GLOBAL MATH PROJECT

Uplifting Mathematics for All

PUZZLES

explained with **EXPLODING DOTS**

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the Global Math Project

Here is a sample of puzzles—some classic, some new—that can be beautifully analyzed and well understood with an Exploding Dots approach. Each piece begins with a discussion on introducing the puzzle to students along with some possible thoughts, questions, and reactions students might have to the puzzles. The solutions are presented in the second half of this document, as well as extensions to explore.

The suggested age levels are very loose! Many elementary grade students will enjoy a number of these puzzles and any puzzle labeled for a certain grade level, when probed more deeply, can unfold to more challenging queries. Do indeed look at the suggested puzzle extensions.

The mathematics of Exploding Dots appears on our website **www.globalmathproject.org.**

Enjoy!

THE PUZZLES

Middle School, High School, and All

High School and All

THE FIVE-CARD MIND-READING TRICK

This is a classic number trick that serves as a perfect start to the entire EXPLODING DOTS story. Students just love this “mind-reading” trick and are fascinated by it. As soon as one’s understanding of machine codes come to light, the magic of this trick suddenly crystalizes. It’s an AHA moment!

EXPLODING DOTS Topic:

Experiences 1 and 2: Understanding machine (binary) codes.

Suggested Grade Level:

Middle School, High School, and All.

THE FIVE-CARD MIND-READING TRICK

Here's a classic trick. On a board write out the following five groups of numbers.

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

Ask your students to each silently think of a number between 1 and 31. They could choose the day of month on which they were born, for instance.

Now perform the mindreading trick by having the following conversation with a student.

"Suzzy. Is the number you are thinking of in group A?"	"Yes."
"Is the number you are thinking of in group B?"	"Yes."
"Is it in group C?"	"No."
"Is it in group D?"	"No."
"Group E?"	"Yes!"
"Ahh … your number is 25."	"Wow! Yes it is!"

Have the same conversation with a few more students, noting each time which groups elicit a "yes" answer from a given student. The secret number they have in mind is simply the sum of the top-left corner numbers in each group with a yes answer. For example, Suzzy answered YES YES NO NO YES. Groups A, B, and E have top left numbers 16, 8, and 1, respectively, and indeed **16+8+1=25.**

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31
YES	YES	NO	NO	YES

Suzzy's responses for the number 25.

As practice, check that if Sameer is thinking the number 13, he will answer NO YES YES NO YES and indeed **8+4+1=13 .**

Do this as many times as your students desire. Invite them to figure out what you are doing, and then, why what you are doing works! (Can you figure out why the top left numbers are key?) Perhaps facilitate thinking by writing a student's YES/NO answers under each group each time as done for Suzzy above.

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. Students might notice that group E contains all the odd numbers.

2. Students might notice that group A contains all the numbers 16 and above.

3. Students might question the number 31. “Why must we choose between 1 and 31. What’s special about the number 31?”

4. Students might notice the power of two in the top left corner of each group. (Though you might prefer to jumble the numbers in each group when you write them on the board to make this trick more mysterious!)

A TWO-BUTTON CALCULATOR

This puzzle asks students to utilise the power of place-value and the effect of multiplying a number by the value of the base in which it is presented. (Multiplying a number written in base ten, for instance, by has the apparent effect of appending a zero to the number.)

EXPLODING DOTS Topic:

Experience 2: Understanding the place-value machines.

Experience 4: Multiplying by the base number in a machine.

Suggested Grade Level:

Middle School, High School, and All.

A TWO-BUTTON CALCULATOR

Present to your students the following three puzzles, either one at a time or altogether at the same time.

Puzzle 1: A calculator, currently displaying **0**, has just two buttons.

+ Pressing this button adds 1 to the number currently on display.

× Pressing this button multiplies the number currently on display by ten.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number **5304** appear on screen?

If so, what is the least number of button presses needed to do it?

Puzzle 2: A calculator, currently displaying **0** , has just two buttons.

+ Pressing this button adds 1 to the number currently on display.

× Pressing this button doubles the number currently on display.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number **244** appear on screen?

If so, what is the least number of button presses you needed to do it?

Puzzle 3: A calculator, currently displaying **0** , has just two buttons.

+ Pressing this button adds 1 to the number currently on display.

× Pressing this button triples the number currently on display.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number **244** appear on screen?

What is the least number of button presses needed to do it?

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. You can always just repeatedly press the + button (and never press the × button). That way you can get any number to appear on display. But it will take 5034 presses to make 5304 to appear in the first puzzle, and 244 presses in each of the remaining two puzzles to make 244 appear.

2. Pressing the × first never does you any good: multiplying zero by ten (or by two or by three) keeps you at zero.

3. You can get to 5000, at least, in the first puzzle by pressing + five times and then × three times.

4. You can get to 5304 in the first puzzle by

    Pressing + five times.
    Pressing ×.
    Pressing + three times.
    Pressing ×.
    Pressing ×.
    Pressing + four times.

    That's fifteen button presses. Can we do it in less?

5. Puzzle 1 is hard enough!

6. In puzzle 1, would it ever be efficient to press the + button ten times in a row?

PEASANT MULTIPLICATION

Sometimes called “Russian Multiplication” or even “Russian Peasant Multiplication,” the curious multiplication technique we discuss here has close connections to an ancient multiplication technique employed more than three-and-a-half millennia ago in Egypt. That older technique is described in the famous Rhind Papyrus and is sometimes called “Egyptian Multiplication” or even “Ethiopian Multiplication,” and we’ll describe it too!

EXPLODING DOTS Topic:

Experience 2: Understanding the $1 \leftarrow 2$ machine.

Suggested Grade Level:

Middle School, High School, and All.

PEASANT MULTIPLICATION

Demonstrate the following multiplication technique, often called "Peasant Multiplication," on the classroom board.

Here's a curious way to perform long multiplication.

To illustrate the technique, let's compute 37×10, whose answer we know.

- Start by heading two columns with the two numbers in the product.
- Repeatedly halve the numbers in the left column and double the numbers in the right – and to make life easy, ignore any fractions that arise. Stop when you see 1 on the left.

37	10
~~18~~	~~20~~
9	40
~~4~~	~~80~~
~~2~~	~~160~~
1	320

10 + 40 + 320 = 370

- Delete any row that has an even number on the left and sum the numbers that survive on the right.
- The sum of the surviving numbers is the desired product!
- **Practice:** Use this technique to show that **10×37** also gives the answer **370.**

Have your students practice this technique with some more examples.

Of course, today's question is: Why does this technique work?

A Nudge: You could perhaps ask your students to examine the products 8 × *N* and 64 × *N* via this technique where N represents some general (unspecified) number.

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. This is weird!

2. Is it really okay to ignore remainders?

3. Working out 10×37 is awkward.

4. You can work out 10×37 in an Exploding Dots way.

~~10~~	~~37~~
5	6|14
~~2~~	~~12|28~~
1	24|56

6|14 + 24|56 = 30|70 = 370

5. To work out $8 \times N$ we end up doubling N three times to get $8N$, and only this survives on the right side. To work out $64 \times N$ we end up double N six times to get $64N$, and only this survives on the right side.

~~8~~	~~N~~
~~4~~	~~2N~~
~~2~~	~~4N~~
1	8N

6. You don't actually have to stop at 1 on the left side: you can keep going to get zeros. But since zero is an even number, these rows get crossed out in any case.

37	10
~~18~~	~~20~~
9	40
~~4~~	~~80~~
~~2~~	~~160~~
1	320
~~0~~	~~640~~
~~0~~	~~1280~~
~~0~~	~~2560~~
⋮	

10 + 40 + 320 = 370

DIVISIBILITY BY 9

There is a well-known divisibility rule for the number 9. In this piece, we explore that rule, and a slightly stronger version of it, and observe some consequences. Students will be able to extend the work here to a divisibility rule for 3 as well, and perhaps to a divisibility rule for 11 as well!

EXPLODING DOTS Topic:

Experience 5: Division in a $1 \leftarrow 10$ machine.

Suggested Grade Level:

Middle School, High School, and All.

DIVISIBILITY BY 9

Many people know a rule for divisibility by nine.

> A number is divisible by 9 only if the sum of its digits is divisible by 9.

For example, 387261 is divisible by 9—apparently—since 3+8+7+6+2+1=27 is. (And if we weren't sure about the number 27 , we could test that it is divisible by 9 by noting that 2+7=9 certainly is.)

> Check: 387261÷9=43029 and indeed there is no remainder.

In fact, this rule can be made a little stronger.

> A number leaves the same remainder upon division by 9 as does the sum of its digits.

For example, 40062 has sum of digits 12, which is 3 more than a multiple of 9, and indeed 40062 is 3 more than a multiple of 9 : 40062=9 × 4059+3. Also, 77 is five more than a multiple of 9 just as 7+7=14, the sum of it digits, is.

Discuss the divisibility rule for 9 with your students, and its stronger version. Have they already heard of the first rule? The second one? Have them test the rules with some examples.

Now the real question is: *Why do these rules work?*

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. Examples do seem to suggest that the rules do hold true.

2. It is actually surprising that these rules work. If we jumble the digits of a number, the sum of digits does not change. Since 387261 is a multiple of 9, this means that 387216 and 783162 and 273613 and all other permutations 387261 are all multiples of 9 too. That's weird, and hard to believe!

Comment: You and the class might choose to dwell on this observation a bit. It is certainly not generally true that you can rearrange the digits of a number and maintain its divisibility by a given factor. For example, changing the order of the digits of 512 won't always keep the number divisible by 2 , and changing the order of the digits of 2864 won't always keep it a multiple of 4. So it seems awfully strange that divisibility-by-9 doesn't "care" about the order of the digits. [Have your students ever noticed it is at least true for two-digit multiples 9? We have that 18 and 81, 27 and 72, 36 and 63, and, 45 and 54 are all multiples of 9.]

3. The second rule implies the first rule. A number is a multiple of 9 only if it leaves a remainder of zero upon division by 9. So, according to the second rule, if the sum of digits leaves a remainder of zero, that is, is a multiple of 9, then so is the original number.

Consequently, explaining the second rule automatically explains the first rule too.

DIVISIBILITY BY 9 AGAIN!

Here we explore a little-known technique for dividing by 9.

EXPLODING DOTS Topic:

Experience 5: Division in a 1←10 machine.

Suggested Grade Level:

Middle School, High School, and All.

DIVISIBILITY BY 9 - AGAIN!

Here's a very strange way to divide a number by 9. We'll illustrate it with a specific example.

To divide 21023 by 9, for instance, write out the partial sums of its digits, computed from left to right

$$
\begin{array}{ll}
2 & = 2 \\
2+1 & = 3 \\
2+1+0 & = 3 \\
2+1+0+2 & = 5 \\
2+1+0+2+3 & = 8
\end{array}
$$

and then read off the answer:

$$21023 \div 9 = 2335 \;\; R \;\; 8$$

(And indeed, $21023 = 9 \times 2335 + 8$.)

In the same way.

$$
\begin{aligned}
1221 \div 9 &= 1 \mid 1+2 \mid 1+2+2 \quad R \quad 1+2+2+1 \\
&= 135 \quad R \quad 6
\end{aligned}
$$

and

$$
\begin{aligned}
20000 \div 9 &= 2 \mid 2+0 \mid 2+0+0 \mid 2+0+0+0 \quad R \quad 2+0+0+0+0 \\
&= 2222 \quad R \quad 2
\end{aligned}
$$

One might have to perform some explosions along the way and deal with extra-large remainders. For instance,

$$
\begin{aligned}
5623 \div 9 &= 5 \mid 5+6 \mid 5+6+2 \quad R \quad 5+6+2+3 \\
&= 5 \mid 11 \mid 13 \quad R \quad 16 \\
&= 623 \quad R \quad 16
\end{aligned}
$$

and a remainder of "16" really corresponds to one extra group of 9 and a remainder of 7. So we actually have $5623 \div 9 = 624 \;\; R \;\; 7$.

Discuss this algorithm with your students. Help them make sense of the examples presented here and check more examples on their own. Then ask: *Why does this curious algorithm work?*

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. Examples do seem to suggest that this algorithm always works.

2. This is just plain weird! Really ... why does it work?

3. What if we start with a multiple of 9. Shouldn't we get a remainder of zero? But this method will never give a remainder of zero!

DIVISIBILITY BY 7

Do you and your students know a divisibility rule for the number 7?

This piece illustrates and explains one. And after exploring it, your students will also be able to create divisibility rules for 11, 13, 17, 19, and more!

EXPLODING DOTS Topic:

Experience 6: Using dots and antidots for division in a 1←10 machine.

Suggested Grade Level:

Middle School, High School, and All.

DIVISIBILITY BY 7

Many people know divisibility rules for dividing by 2, 3, 4, 5, 8, 9, and 10, and perhaps 11 (and, hence also for 6 by using the rules for 2 and 3 together, for 12 by using the rules for 3 and 4 together, and for 15 and the like). But do you and your students know this curious divisibility rule for the number 7?

> **To determine whether or not a number is divisible by 7, delete its last digit and subtract double that deleted digit from what remains. The original number is divisible by 7 only if this new number is. This procedure can be repeated until one has a number sufficiently small to be easily recognized as a multiple of 7 or not.**

This is a bit hard to parse. An example helps. Let's check if 39872is a multiple of 7.

$$\begin{array}{r} 3987\cancel{2} \\ -\quad 4 \\ \hline 3983 \end{array} \qquad \begin{array}{r} 398\cancel{3} \\ -\quad 6 \\ \hline 392 \end{array} \qquad \begin{array}{r} 39\cancel{2} \\ -\quad 4 \\ \hline 35 \end{array}$$

Delete the final digit, the 2, and subtract double this, 4, from the 3987 that remains. This gives 3983. I can't readily tell if this is a multiple of 7, so let's do this procedure again: delete the final digit 3 and subtract double this from the remaining 398 to yield 392. It is still hard to tell whether or not we have a multiple of 7. One more run through of the procedure yields 35. This is a multiple of 7, and this apparently means that our original number was too!

Check: 39872 5696 7 = × . Yes!

> **By the way:** Suppose I didn't know that 35was a multiple of 7. If I run the procedure through one more time, might I get a number I could surely recognize is divisible by 7?

Have your students practice the procedure a few times, with some beginning numbers that they know to be multiples of 7 and some known not to be.

The algorithm always seems to work. But the real question, of course, is: *Why does it work?*

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. If you really want to know if a number is divisible by 7 or not, just get out a calculator!

2. This is just plain weird! Really … why does it work?

3. This rule doesn't actually tell us what the answer to the division problem is. It just seems to be a YES/NO procedure.

4. If the beginning number is not a multiple of 7, does this procedure at least tell us what the remainder is upon division by 7? (Our divisibility rules for 9 did.)

DIVISIBILITY BY 37

We've explored the divisibility rules for 3 and 9, and 7 in other essays, and in them we also paved the way for creating divisibility rules for 11, 13, 17, 19, and more.

In this essay we adopt a different approach for discovering more divisibility rules for awkward numbers. In particular, we present in detail a divisibility rule for the number 37.

EXPLODING DOTS Topic:

Experience 6: Using dots and antidots for division in a $1 \leftarrow 10$ machine.

Suggested Grade Level:

Middle School, High School, and All.

DIVISIBILITY BY 37

Do you and your students know this curious divisibility rule for the number 37?

> To determine whether or not a number is divisible by 37, delete its first digit and add it to the digit three places further in. (Perform some carries, if necessary.) The original number is divisible by 37 only if this new number is.
>
> This procedure can be repeated until one has a number sufficiently small to recognize as a multiple of 37 or not.

An example helps make sense of this. Let's check if 348022 is a multiple of 37.

$$\begin{array}{lr} & \cancel{3}48022 \\ + & 3 \\ \hline = & 48322 \end{array} \qquad \begin{array}{lr} & \cancel{4}8322 \\ + & 4 \\ \hline = & 8362 \end{array} \qquad \begin{array}{lr} & \cancel{8}362 \\ + & 8 \\ \hline = & 370 \end{array}$$

As 370 is clearly a multiple of 37, the original number 348022 must have been too!

Check: $348022=37\times9406$.

As another example, this method shows that 3499 is not a multiple of 37:
we recognize that 4 | 9 | 12=4 | 10 | 5=505 is not a multiple of 37. (How?)

This algorithm is not very practical or helpful (are you able to readily tell if a three-digit number is a multiple of 37?). But the interesting part of this is not its practical use.
We want to know:

Why does the algorithm work?

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. If you really want to know if a number is divisible by 37 or not, just get out a calculator! Clearly this is not a practical or efficient technique.

2. The algorithm is just weird.

3. It seems that that the algorithm doesn't actually give us the result of dividing a number by 37. It's just answering a YES/NO question.

orem ipsum

A STRATEGY FOR NIM

The classic game of NIM is an ancient game played with piles of pebbles. Its origin is unknown. But it was only relatively recently in 1901 that the full mathematics behind the game was explored and explained. This was done by American mathematician Charles Bouton, who also coined the name "NIM."

A winning strategy for the game relies on writing counts of pebbles in binary and today's puzzle is to explain why this curious winning strategy works.

EXPLODING DOTS Topic:

Experience 2: Understanding Binary

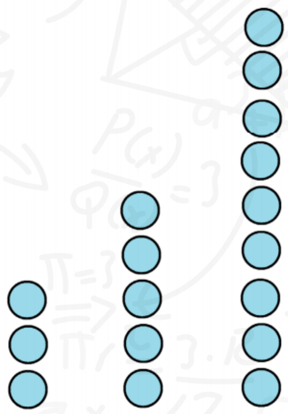
Suggested Grade Level:

Middle School, High School, and All.

A STRATEGY FOR NIM

Introduce the game of NIM to your students. Use counters, buttons, or some such for students to practice playing the game.

The game of NIM starts with three piles of counters on the tabletop between two seated players. One pile contains 3 counters, one 5 counters, and the third 9 counters.



Players take turns removing one or more counters from a single pile. (One is permitted to remove all the counters from a pile. One is required to take at least one counter.)

The person who takes the last counter—thereby leaving the tabletop empty—is the winner.

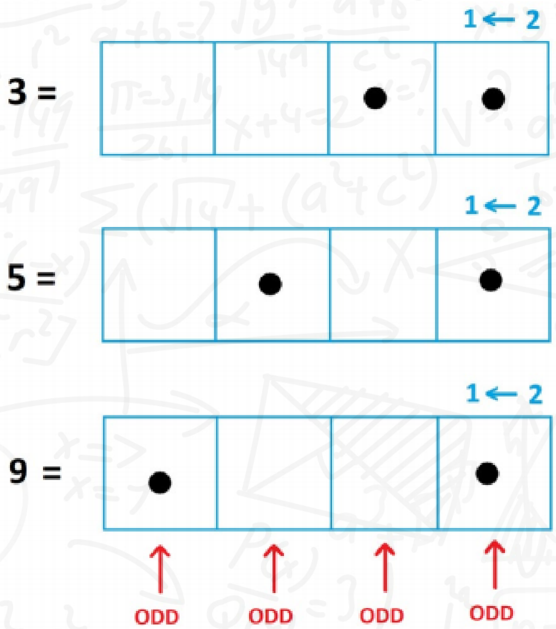
Play a few rounds of NIM with a partner just to get a feel for the game. Switch who has the first move a few times.

One can, of course, play this game with any number of piles, each containing any number of counters. To keep the conversation consistent, we'll just discuss this 3-5-9 game of NIM for now.

When the time is right, introduce material of this next section that describes a winning strategy.

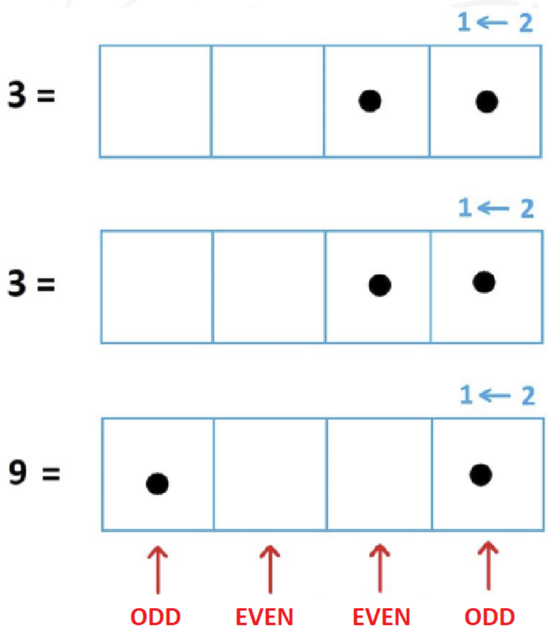
In 1901, American Mathematician Charles Bouton found a curious winning strategy for playing NIM. We present his approach here in terms of codes in a 1⟵2 machine (aka binary codes).

Represent the count of pebbles in each pile in a 1⟵2 machine with the machines stacked on top of one another as shown.



At present, each column in this picture contains an odd number of dots.

A NIM move will change this diagram and change the even/odd-ness of some or of all of the columns. For example, removing two pebbles from the middle pile produces this picture.



a) Find a NIM move from the original 3-5-9 diagram that the first player can take that, instead, gives a new diagram with each column having an even count of dots.

b) From your new diagram with each column possessing an even count of dots, explain why, in whatever NIM move the second player now takes, she is sure to create a diagram with at least one column containing an odd number of dots.

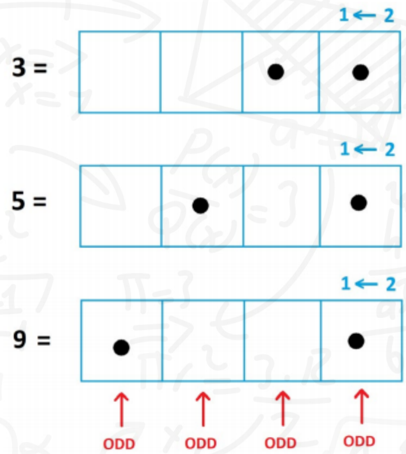
c) Prove, when presented with a diagram containing at least one column with an odd count of dots, it is always possible to make a single NIM move that gives a diagram with each column containing an even count of dots.

d) Prove, when presented with a diagram with all columns containing an even count of dots, every NIM move is sure to create a diagram with at least one column possessing an odd count of dots.

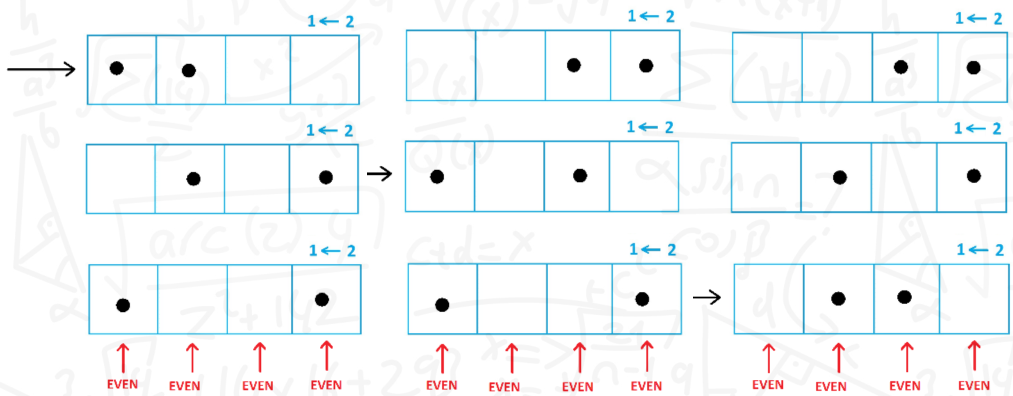
e) Explain why the first player of the 3-5-9 NIM game can be sure to win the game.

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. I don't know where to start. These questions seem hard!
2. So, what are we meant to do? Take this picture, change one of the rows (which means we're changing one of the piles), and get an even number of dots in each column? That doesn't seem possible!



3. Wouldn't any of these work, changing either the first, second, or third row as shown?



Oh! The first two have made piles bigger. The last one changes 3-5-9 to 3-5-6. It corresponds to taking three pebbles from the 9 pile.

So maybe it can always be done by removing pebbles from the biggest pile?

4. Part b) seems really hard: too many cases to check!

TWO-PAN BALANCE PUZZLES

We start this classic weighing puzzle with a simple version that makes use of $1 \leftarrow 2$ machine codes (that is, binary codes) of numbers. This, in-and-of-itself, can serve as a lovely introduction to binary.

We then move to the main puzzle itself and some extensions.

EXPLODING DOTS Topic:

Experiences 2 and 9: Using a $1 \leftarrow 3$ machine but with the digits 0,1, and -1. (These are called "balanced ternary" codes.)

Suggested Grade Level:

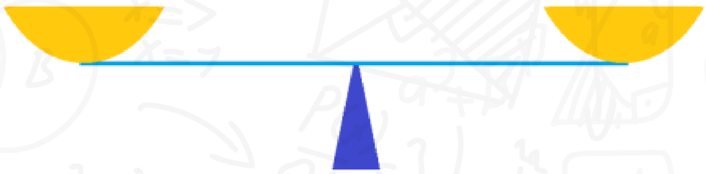
High School, and All.

TWO-PAN BALANCE PUZZLES

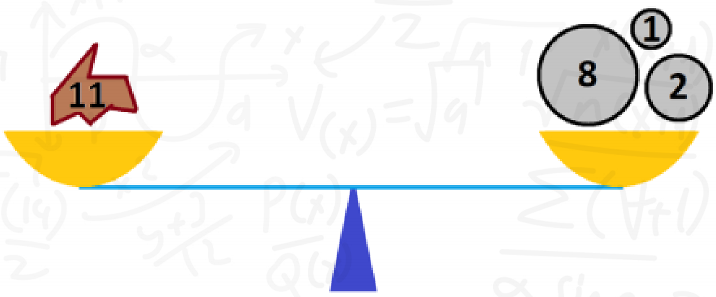
Here's a warm-up puzzle you can share with your students. (Or, for young students, this can be a lovely challenge in-and-of itself.)

Perhaps draw pictures on the board as you explain the puzzle.

You have four rocks. One weighs exactly 1 kilogram, one exactly 2 kilograms, one exactly 4 kilograms, and one exactly 8 kilograms. You also have a simple two-pan balance, with pans each big enough to hold up to four rocks.



If someone puts a rock of weight 11 kilograms, say, on the left pan, you could balance it perfectly by placing your 1 kg, 2 kg, and 8 kg rocks on the right pan.



Which rock weights in the left pan could you balance with some combination of your four rocks?

Your students will likely realize that you can balance a rock of whole number of kilograms weight from 1 kg all the way up to 15 kg with combinations of the four special rocks. From studying the codes for numbers from a 1←2 machine, we see that every number from 1 up to 15 can be represented as a sum of some collection of the numbers 1, 2, 4, and 8.

Now add to the story …

You decide to open a rock-weighing business. You advertise:

Do you have a rock? And do you want to know its weight? If it weighs a whole number of kilograms between 1 kg and 15 kg, come to me! I'll figure out its weight for you.

With trial-and-error you can figure out which combination of your four rocks balance with the customer's rock, and hence determine its weight.

Business is good for a while, and then it suddenly halts! You notice down the road that Poindexter Farklesnark has just opened a competing business. He advertises:

Do you have a rock? And do you want to know its weight? If it weighs a whole number of kilograms between 1 kg and 40 kg, come to me! I'll figure out its weight for you.

Customers are flocking to his business.

You see that Poindexter is also using a two-pan balance and four special rocks, but he is placing combinations of his rocks on both sides of the balance with the customer's rock. And that is how he is balancing a larger range of weight values.

Is Poindexter using four rocks of weights 1 kg, 2, kg, 4 kg, and 8 kg? Or is he using a different set of four special rocks? (If so, what are the weights of his four special rocks?)

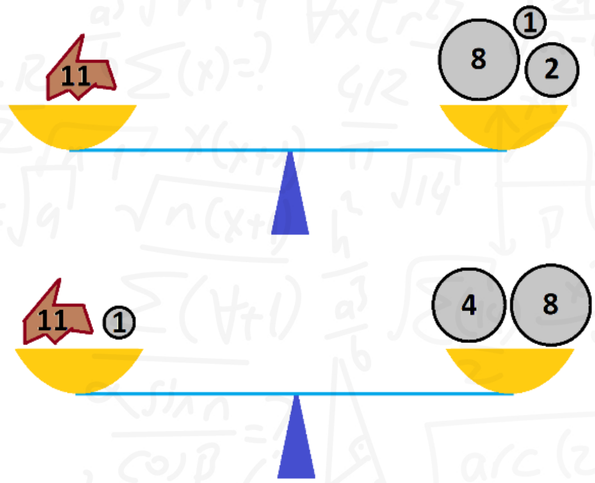
Check that Poindexter really is able to balance each of the weights 1kg up to 40 kg, inclusive, with four special rocks.

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. With rocks of weights 1 kg, 2kg, 4 kg, and 8 kg in the first business, students might wonder: Is it actually possible, "with trial-and-error," to figure out the weight of an unknown rock?

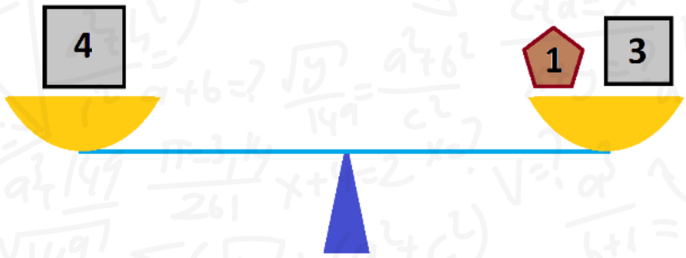
   This might be worth dwelling on. Ask: How could you tell if a rock weighs more or less than 8 kg? How might you tell if a rock weighs more less than 12 kg? Could there be a general strategy to follow to determine the unknown weight of a rock?

2. With rocks of weights 1 kg, 2kg, 4 kg, and 8 kg, it is possible to balance scales in more than one way with a given customer's rock – if you allow use of both pans of the scale.



3. Students might notice that even with use of both pans, one cannot balance a rock of weight more than 15 kg using four rocks of weights 1 kg, 2 kg, 4 kg, and 8 kg. Poindexter has four special rocks of different weights.

4. Some students might argue that Poindexter must have at least one special rock of weight 1 kg: “How else can he balance a customer’s rock of 1 kg?” Other students might realize that realize that he need not! For example, with a 3 kg rock and a 4 kg rock, Poindexter can still detect a rockweight of 1 kg.



5. Some students might decide to start with the assumption that Poindexter has a 1 kg rock, just for ease, and realize that he doesn’t actually need a 2 kg rock next if he works with a 3kg rock instead. With 1 kg and 3 kg at hand, Poindexter can balance all the weights 1 kg, 2 kg, 3 kg, and 4 kg. They might wonder what the next biggest rock weight could be to get 5 kg, 6kg, 7kg, ….

6. Students might say that having a rock in the same pan as the customer’s rock is like having an “anti-rock”: it subtracts from the effective weight of the customer’s rock.

ANOTHER TWO-PAN BALANCE PUZZLE

Here's a surprising application of a 1←2 machine and the negabinary codes that result from them. These codes have a natural representation in a two-pan balance setting.

EXPLODING DOTS Topic:

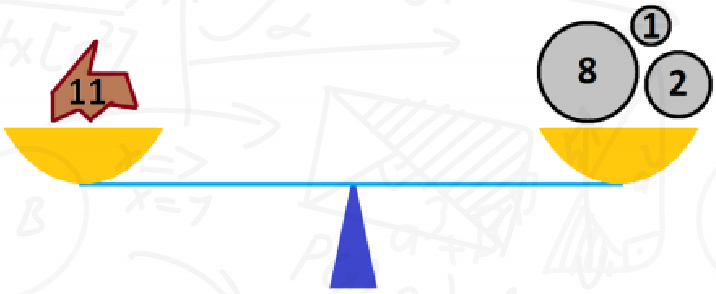
Experiences 2 and 9: Having fun in bases two and negative two!

Suggested Grade Level:

High School, and All.

ANOTHER TWO-PAN BALANCE PUZZLE

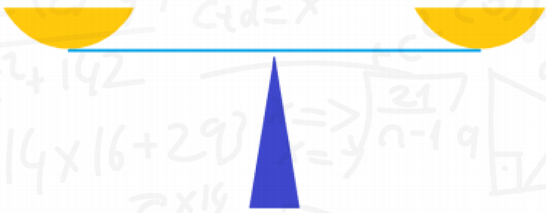
This puzzle assumes that students are familiar with the fact that every number can be written as sum of powers of two. Consequently, given a collection of rocks of weights running through the powers of two, any rock of integer weight can be placed on one side of a two-pan balance and be balance by a set of rocks from that collection

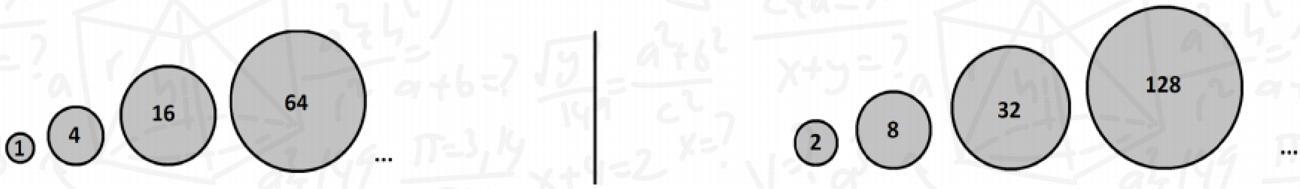


If this might not be evident to your students, present content of the "TWO-PAN SCALE PUZZLES" essay first. Then, when ready, try sharing this next challenge.

You have an infinite collection of rocks, one weighing 1 kilogram, one weighing 2 kilograms, one weighing 4 kilograms, one weighing 8 kilograms, and so on, with one rock matching each power of two in weight.

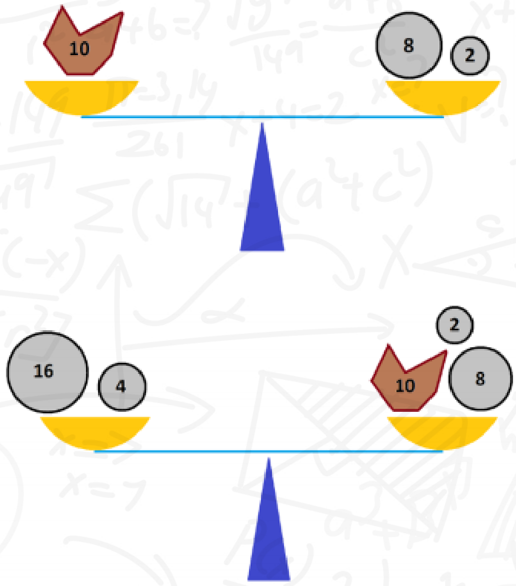
You also have a two-pan balance.





For reasons that really can't be explained you will place rocks of weights 1, 4, 16, 64, … kilograms on the left side of the balance (and never on the right) and rocks of weights 2, 8, 32, 128, … kilograms on the right side of the balance (and never on the left).

If someone places a rock of weight 10 kilograms on either side of the two-pan scale you realise that you can balance the system with your rocks while following these rules.



But now one wonders: Will this always be possible? Using the rocks of weights the powers of two, under the rules given, can one always create balanced system if a rock of random integer weight is placed in one of the two pans?

a) Suppose a rock of weight 20 kilograms is placed in the right pan. Can you use the rocks of weights the powers to two to balance matters? What if the rock is placed, instead, in the left pan?

b) Suppose a rock of weight 50 kilograms is placed in the left pan. Can you use the rocks of weights the powers to two to balance matters? What if the rock is placed, instead, in the right pan?

Let's just give the answer away.

c) Develop some convincing reasoning that if a rock of any integer weight is placed in either pan, then it is for certain possible to place rocks of weights the powers of two in the pans, while following the stated rules, to create a balanced system.

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. Via trial-and-error we see that rocks of weights 4 and 16 in the left pan balance with the rock of weight 20 in the right pan. Also, rocks of weight 8 and 32 in the right pan balance with rocks of weight 20 and 4 and 16 in the left pan.

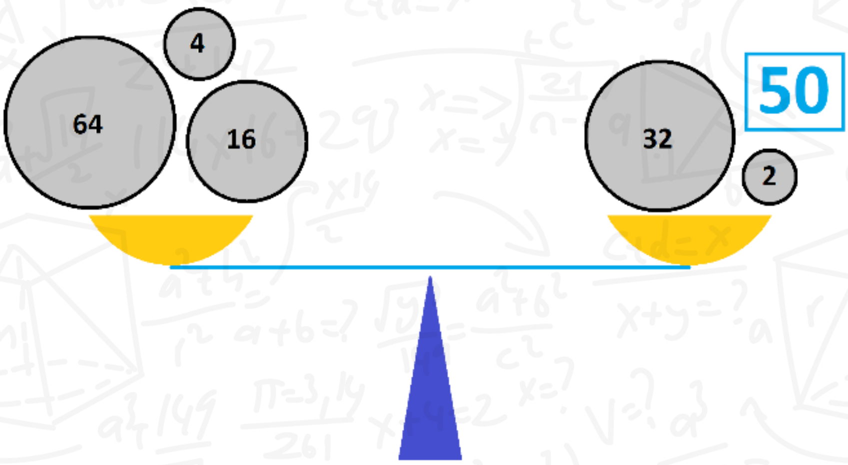
2. Dealing with a rock of weight 50 kilograms in either pan seems annoying and hard.

3. As a sum of powers of two, 50=32+16+2. But with the 16 rock on the left and the 2 and 32 rocks on the right, this doesn't seem to mean anything?

4. You can get the equivalent of "32 on the left" by placing the 64 rock on the left and the 32 rock on the right.

   Ooh! And "2 on the left" is equivalent to placing the 4 rock on the left and the 2 rock on the right.

   This leads to a balanced solution with a 50-kilogram weight on the right.



5. With persistent trial-and-error, we see that

$$50+16+64=2+128.$$

This gives a balanced solution with the 50-kilogram weight on the left.

6. But this is all random trial-and-error. Is there a systematic approach?

7. The examples explored so far, 10 kilograms and 50 kilograms, are both weights an even number of kilograms. Might weights an odd number of kilograms be problematic?

8. Let's just read the solution and see if we can make sense of that!

DIVIDING BY 101 PUZZLE

This piece is based on an American Mathematics Competition problem that asks us to determine when certain numbers are divisible by 101. It does require making use of dots and antidots within the 1←10 machine.

Upper high-school students could see a connection to the Factor Theorem in polynomial algebra.

EXPLODING DOTS Topic:

Experiences 5 and 6: Using dots and antidots for division in a 1←10 machine.

Suggested Grade Level:

High School, and All.

DIVIDING BY 101 PUZZLE

Here's a question that comes from the American Mathematics Competitions (the 2018, AMC 10B competition as question 13, in fact).

> **How many of the first 2018 numbers in the sequence 101, 1001, 10001, 100001, ... are divisible by 101?**

Perhaps write this question on the board and have a class discussion with your students about it.

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. Why the first 2018 numbers in the sequence? (If you don't mention the source of the question—that it comes from a competition that was held in the year 2018—this is indeed a strange number to pull out of the air!)

2. What is the 2018th number in the sequence?

3. Students will likely notice that *n*th number in the sequence is the number created by a pair of 1s with n zeros between them.

4. The very first number in the list, at least, is divisible by 101. If we get out a calculator we can find other numbers in the list are divisible by 101 too. (Students working this way may well detect a pattern as to which ones are.)

5. Our calculators can't handle numbers with 2018 zeros between a pair of ones.

A POURING PUZZLE

This puzzle leads on into a theory of infinitely long processes to be represented by (potentially) infinite decimals in unusual bases.

The standard curriculum, of course, assume students have fully mastered thinking about decimals in early grades. But this puzzle demonstrates that there is no such thing as being "done" with a topic!

EXPLODING DOTS Topic:

Experience 8: Decimals in alternative bases

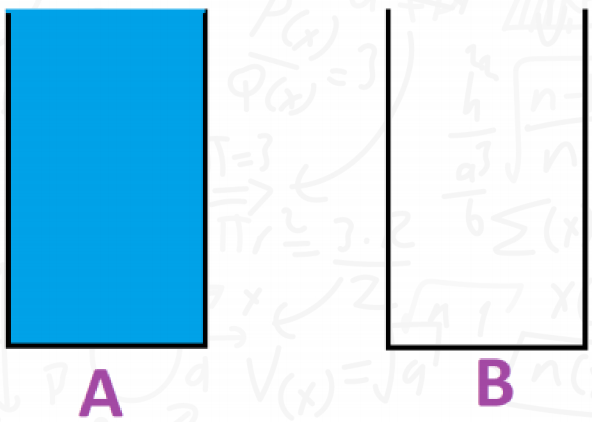
Suggested Grade Level:

High School, and All.

A POURING PUZZLE

Here is a curious puzzle that leads into thinking about infinite processes. Share it with your students.

We have two identical containers A and B. Currently A is completely full of water and B is completely empty. We will be pouring water back and forth between the two containers.



In fact, we'll be performing very specific "pouring moves:" at any time we may either pour $\frac{2}{3}$ of the content of container A into container B, or pour $\frac{2}{3}$ of the content of container B into container A.

a) After a finite sequence of pouring moves is it possible to end up with contaner B precisely one-quarter full?

If the answer is NO, then …

b) Is it possible to see an amount of water in container B that is so close to one-quarter its volume that the human eye couldn't tell the difference?

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. The fractions two-thirds and a quarter just feel "incompatible."
The answer to part a) is probably NO!

2. Given that the author of this question went on to offer part b) probably means that the answer to a) is indeed NO! (Going on pure psychology here!)

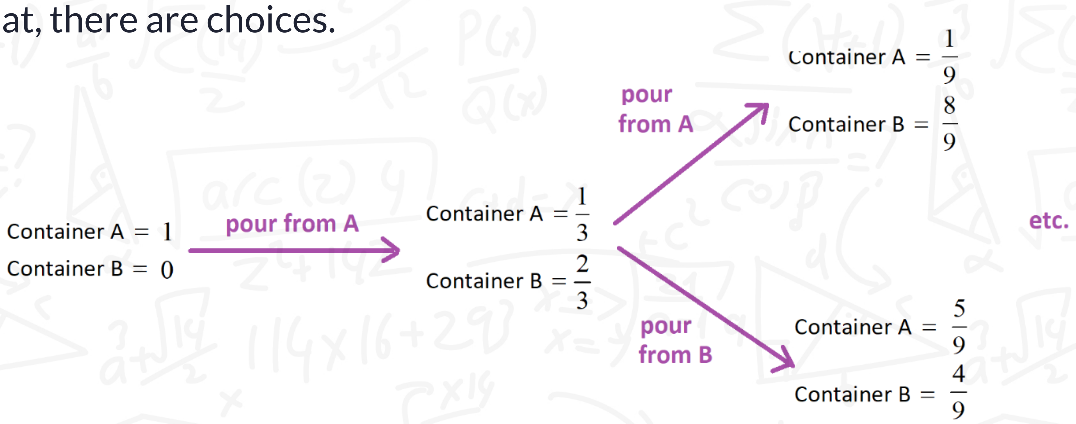
3. The author does not say one must alternate pouring directions. One could pour from the same container multiple times in a row.

4. The first move one makes must be to pour $\frac{2}{3}$ of the content from A into B to get

Container A = $\frac{1}{3}$ full

Container B = $\frac{2}{3}$ full.

After that, there are choices.



(In the diagram, the numbers how full each container is as a fraction of its volume.)

Since we're only ever taking thirds and two-thirds of fractions with denominators that are powers of three, we will never see the fraction $\frac{1}{4}$ arise. The answer to part a) is definitely NO!

A MANGO SHARING AND EATING PUZZLE

This piece is based on a USA Mathematical Talent Search question.

It's a long and tricky question to read and make sense of, and this piece is chiefly about what to do when in such a challenging situation. The goal is not to solve the puzzle—though we do—but to learn how to honor one's emotional reactions and learn how to make a step of progress nonetheless. This is an important life skill!

EXPLODING DOTS Topic:

Experiences 1 and 2: Recognizing a $1 \leftarrow 2$ machine in disguise.

Suggested Grade Level:

High School, and All.

A MANGO SHARING-AND-EATING PUZZLE

Consider sharing the following scary-looking problem with your students. Let them know that the point of this exercise is not to solve the puzzle—though, it will turn out we will—but to simply have an emotional reaction to it!

A group of100friends stands in a circle. Initially, one person has 2019 mangos, and no one else has mangos. The friends split the mangos according to the following rules:

- *Sharing:* to share, a friend passes two mangos to the left and one mango to the right.

- *Eating:* the mangos must also be eaten and enjoyed. However, no friend wants to be selfish and eat too many mangos. Every time a person eats a mango, they must also pass another mango to the right.

A person may onlyshareif they have at least three mangos, and they may onlyeatif they have at least two mangos. The friends continue sharing and eating, until so many mangos have been eaten that no one is able to share or eat anymore.

Show that there are exactly eight people stuck with mangos, which can no longer be shared or eaten.

Source: USA Mathematical Talent Search
Year 31 : Academic Year 2019-2020
Round 1 : Problem 4/1/31
https://www.usamts.org/Tests/Problems_31_1.pdf

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. This question is hard to understand: there is too much to take in!
2. The puzzle feels awfully contrived. Is it worth our while?
3. No one sits with 2019 mangos! There are never 100 people standing in a circle!
4. Ooh! I like puzzles like these.

A Discussion to Possibly Have

Mathematics is an intensely human subject. It was invented/created for humans, by humans, to be experienced, used, and enjoyed by humans. As such, we should each be our honest human selves when doing mathematics.

There are two fundamental steps to solving any problem in math – or life, for that matter. The first:

Step 1: Have an emotional reaction to the problem.
Openly acknowledge your reaction.

If a problem looks hard or scary, say so! If it seems artificial and contrived, say “Why should I care?” If you don’t know what to do, acknowledged that you are flummoxed. And if you are intrigued, be intrigued. Whatever your reaction may me, consciously acknowledge your emotional response.

Then, when you are ready, take a deep breath and move to

Step 2: Do something. ANYTHING!

Turn the page upside down. Read the question again. Read the question backwards. Circle some words. Draw a picture. Draw a picture of a fish. Put the question aside, go for a walk, and come back later.

The point is to do something either relevant or irrelevant to the problem and work to get past an emotional impasse.

Many people just "shut down" when confronted with something confusing or scary. But by deliberately acknowledging one's emotions and taking a step of any kind to work with them, one often finds that a first step to handling the problem falls into place, even if your first step seemed irrelevant to the task at hand.

Discuss with your students some first steps they could take with this problem.

Some directly relevant ideas they suggest could include the following.

- Read the question again.
- Read the question again and pausing after each sentence to ask:
  Does that sentence seem important?
- Try the question starting with 1 mango, not 2019. Then maybe 2 mangos. Then 3.
- Draw a picture of some kind demonstrating how mangos "move."
- Start with answer 8 and somehow work backwards.
- Google the source of the problem and see if a solution was provided.
  (It was, but it too is very hard to parse!)

Listing such ideas is usually always fruitful. In this case, playing with small counts of mangos and drawing diagrams not only gives us a feel for what is going on, but proves to be enlightening!

By the way: Feel free to check out the interactive web app created by WildThinks Blog to enact this puzzle with small counts of mangos.
https://wildthinkslaboratory.github.io/smartblog/usamts/2020/01/29/usamts2.html

THE JOSEPHUS PROBLEM

A horrific story dating back some 2000 year tells of a soldier Flavius Josephus (37 CE – ca.100 CE) being trapped in a cave with 40 fellow soldiers about to be captured by Roman forces. Rather than face the fate of being Roman slaves—or worse—they decided to commit collective suicide. They stood in a circle and counted off every third person, who was either to kill himself on the spot or, if incapable of conducting the act, be killed by his neighbors. A final soldier would be left standing who would have to commit suicide with no aid.

When the soldiers conducted this, Josephus and just one other man found themselves still standing. They decided they could not continue on and both surrendered to the Romans.

Horrors aside, this question leads to an interesting mathematical question. Where should one stand in the circle to be the last person standing in this counting act?

In this essay we take a less gruesome tact to this story and adjust the puzzle by counting off every second person.

EXPLODING DOTS Topic:

Experience 2: Understanding the binary codes of the 1←2 machine.

Suggested Grade Level:

High School, and All.

THE JOSEPHUS PROBLEM

Present the following problem to your students. (You, of course, have the option to explain the historic origin of this problem if you wish, but it could be best to leave that aside.)

A number of students, N of them, numbered 1 through N, sit in order in a circle. Walking around the circle many times the teacher taps each student on the shoulder alternately saying the words "in" and "out" as he does each tap. Any student tapped with the word "in" stays in place and those tapped with the word "out" must leave the circle and are out of the game. Even though the circle thins out as this game is played, the teacher keeps strolling around the chairs, tapping each remaining student on the shoulder—in, out, in, out, in, out, ... —until one student remains. That lucky student wins a lifetime supply of really cool math books.

Each time the teacher plays this game with a group of students, he always starts by tapping the shoulder of student 1 with the word "in."

As practice, consider the situation with N=5 students in a group. The teacher keeps student 1 in, sends student 2 out, keeps student 3 in, sends student 4 out, keeps student 5 in, sends student 1 out, keeps student 3 in, sends student 5 out to then leave student 3 as the winner.

Write W (N) for the number of the winning student in a game played with N students.

We have W (5) = 3

a) Check that W (12) = 9 and W (16) = 1.

b) If you like, complete the following table and look for patterns.

Can you explain any patterns you see?

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
W(N)	1				3							9				

But here is the real question.

> Write the number N in its binary 1←2 machine code.
> Move the leading 1 from the front of the code to the back.
> Then W (N) is the number with that binary code.

For example, N=12 has code **1100.** Move the front **1** to the back and get **1001**, which is the code for **9**. And lo and behold, W **(12)=9** ! Also N=**5** has code **101** , move the front **1** to the back and get **011** which is the code for **3**. And indeed, W **(5)=3**. Whoa!

c) Can you explain this mysterious connection to binary codes?

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. No!

   And this could be meant in one of two ways: that they can't explain what is going on, or that they simply reject the request to try to explain what is going on!

2. This feels like a really weird and hard puzzle.

3. "I don't get it."

   The thing to do here is to conduct some more practice examples. One can write out on paper what is going on in any particular example and collect data swiftly. For example, each line here shows which student is sent out in turn for N=5 students. (In practice, one wouldn't draw a separate new line each time a student number is crossed out.)

   1 ~~2~~ 3 4 5
   1 ~~2~~ 3 ~~4~~ 5
   ~~1~~ ~~2~~ 3 ~~4~~ 5
   ~~1~~ ~~2~~ 3 ~~4~~ ~~5~~ 3 wins!

   It doesn't take too much effort to get some more data values.

N	1	2	3	4	5	6	7	8	9	10
W(N)	1	1	3	1	3	5	7	1	3	5

One can see that eight, for example, is 1000 in binary and moving the front one to the back gives 0001, which is the code for 1. And indeed $W(8)=1$.

We can check that the claim of the puzzle keeps working!

4. Every student with an even number, 2, 4, 6, 8, 10, …, is eliminated right away. Only odd-numbered students have a hope of winning. (Indeed, it does look like that $W(N)$ is always an odd value.)

5. In binary, the odd numbers are the ones with final digit 1.

6. “I am intrigued. But I don’t know how or where to begin thinking about this!”

AMC 10A 2020 PROBLEM 21

Every now and then the American Mathematics Competitions present a problem ideally suited for Exploding Dots thinking. This essay is about one such problem.

EXPLODING DOTS Topic:

Experiences 6 and 2: Division of polynomials thinking in base two.

Suggested Grade Level:

High School, and All.

AMC 10A 2020 PROBLEM 21

Here's a scary-looking question from the American Mathematics Competition 10A, year 2020. Perhaps copy the question onto the board or project of copy of it on a screen.

Question 21 from AMC10A, 2020.

There exists a unique strictly increasing sequence of non-negative integers $a_1 < a_2 < \ldots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}.$$

What is ***k***?

Before attempting to make sense of the problem, yet alone answer, perhaps discuss with your students the emotions this problem might evoke.

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. This problem does look scary.

2. The problem is very "mathy." It seems technical and tricky.

3. It doesn't look fun. Who would want to do this?

4. It's a problem designed for tenth-graders, those who have done some geometry and algebra. Geometry probably doesn't apply, but algebra likely does. The author of this question must think that tenth-graders have the tools and means to solve it.

5. It might not be as scary as it actually looks.

6. Those numbers 17 and 289 look suspicious.

7. I don't get that "$a_1 < a_2 < \ldots < a_k$" part.

8. It's all about powers of two.

DIVISIBILITY BY 5 IN BASE ONE AND A HALF

This puzzle is really "out there." It assumes some familiarity with the 2←3 machine and how the codes that arise from it are representations of numbers in base one-and-a-half with the digits 0, 1, and 2. (This is Experience 9 of the Exploding Dots story.)

Very little is known about the codes of numbers from this base, including basic divisibility rules. But one can, at least, give a curious divisibility rule for 5!

EXPLODING DOTS Topic:

Experience 9: Understanding the codes of the 2←3 machine.

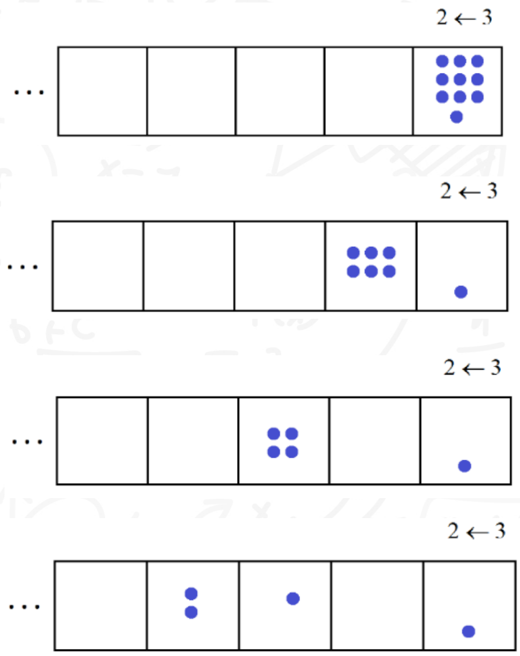
Suggested Grade Level:

High School, and All.

DIVISIBILITY BY 5 IN BASE ONE-AND-A-HALF

If you and your students have played deeply with Exploding Dots and talked about the mysteries of the codes that arise in base one-and-a-half from a 2⟵3 machine, perhaps try presenting this puzzle.

In a 2⟵3 machine, three dots in any one box disappear – EXPLODE!—to be replaced with two dots just to their left. For example, placing ten dots in the machine yields the code 2101 for ten.



Here are the codes for the number zero to forty from a 2⟵3 machine.

0			
1	2102	21220	212021
2	2120	21221	212022
20	2121	21222	212210
21	2122	210110	212211
22	21010	210111	212212
210	21011	210112	2101100
211	21012	212000	2101101
212	21200	212001	2101102
2100	21201	212002	2101120
2101	21202	212020	2101121

Prove that a number is divisible by 5 only if the alternating sum of the digits of its 2⟵3 machine code is a multiple of five.

For example, twenty has code 21202 and 2−1+2−0+2=5 is a multiple of five. And forty has code 2101121 and 2-1+0-1+1-2+1=0 is a multiple of five. And eleven has code 2102 and 2-1+0=2=-1 is not a multiple of five.

SOME THINGS STUDENTS MIGHT NOTICE OR QUESTION

1. What??!!!

2. This feels like the divisibility rule for eleven in ordinary base ten.

3. The $2 \leftarrow 3$ machine code $a \mid b \mid c \mid \ldots \mid d \mid e$ (with each digit 0, 1, or 2) for a number N means we're writing N as

$$a\left(\frac{3}{2}\right)^{k} + b\left(\frac{3}{2}\right)^{k-1} + c\left(\frac{3}{2}\right)^{k-2} + \cdots + d\left(\frac{3}{2}\right) + e\,.$$

   Do we really have to mess with sums of fractions like these?

4. This is horrible!

SOLUTIONS