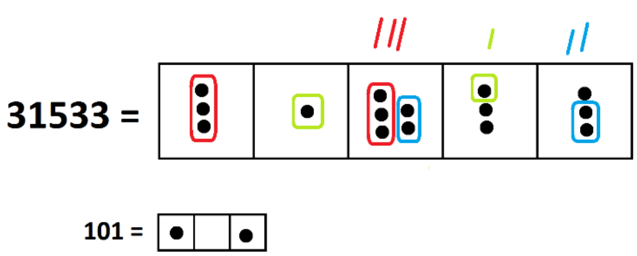
**Practice 24:** 3900 ÷ 12 = 325.
We need some unexplosions along the way. (And can you see how I am getting efficient with my loop drawing?)



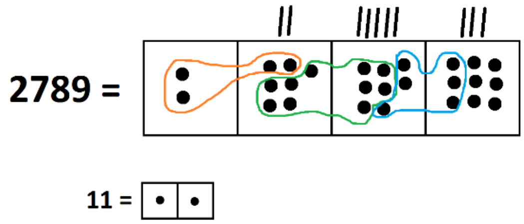

**Practice 25:** 46632 ÷ 201 = 232.



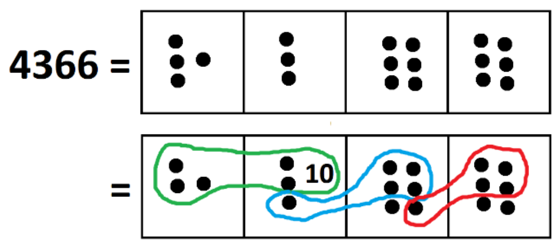

**Practice 26:** 31533 ÷ 101 = 312 with a remainder of 21. That is, $31533 \div 101 = 312 + \frac{21}{101}$



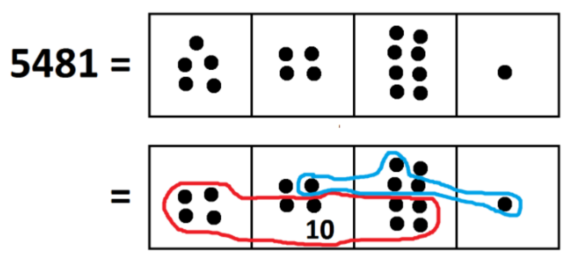
**Practice 27:** We have 2789 ÷ 11 = 253 with a remainder of 6. That is, $2789 \div 11 = 253 + \frac{6}{11}$.



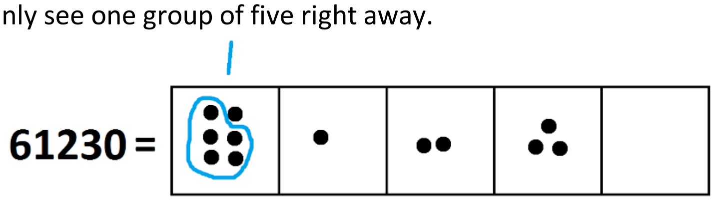

**Practice 28:** $4366 \div 14 = 311 + \frac{12}{14}$.



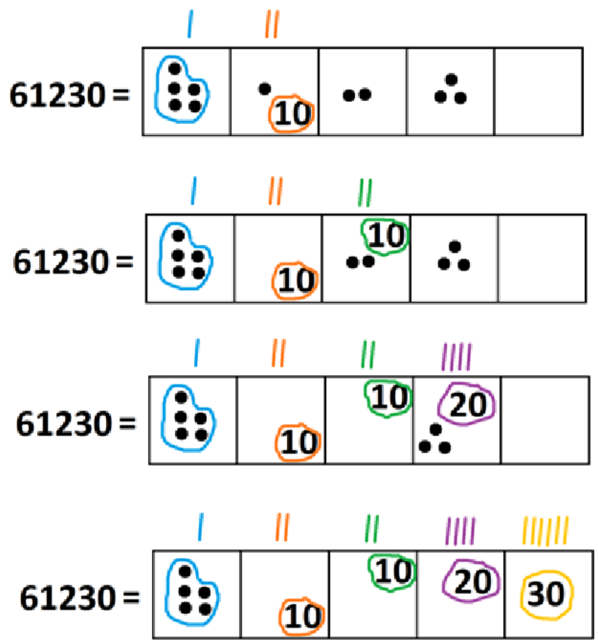

**Practice 29:** $5481 \div 131 = 41 + \frac{110}{131}$.




**Practice 30:** We certainly see one group of five right away.




Let's perform some unexplosions. (And let's write numbers rather than draw lots of dots. Drawing dots gets tedious!)




We see $61230 \div 5 = 12246$.

**Practice 31:** "Six tenths" says exactly what the number is.

**Practice 32:** Indeed both students are correct. Performing some explosions on Subra's picture gives JinJin's picture. (And performing some unexplosions on JinJin's picture gives Subra's!)

**Practice 33:** a) B  b) C

**Practice 34:** We have, in order, $\frac{1}{20}$, $\frac{1}{5}$, $\frac{4}{5}$, and $\frac{1}{250}$.

**Practice 35:** have, in order, $\frac{4}{10} = 0.4$, $\frac{4}{100} = 0.04$, $\frac{5}{100} = 0.05$, $\frac{5}{1000} = 0.005$, and $\frac{8}{10000} = 0.0008$.

**Practice 36:** a) B  b) D

**Practice 37:** In order, we have $0.35$, $0.64$, $0.602$, $0.34$, and $0.75$.

**Practice 38:** a) As a mixed number it is $2\frac{3}{10}$. Or we could write $2 + \frac{3}{10} = \frac{20}{10} + \frac{3}{10} = \frac{23}{10}$.

b) $17+\dfrac{4}{100}=17\dfrac{1}{25}$.

c) $1003\dfrac{1003}{10000}$.

**Practice 39:** a) Unexploding one dot in the tenths box does “add” ten dots to the hundredths box.

b) Unexploding one dot in the tenths box does now give us 19 dots in the hundredths box. Unexploding these gives 190 dots in the thousandths box.

c) There isn’t much to say here. One can see that one can convert any one picture to any other by explosions and/or unexplosions.

d) Yes! All the pictures are the same, so 0.19 and 0.190 are the same number.

**Practice 40:** a) $5.3$

b) $7+\dfrac{2}{10}=7.2$

c) $13.5$

d) $106+\dfrac{15}{100}=106.15$

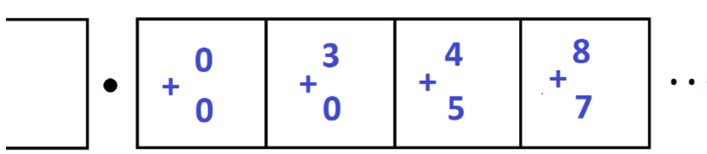
e) $3+\dfrac{3}{25}=3+\dfrac{12}{100}=3.12$

f) $2+\dfrac{1}{4}=2+\dfrac{25}{100}=2.25$

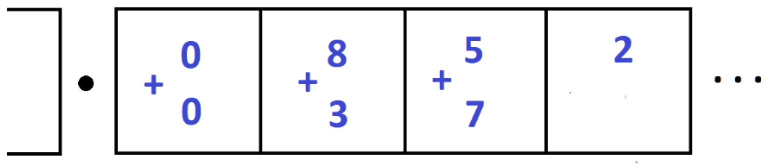
g) $3+\dfrac{11}{40}=3+\dfrac{275}{1000}=3.275$

**Practice 41:** $0.05+0.006=0.056$ and $0.05-0.006=0.0\,|\,5\,|\,{-6}=0.044$. Drawing the dots and boxes makes these answers clear.

**Practice 42:** If I draw dots and boxes, then I agree with Agatha.




But I don’t agree with Percy.




**Practice 43:** Multiplying by $100$ is the same as multiplying by ten and then multiplying by ten again.

**Practice 44:** We have $$0.04\times 0.5=\frac{4}{100}\times\frac{5}{10}=\frac{20}{1000}=\frac{2}{100}=0.02.$$

(We could have seen this right away too if we noted that $0.5$ is a half, and one half of $0.04$ is $0.02$.)

We have $1000\times 0.0385=1000\times\left(\frac{3}{100}+\frac{8}{1000}+\frac{5}{10000}\right)=30+8+\frac{5}{10}=38.5$.

**Practice 45:** We have $\frac{0.9}{10}=\frac{9}{100}=0.09$.

We have $\frac{2.34}{1000}=\frac{1}{1000}\times\left(2+\frac{3}{10}+\frac{4}{100}\right)=\frac{2}{1000}+\frac{3}{10000}+\frac{4}{100000}=0.00234$ .

We have $\left(40+\frac{4}{100}\right)\times\frac{1}{100}=\frac{4}{10}+\frac{4}{10000}=0.4004$.

**Practice 46:** $\frac{0.75}{25}=\frac{75}{2500}=\frac{3}{100}=0.03$.

**Practice 47:** This is technically a yes/no question, and the smart answer is no.

For those who answered yes, here are my approaches to the problems.

*a)* $$0.3\times(5.37-2.07)=0.3\times(3.3)=\frac{3}{10}\times\left(3+\frac{3}{10}\right)=\frac{9}{10}+\frac{9}{100}=0.99$$

*b)* $$\frac{0.1+(1.01-0.1)}{0.11+0.09}=\frac{0.1+(0.91)}{0.2}=\frac{1.01}{0.2}=\frac{101}{20}=\frac{505}{100}=5.05.$$

*c)*

$$\frac{(0.002+0.2\times 2.02)(2.2-0.22)}{2.22-0.22}=\frac{(0.002+0.404)(1.98)}{2}$$

$$=\frac{(0.406)(1.98)}{2}$$

$$=\frac{\left(\frac{4}{10}+\frac{6}{1000}\right)\left(1+\frac{9}{10}+\frac{8}{100}\right)\times 1000\times 100}{2\times 1000\times 100}$$

$$=\frac{(400+6)(100+90+8)}{2\times 1000\times 100}$$

$$=\frac{406\times 198}{2\times 1000\times 100}$$

$$=\frac{80388}{2\times 100000}$$

$$=\frac{40194}{100000} \qquad = 0.40194$$

Whoa!

**Practice 48:** Do it!

**Practice 49:** Do it too!

**Practice 50:** Yep. Do it!

**Practice 51:** You guessed it. Do it!

**Practice 52:** In order, we have $\frac{3}{4}$, $\frac{5}{8}$, $\frac{4}{25}$, $\frac{17}{20}$, and $\frac{5}{80}$.

**Practice 53:** We get $\frac{4}{7}=0.571428\ 571428\ 571428\ 571428\ ...$ .

**Practice 54:** We get $\frac{1}{11}=0.09\ 09\ 09\ 09\ ...$ .

**Practice 55:** The fractions $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, and $\frac{1}{9}$ have infinitely long decimal expansions.

**Practice 56:** $\frac{133}{6} = 22.166666...$ .

**Practice 57:** $\frac{255}{11} = 23.18181818....$ .

**Practice 58:** Do it!

**Practice 59:** Do it too!

**Practice 60:** Yep, do it!

**Practice 61:** Let's do all our writing in a $1 \leftarrow 7$ machine. So the number seven is $10$ and one less than this number is $6$. Also, notice that multiplying a number by seven in a $1 \leftarrow 7$ machine has the effect of addending a zero at the end of the number. (Can you see why? It is the same reason why multiplying by ten in a $1 \leftarrow 10$ machine has the effect of addending a zero.) Multiplying by seven also "shifts the decimal point" one place.

> Set $0.333.... = x$.
>
> Multiply each side by seven (which looks like $10$). This gives
>
> $$3.3333... = 10x .$$
>
> The left side is $3 + 0.3333...$, that is, $3$ plus the original number.
>
> $$3 + x = 10x$$
>
> Subtract $x$ from both sides.
>
> $$3 = 6x$$
>
> So $x$ must equal a half!

**Practice 62:** Can you see how to do it?

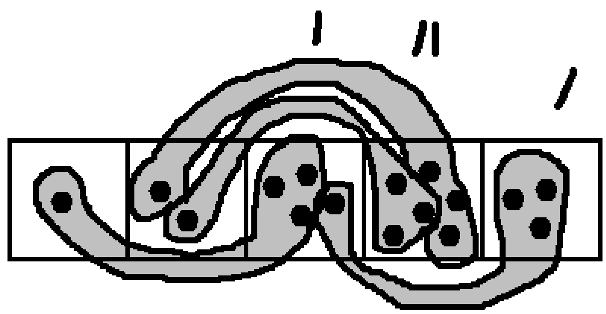
**Practice 63:**

a) $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1) = \; x^3 + x^2 + 2x + 1$

b) $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1) = x^2 + 2x + 3$

And if $x$ happens to be 10, we've just computed 23541 ÷ 21 = 1121 and 13653 ÷ 111 = 123.

**Practice 64:** We can do it. The answer is $x^2 + 2x + 1$.



**Practice 65:**

a) For $x = 10$ it says 14641 ÷ 11 = 1331

b) For $x = 2$ it says 81 ÷ 3 = 27

c) For $x = 3$ it says 256 ÷ 4 = 64
For $x = 4$ it says 625 ÷ 5 = 125
For $x = 5$ it says 1296 ÷ 6 = 216
For $x = 6$ it says 2401 ÷ 7 = 343
For $x = 7$ it says 4096 ÷ 8 = 512
For $x = 8$ it says 6561 ÷ 9 = 729
For $x = 9$ it says 10000 ÷ 10 = 1000
For $x = 11$ it says 20736 ÷ 12 =1728

d) For $x = 0$ it says 1 ÷ 1 = 1.

e) For $x = -1$ it says 0 ÷ 0 = 0. Hmm! That's fishy! (Can you have a 1 ← 0 machine?)

**Practice 66:** $\frac{x^3-3x^2+3x-1}{x-1} = x^2 - 2x + 1.$

**Practice 67:** $\frac{4x^3-14x^2+14x-3}{2x-3} = 2x^2 - 4x + 1.$

**Practice 68:** $\frac{4x^5-2x^4+7x^3-4x^2+6x-1}{x^2-x+1} = 4x^3 + 2x^2 + 5x - 1.$

**Practice 69:** We know that $(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$, so I bet $(2x^2 + 7x + 7) \div (x + 2)$ turns out to be $2x + 3 + \frac{1}{x+2}$. Does it? Can you make sense of remainders?

**Practice 70:** $\frac{x^4}{x^2-3} = x^2 + 3 + \frac{9}{x^2-3}$.

**Practice 71:** $5x^2 - 2x + 21 + \frac{-14x^2+86x-14}{x^3-4x+1}$.