MATHEMATICS FOR HUMANS

PLACE VALUE

The Gateway to Powerful Mathematics

GLOBALMATHPROJECT

www.globalmathproject.org

Photo: Erick Mathew, Tanzania

A PDF of all this guide and all the videos that accompany it are available at

gdaymath.com/courses/exploding-dots/

INSTRUCTIONS for Students

Just start at the beginning of this book and work your way through the sections! (Actually, it could be really fun to read it in a group or as a whole class and talk about and work through all the math together. Make it a community experience!)

Try as much or as little as you like from each section and don't stress over things. Just have fun with thinking about the ideas here.

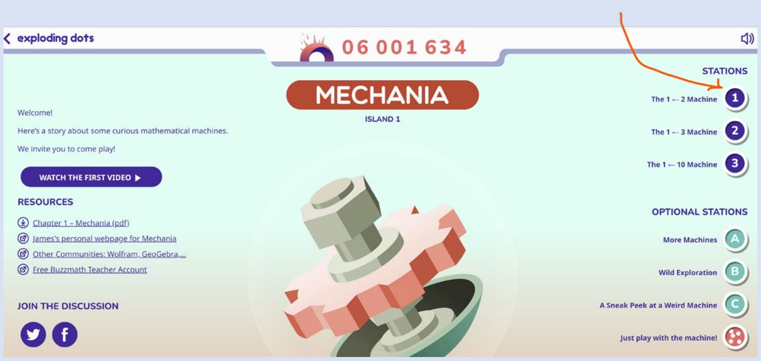
Each section has a video to watch if you like. They're fun!

Some people—over 7 million people, in fact! —like to play with our web app.

www.explodingdots.org

Start on ISLAND 1, Station 1.

This app has its own, different, collection of videos and practice materials.



Some Comments for Teachers

The invention of place value is one of humankind's greatest intellectual achievements. It is surprisingly nuanced, and it permeates the entire K-12 curriculum (and beyond!). So many mathematics standards rely on a deep understanding of place value.

This workbook is a curriculum supplement designed to deepen upper-elementary and middle-school students' understanding of, appreciation for, and joy in mathematics. It focuses on the core of arithmetic and lays groundwork for algebra.

The tool developed here is fun, appealing, and highly visual. It is called *Exploding Dots* and has been used by well over 7 million students and teachers across the planet. You will no longer hear students say: "Math is not for me."

This short video gives the story of this global phenomenon.

https://youtu.be/q51_X8U9JkQ

This text is not a textbook. It is not designed to teach each mathematics topic in its fullest degree, nor in a manner designed as sole first introduction to the topic.

But you will find that this material will—for your students, and for you! —shed stunning light on the curriculum materials you do use and help students gain profound, personal understanding of those resources as you explore them in full, complete detail.

This volume will save you effort and time, and the set the stage for your students' joyful understanding of math, development of powerful meta-cognition skills, and establishing the confidence for them to be empowered, thinking citizens of the world.

How Educators use this Book

The first week of the school year is always informal and quirky. Some teachers follow this work as the start-of-the-year activity and continue it during their regular "Fun Fridays" or for between-unit breaks. Many classes read each section out loud together, pausing to try the practice problems, with the educator stepping in and illustrating ideas as needed.

Each section has a video covering the ideas discussed in that section. They are fun and many classes like to watch each video first before dividing into the content of that section.

There is a web app too for your students to play with if they like.

<u>One small comment</u>: The bulk of the book avoids using exponent notation as not all students are familiar with it. But if your students are, feel free to make use of it! (They might even naturally do so themselves.) Exponent notation is briefly discussed in the final sections of the guide when it is natural to start using it.

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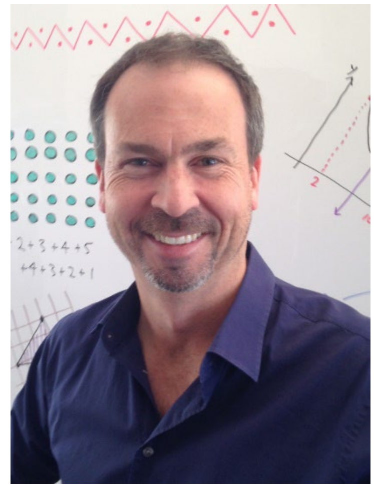
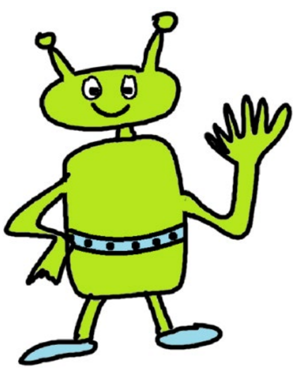
0. Introduction: A Story that is True



G'Day!

I'm James, and here is a story that is true. It is about me realizing as a young lad that mathematics is very powerful, likely more powerful than I as a human being can fully take in.

When I was 12 or so I became fascinated with thinking about the universe and whether or not there could be other civilizations "out there." And I remember wondering if it would be possible to communicate with such aliens (assuming they exist). What would it take to do so?



I really thought hard about this and became very logical and rational in my thinking about it.

First, I realized that communication of any kind requires each person involved (or should I say entity?) being aware of something to communicate with. It must have a sense of itself and a sense of something more than just itself. If the entity is only aware of itself, all will be pointless: it won't realize there is something to talk to.

Such a being, having a sense of self and something more, I then reasoned, is likely to have a sense of not just one thing and two things, but maybe also three things. And four things. And so on. We might both understand the *counting numbers*: 1, 2, 3, 4, … .

One thing *Two Things* *Three Things*

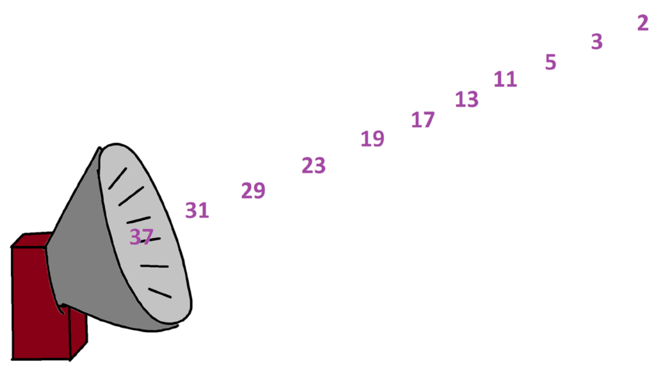
I decided that my best bet in communicating with an alien is to assume we each know how to count things and thus to communicate via the counting numbers. (You can see how I was led to believing that mathematics is universal.)

So, I decided we should send blips of sound or light out into space, in patterns of different counts, to somehow say "Hello! I am here."

But what counts of these blips should we send? What pattern of counting numbers would be interpreted as deliberate and "intelligent" and as coming from someone trying to communicate?

I decided we should send blips that match the first few prime numbers—**2** blips, pause, **3** blips, pause, **5** blips, pause, **7** blips, pause, **11** blips, pause, **13** blips, pause, **17** blips, pause, **19** blips, pause, **23** blips, pause, **29** blips, pause, **31** blips pause, **37** blips, pause, **41** blips, pause, **43** blips, pause, **47** blips, pause, **53** blips, pause, **59** blips, pause, **61** blips, pause, **67** blips. (Maybe that's enough?)

These numbers are special. They seem random, and so are unlikely to be generated by natural phenomena in space. But they are not at all random when you examine them closely. This sequence of numbers would be seen as "intelligent" and like a message.



And this idea of mine, I later realized, is not a bad one: several science fiction writers had come up with the same suggestion of sending out the sequence of prime numbers to grab someone's attention.

Looking back, it was clear as a young lad I was enamored with mathematics. I could sense power to it. I could sense universality to it. And I felt that mathematics went beyond my humanness. I found that thrilling, and inspiring, and somehow comforting.

My young mind could start to see a marvelous journey to be had by simply contemplating simple counting. I sensed the power of it.

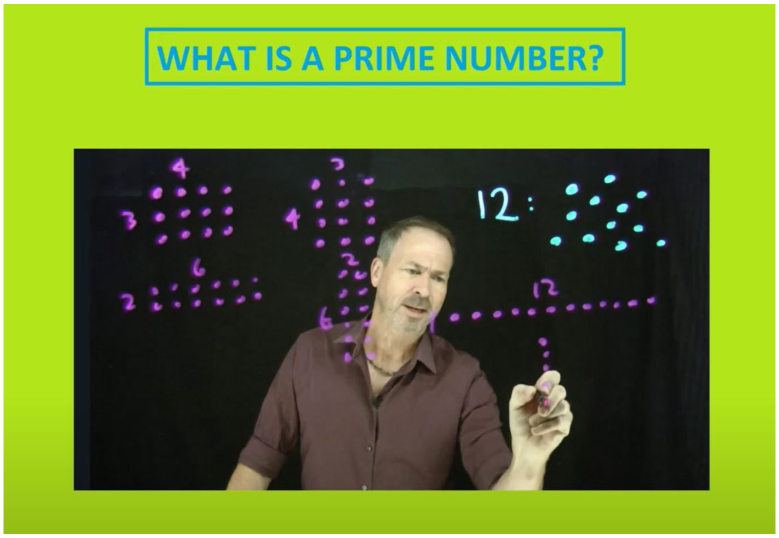
I invite you to go on a journey of powerful mathematics with me, right now! It will be mighty fun!

Question: Do you think there could be other civilizations in the universe? If so, might there by a better way to try to communicate with them than the ideas I came up with as a young lad?

Write your thoughts here.

Question: Do you know what a **prime number** is?

Here's a video of me talking about them if you are curious.

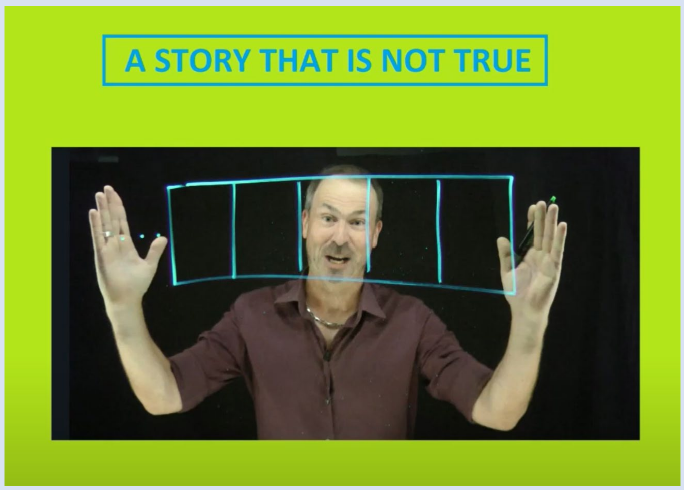


https://youtu.be/YKYHc7yAVOk

1. A Story that it is not True

Let’s start our mathematical journey together with another story, one that is definitely not true!

Here’s the video for this section.



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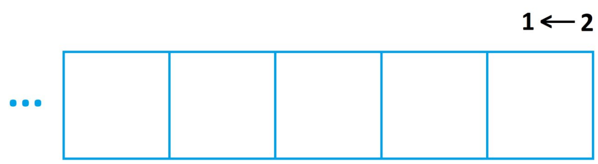
When I was a child, I invented a machine – not true.

This machine was nothing more than a row of boxes that went as far to the left as I ever wanted. For example, here is a machine with five boxes going to the left. But I could have drawn 7 boxes, if I wanted, or 100 boxes, or 1007 boxes.

And, in this untrue story, I gave this machine a name. I called it a "two-one machine," writing "$1 \leftarrow 2$" in a funny backwards kind of way. (I knew no different as a child.)



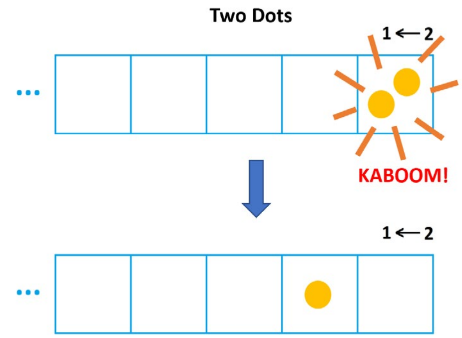
And what do you do with this machine? You put in dots. Dots always go into the rightmost box.

Put in one dot, and, well, nothing happens: it stays there as one dot. Ho hum.

One Dot

But put in a second dot – always in the rightmost box – and then something exciting happens.

Whenever there are two dots in a box they explode and disappear – KAPOW! – to be replaced by one dot, one box to the left.

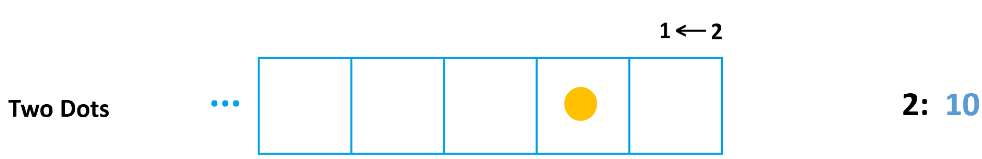


(I always make an exploding sound when an explosion like this happens.)

Question 1.1: Do you see now why I called this a "1 ← 2 machine" written in this funny way?

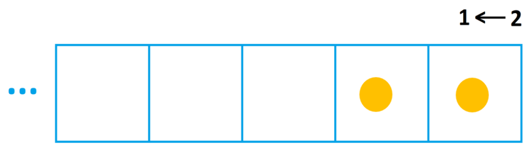
Write your thoughts here.

We see that two dots placed into the machine yields one dot followed by zero dots. I wrote "1" for the dot and "0" for the empty space to its right and created the code "10" for the number two. (This looks like the number ten, I know, but I am thinking of this code as "one zero.")



Putting in a third dot – always the rightmost box – gives the picture one dot followed by one dot. I wrote the code “11” for three.

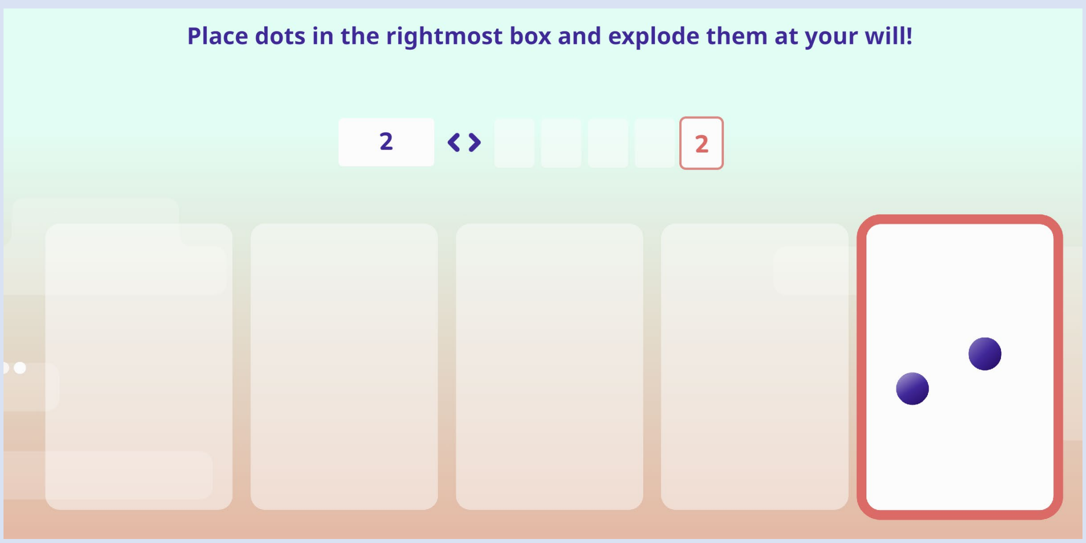
Three dots



3: 11

If you want to play with a 1 ← 2 that is already built, go to

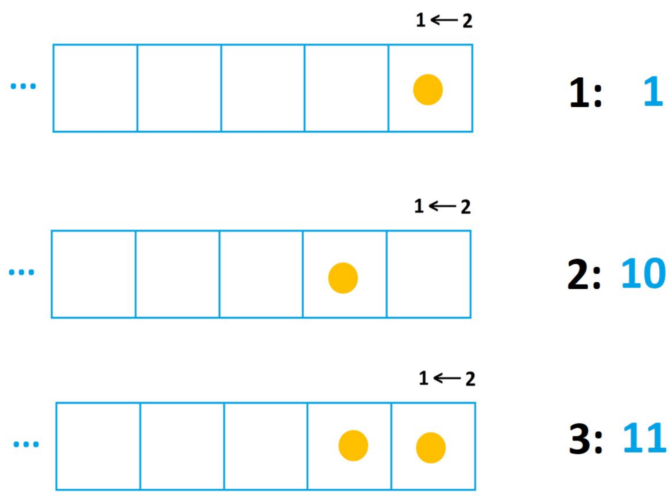
www.explodingdots.org/station/OpenMachinesMechania



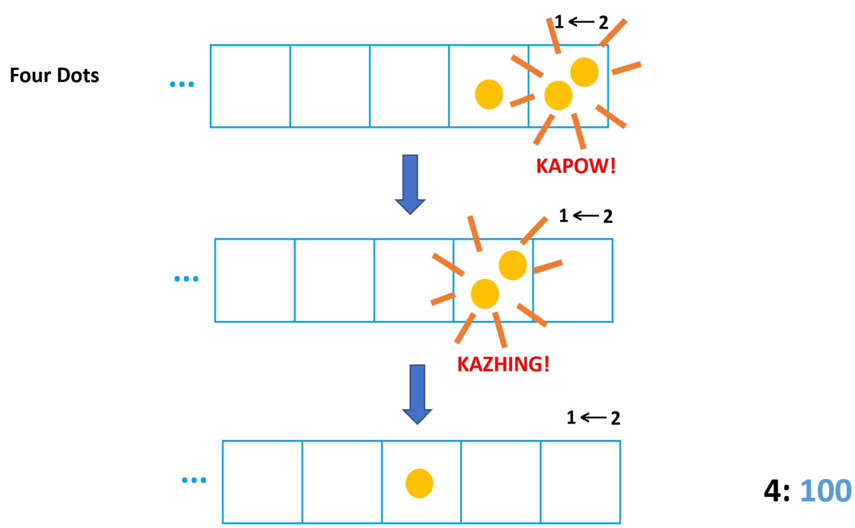
To explode dots, just drag them to the box one place to their left.
To delete a dot, just drag it outside of the machine.

I realized that this machine, in my untrue story, was giving codes for numbers.

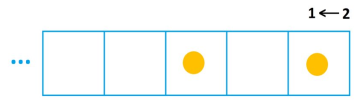
I had the codes 1 and 10 and 11 for the number one, two, and three.



Adding a fourth dot into the machine after having exploded three dots is particularly exciting: we are in for multiple explosions! The 1 ← 2 code for four is 100.

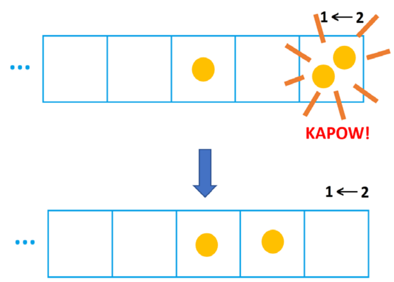


The $1 \leftarrow 2$ machine code for five is 101.



5: 101

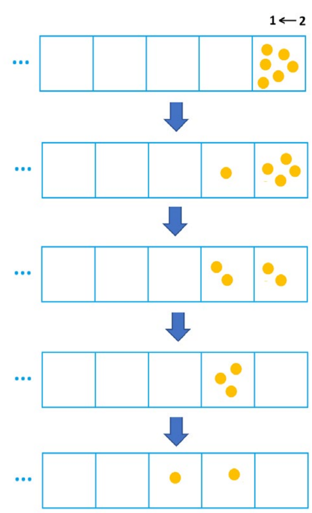
And the code for six? Adding one more dot to the code for five gives 110 for six.



6: 110

We can also get this code for six by clearing the machine and putting all six dots in at once. Pairs of dots will explode in turn to each produce one dot, one box to their left.

Here is one possible series of explosions. (Sound effects omitted!)



Question 1.2: Do you get the same final code of 110 if you perform explosions in a different order? (Try it!)

Did you?

Question 1.3: What is the $1 \leftarrow 2$ machine code for the number thirteen? (It turns out to be 1101. Can you get that answer?)

Can you show how to get the code 1101 for the number thirteen?

Here are the $1 \leftarrow 2$ machine codes for the first ten numbers.

1: 1	**4: 100**	**7: 111**	**10: 1010**
2: 10	**5: 101**	**8: 1000**	
3: 11	**6: 110**	**9: 1001**	

Question 1.4: Do you want to work out the codes for all the numbers up to twenty? (The answer can be NO!)

More codes!

Challenge 1.5: Which number has code 10011 in a $1 \leftarrow 2$ machine?

Can you figure it out?

There are hours of fun to be had playing with codes in a $1 \leftarrow 2$ machine.

SELF-CHECK

Would you like to try these three quick questions to check your own understanding? (The answer can be no!)

Self-Check 1.1 How many 1s appear in the $1 \leftarrow 2$ machine code for the number five?

a) One
b) Two
c) Three
d) Four

Self-Check 1.2 Which number has code 1000 in a $1 \leftarrow 2$ machine?

a) Seven
b) Eight
c) Nine
d) Do I really have to answer this?

Self-Check 1.3 Could a number have the code 1020 in a $1 \leftarrow 2$ machine?

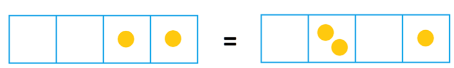
a) The two dots in a box would explode and make one dot, one place to their left. So … NO!
b) If you don't like explosions, I suppose you could leave the two dots there in the one box and not explode them. So, YES … maybe?
c) But most people do choose to do explosions, and so would probably answer NO.
d) Everything said in a), b), and c) is reasonable.

ANSWERS:

1.1
b) The code for five is 101. There are two 1s.

1.2
b) Try putting eight dots into the machine. It gives the code 1000.
(Also, feel free to answer d).)

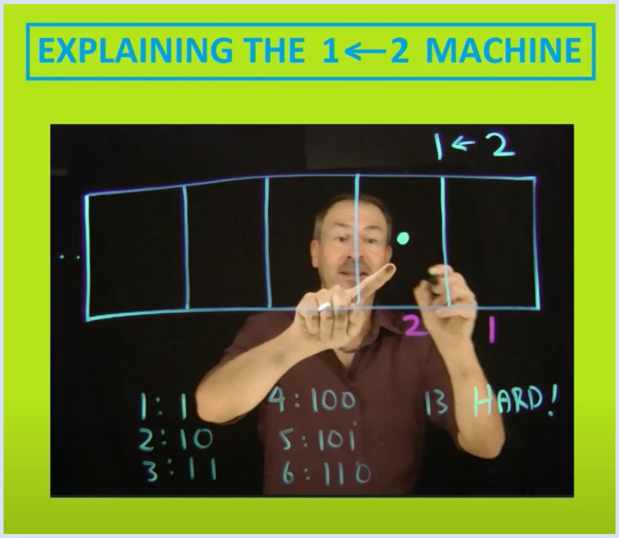
1.3
All answers here are reasonable. Most people do explode dots and say that 1020 is really 1100.



2. Explaining the $1 \leftarrow 2$ machine

Let's make sense of matters.

Here's the video for this section.

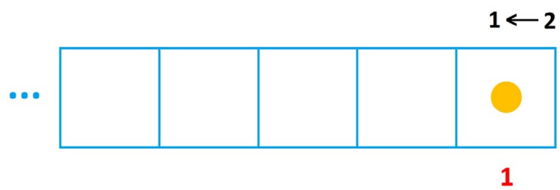


https://youtu.be/t5weGOs5Msc

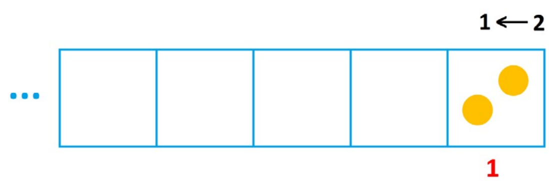
We set up the 1 ← 2 machine so that

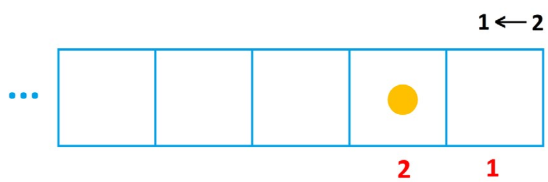
> *Whenever there are two dots in any one box, they "explode," that is, disappear, to be replaced by one dot, one place to their left.*

Also, the machine is set up so that dots in the rightmost box are always worth one.

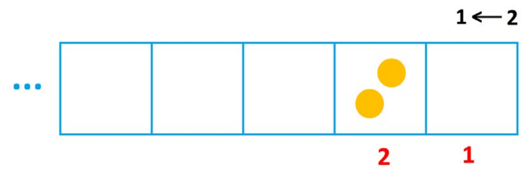


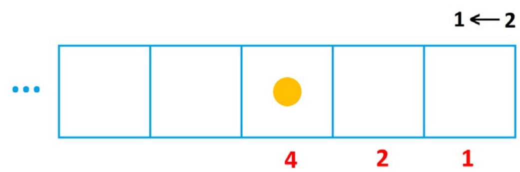
With an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1s, that is, 2.



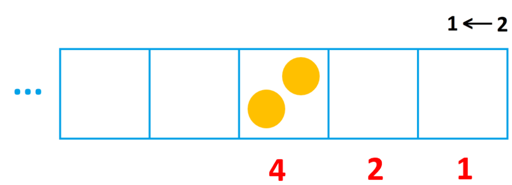


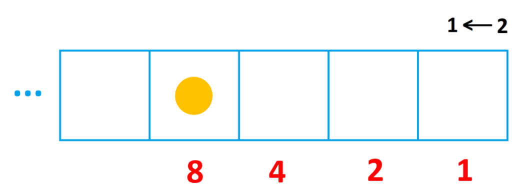
And two dots in this second box is equivalent to one dot, one place to the left. Such a dot must be worth two 2s, that is, worth 4.





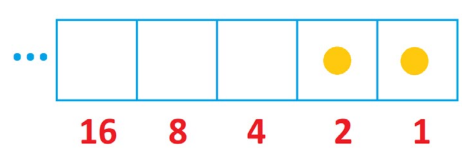
And two 4s makes 8 for the value of a dot the next box over.





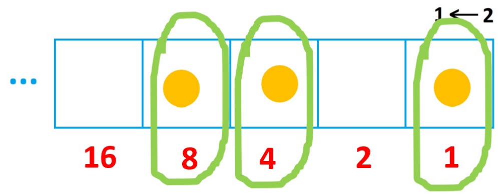
And two 8s make 16, and two 16s make 32, and two 32s make 64, and so on.

Earlier we saw that the $1 \leftarrow 2$ machine code for the number three is 11. We can now see that one dot and one dot in each of the last two boxes does indeed make three: $2+1=3$. Neat!



Last section, I gave the challenge of finding the $1 \leftarrow 2$ machine code for the number thirteen. It was hard! Did you try it? Did you get the answer 1101?

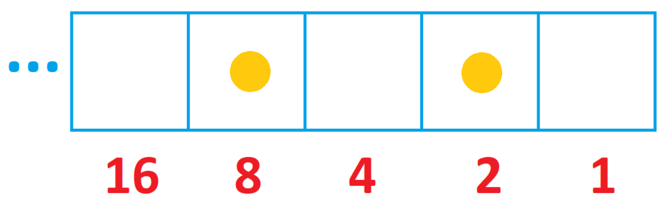
Now that we know how this machine works we can now see—literally see!—that this answer 1101 is absolutely correct: one 8 and one 4 and no 2s and one 1 does indeed make thirteen. Wowza!



$$13 = 8 + 4 + 1$$

Example: Which number has the $1 \leftarrow 2$ machine code 1010?

Answer: We can see that the answer is ten.



$$8 + 2 = 10$$

Question 2.1: We have that $16 + 1 = 17$. What then is the $1 \leftarrow 2$ machine code for the number seventeen?

(If you are playing with the web app, try putting seventeen dots into the machine and watch them all explode to get your answer.)

What's your answer? (Is it helpful to also draw a dots-and-boxes picture showing seventeen?)

Question 2.2: Write the number thirty as a sum of some or all of the numbers 16, 8, 4, 2, and 1. What is $1 \leftarrow 2$ machine code for the number thirty?

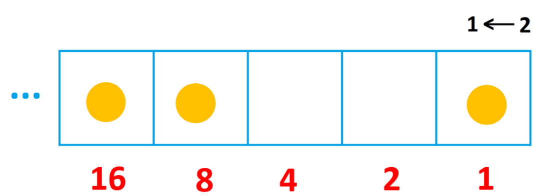
Write your two answers here. (Perhaps also draw a dots-and-boxes picture showing thirty?)

Question 2.3: *Which number has $1 \leftarrow 2$ machine code 11001?*

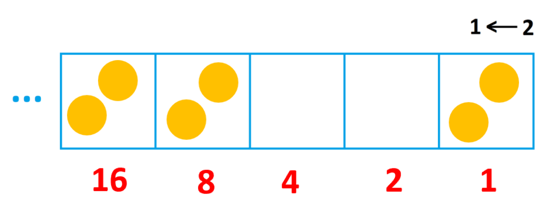
What is your answer?

Question 2.4: What is the $1 \leftarrow 2$ machine code for the number fifty?

Before you answer this question … Let me mention that Lela answered the previous question and saw that twenty-five has code 11001.



She then said, "Well, fifty is double twenty-five" and so she just doubled the number of dots in the picture.



What do you think of Lela's approach? Does it lead here to the $1 \leftarrow 2$ machine code of the number fifty?

Question 2.5: *What is the largest number with a five-digit $1 \leftarrow 2$ machine code?* That number would have code "11111." What number is that?

Write your answer here.

Question 2.6: *What is the smallest number with a five-digit $1 \leftarrow 2$ machine code?*

Write your answer here.

When Ari answered this question, he first said that that the smallest five-digit code is 00001, which corresponds to the number one. But then he realized that 01 and 001 and 0000000000001 all represent the number one, and that you can always put zeros in front of a code and it won't affect the number. So, he deduced:

People don't usually write zeros at the beginnings of their codes.

He then changed his mind and said, "I bet 10000 is the smallest five-digit code."

What do you think?

For example, do you think 101 and 0101 and 00101 are all the same number?

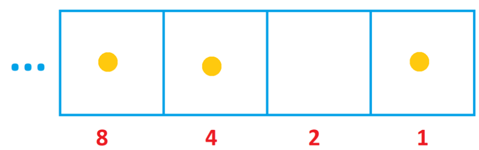
Do we need extra zeros at the front? Is it ever worth writing them?

Some space in case you want to write some thoughts.

People call the 1 ← 2 codes for numbers the **binary** representations of numbers (with the prefix *bi-* meaning "two"). They are also called **base two** representations. One only ever uses the two symbols 0 and 1 when writing numbers in binary.

Computers are built on electrical switches that are either on, or off. So, it is very natural in computer science to encode all arithmetic in a code that uses only two symbols: say 1 for "on" and 0 for "off." Thus, base two, binary, is the right base to use in computer science.

BASE TWO



thirteen = 1101

Some more practice if you would like.

Question 2.7 Which number has code 10011 in a $1 \leftarrow 2$ machine?

Write your answer here.

Question 2.8

a) What is the $1 \leftarrow 2$ machine code (binary code) for the number 15?
b) What is the binary code for the number 30?

Write your answers here.

Question 2.9 Here's a fun question.

The codes in the $1 \leftarrow 2$ machine are called the *binary* codes of numbers. The prefix "bi" means two. Can you guess what each of these English words have to do with the number two?

bicycle **binoculars** **bisect** **biped**

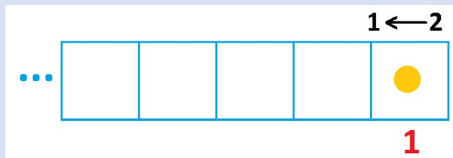
bivalve (an oyster and a clam are examples of bivalves)

Write your answers here.

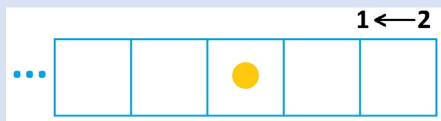
SELF-CHECK

Would you like to try these three quick questions to check your own understanding?

Self-Check 2.1 A dot in the rightmost box of a $1 \leftarrow 2$ machine is worth 1.



What is a dot in the box shown worth?



a) One
b) Two
c) Three
d) Four
e) It is impossible to know.

Self-Check 2.2 The number eighteen equals $16+2$. What is the code for eighteen in a $1 \leftarrow 2$ machine?

a) 10001
b) 10010
c) 10100
d) Something else
e) I prefer not to answer this question.

Self-Check 2.3 Which number has code 11011 in a $1 \leftarrow 2$ machine?

a) Twenty-seven
b) Not twenty-seven

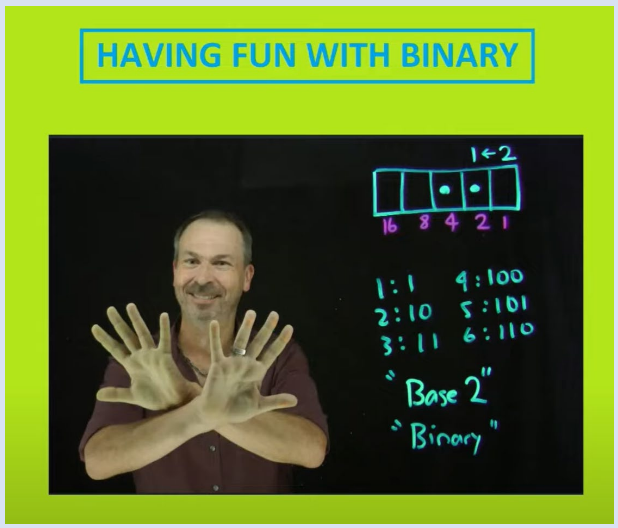
ANSWERS:

2.1 It's worth four. **2.2** It's 10010 (if you didn't choose e). **2.3** It is twenty-seven.

3. Having Fun with Binary

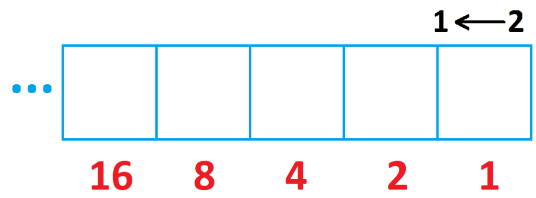
We’re ready now to have some quirky fun!

Here’s the video for this section.

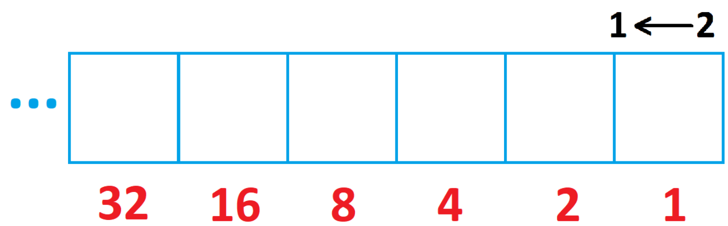


https://youtu.be/kqHtWvJuciQ

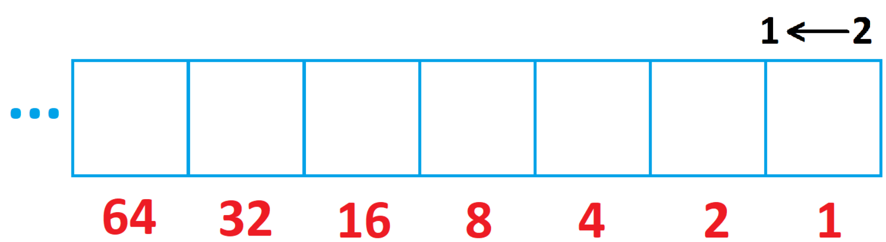
We saw that dots in the boxes of a $1 \leftarrow 2$ machine are worth 1, 2, 4, 8, and 16. These are numbers that are doubling. (Some people call them **the doubling numbers**.)



If we drew a sixth box in the machine, a dot there would be worth double sixteen, which is 32. Why? Because two dots each worth 16 in the 16 box would explode and make one dot worth 32, one place to the left.



And two dots each worth 32 would explode to make one dot worth 64.



And we can keep on going!

Question 3.1: What's the value of a dot if we add yet another box to the machine?

Write your answer here.

Question 3.2: What number has code 101101 in a $1 \leftarrow 2$ machine?

Write your answer here. (Does drawing a picture help?)

Question 3.3: What number has code 1100100 in a $1 \leftarrow 2$ machine?

Write your answer here.

Question 3.4: What's the $1 \leftarrow 2$ machine code for the number two-hundred?

Show your thinking, and answer, here.

The doubling numbers keep on going forever and get very large, very quickly!

1 2 4 8 16 32 64 128 256 512 1024 2048 4096 ...

Optional Challenge 3.5: What is the first doubling number larger than a million?

If you are willing to try this, write your answer here.

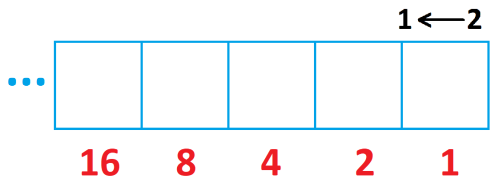
Optional Challenge 3.6: What number has code 1111111111 (that's ten ones in a row) in a $1 \leftarrow 2$ machine?

If you are in the mood to try this, write your answer here.

Optional Extra Hard Challenge 3.7: Is there a doubling number that begins with a 7?

Is there one?

I first showed a $1 \leftarrow 2$ machine with just five boxes, using the doubling numbers 1, 2, 4, 8, and 16.



Here's something fun you can try with five fingers on one hand.

Write the numbers 1, 2, 4, 8, and 16 on your fingers as shown.

Now use your fingers to represent numbers in binary!

24 = 11000

25 = 11001

It's really fun to run through the numbers 1, 2, 3, 4,5, ..., 30, 31 in turn on your fingers. Try it!

Check this box if it is true.

I made all the numbers 1 through 31 in binary on my hand. ☐

Optional Question 3.8: What if you used all ten fingers on two hands with the doubling numbers 1, 2, 4, 8, 16, 32, 64, 128, 256, and 512?

How do you make the number five-hundred on your two hands?

How do you make the number nine-hundred-and-ninety-nine on your two hands?

What's the biggest number you can make on your two hands?

Share your answers here.

SELF-CHECK

Self-Check 3.1 The doubling numbers are kinda cool.

a) Yes
b) No

Self-Check 3.2 It's totally amazing that one can count to 1023 with the fingers on your hands.

a) It is!
b) It isn't!
c) Hang on! What's this number 1023?

Self-Check 3.3 The binary code for one-thousand-and-twenty-three is 1111111111. (That's ten ones in a row.)

a) Yes it is.
b) No it isn't.

ANSWERS:

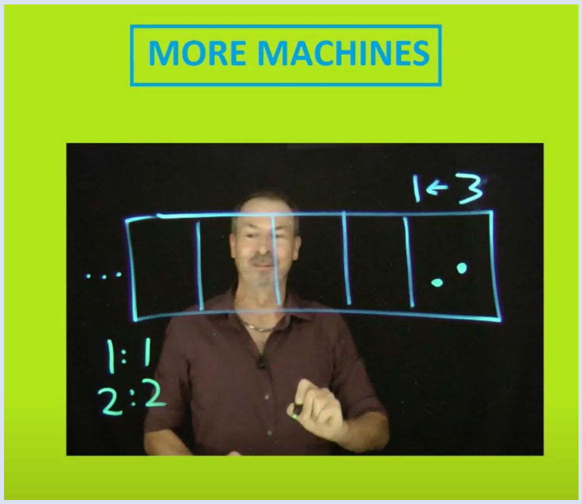
3.1 and **3.2**: I don't know what you chose to answer.

3.3: Yes, the code for 1023 is indeed 1111111111. It corresponds to having all ten fingers up on your two hands, and, if you check, $512+256+128+64+32+16+8+4+2+1$ does add up to 1023.

4. More Machines

Let me continue the story of the machines.

Here's the video for this section.

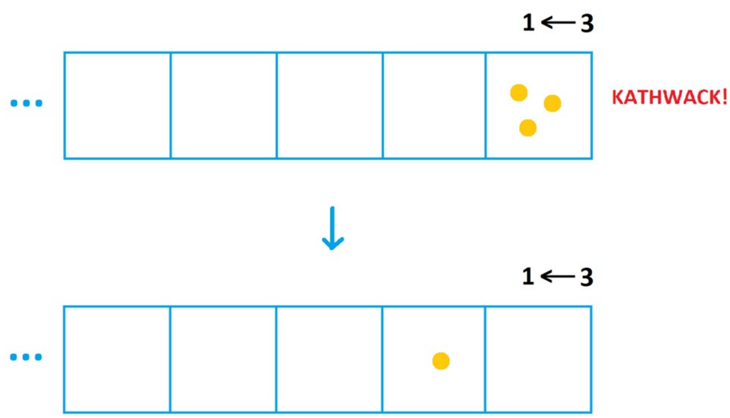


https://youtu.be/FaFjvUDu-uc

After playing with the 1 ← 2 machine for a while, I suddenly had a flash of insight. Instead of playing with a 1 ← 2 machine, I realized I could also play with a 1 ← 3 machine. Whoa!

Again, this "1 ← 3 "notation is written and read backwards: a "three-one "machine.

Now, whenever there are *three* dots in a box, they explode away to be replaced with one dot, one box to the left.



Question 4.1: Work out the codes for the numbers one, two, three, four, five, and six in a $1 \leftarrow 3$ machine. Double-check that my list here is correct.

1:	1	4:	11
2:	2	5:	12
3:	10	6:	20

Am I right?

Question 4.2 What's the code for thirteen in a $1 \leftarrow 3$ machine?
(Try putting 13 dots in all at one in the rightmost box. Use the web app if you like.)

Write your answer here.

Question 4.3 Is the code 20142 "stable" in a $1 \leftarrow 3$ machine? (You can probable guess what I mean by "stable" here and that the answer to the question is: NO, you can still do an explosion. If you do conduct that explosion, what's the stable version of the code you get?)

Write your answer here.

Question 4.4 The code for sixteen in the $1 \leftarrow 3$ machine is 121. (Check this!)

a) Amit wrote that the code for sixteen as 0121. Is he also correct? What do you think?
b) Shania write the code for sixteen as 1210. Is she also correct? What do you think?

Write your answers here.

Optional Challenge 4.5 Which number has $1 \leftarrow 3$ machine code 222 ?

Write your answer here if you choose to try this question.

And hours of fun are to be had playing with numbers in a 1 ← 3 machine.

But then …
Another flash of insight!

Instead of doing a 1 ← 3 machine, I realized I could do a 1 ← 4 machine!

And then …
Another flash of insight!

Instead of doing a 1 ← 4 machine, I could do a 1 ← 5 machine!

And then …
Another flash of insight!

Instead of doing a 1 ← 5 machine, I could do a 1 ← 6 machine!

And then …

I decided to go wild!

SELF-CHECK

Self-Check 4.1 How many dots in a box explode to make one dot, one place to the left, in a $1 \leftarrow 4$ machine.

a) Err …. Four!

Self-Check 4.2 What's the code for ten in a $1 \leftarrow 5$ machine?

a) It's 20.
b) It's 20.
c) It's 20.
d) It's not 20.

Self-Check 4.3 Would the code 30721 be stable in a $1 \leftarrow 6$ machine?

a) No. In the group of 7 dots in the middle, six of them would explode to make one dot, one place to the left.
b) Answer a) is correct.

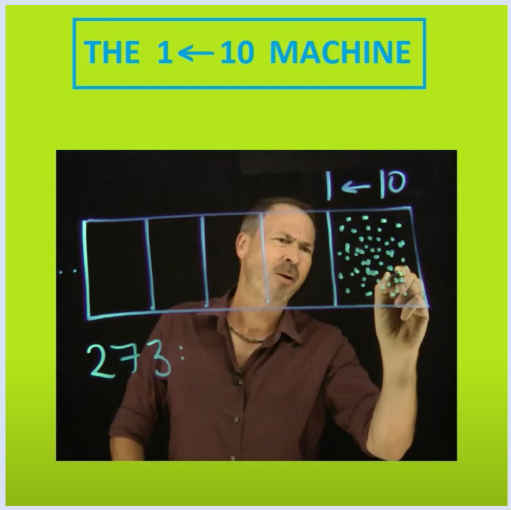
ANSWERS

4.1 a) s correct. **4.2** a) b) c) are all correct. **4.3** Both a) and b) are correct.

5. The $1 \leftarrow 10$ Machine

Okay. Let's go all the way up to a $1 \leftarrow 10$ machine. Crazy!

Here's the video for this section.

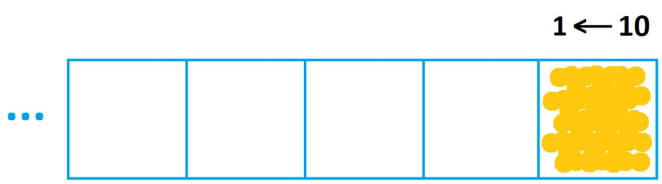


https://youtu.be/v7ddCzzSQmM

And let's put in 273 dots. Crazier!

What is the secret $1 \leftarrow 10$ machine code for the number 273?

Here goes …



273:

I thought my way through this by asking a series of questions.

Will there be any explosions? Are there any groups of ten that will explode? Certainly!

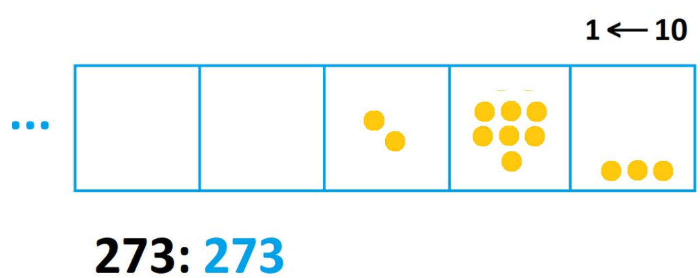
How many explosions will there be initially? Twenty-seven.

Any dots left behind? Yes. Three.

Okay. So, there are twenty-seven explosions, each making one dot one place to the left, leaving three dots behind.

Any more explosions? Yes. Two more.

Any dots left behind? Seven left behind.



Look at what we have:

The $1 \leftarrow 10$ code for two hundred seventy-three is … 273.

Whoa!
Something curious is going on!

SELF-CHECK

Self-Check 5.1 Draw twenty-four dots in the rightmost box of a $1 \leftarrow 10$ machine. Do all the explosions and see the code "24" appear for the number twenty-four.

a) I tried this and I did get indeed get the code 24.
b) I didn't actually draw it, but I can see in my mind's eye how this would work and would give the code 24.
c) I did try it and didn't get the code 24, but I will try again.

Self-Check 5.2 Draw three-hundred-eighty-two dots in the rightmost box of a $1 \leftarrow 10$ machine. Do all the explosions and see the code "382" appear for the number three-hundred-eighty-two.

a) I tried this and I did get indeed get the code 382.
b) That's too much work. I am not going to do that.

Self-Check 5.3 If you had to make a guess for the $1 \leftarrow 10$ machine code for the number one-thousand, eight-hundred, and forty-nine, what would you guess?

a) 1849

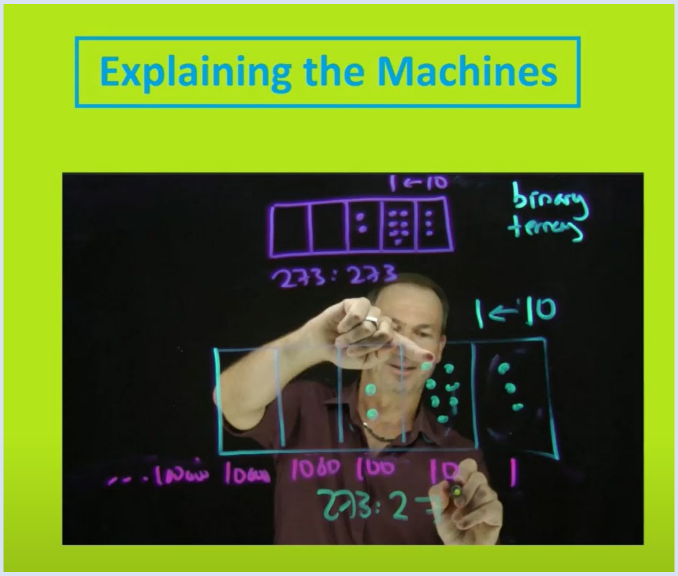
All answers offered could be correct.

ANSWERS

6. Explaining the Machines

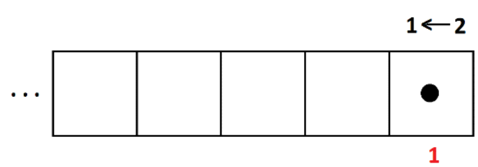
All right. Let's explain what is going on with everything.

Here's the video for this section.



https://youtu.be/6Mc91XIvRo8

We already figured out how the 1 ← 2 machine works. Dots in the rightmost box are always worth one.



And as we keep adding dots to the machine, we follow the rule

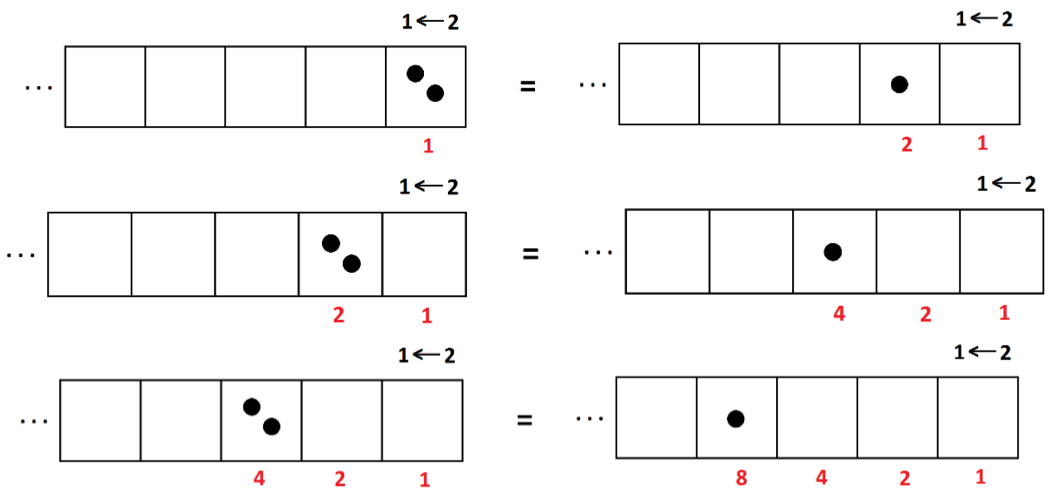
> *Whenever there are two dots in any one box, they "explode," that is, disappear, and are replaced by one dot, one place to their left.*

This means, with an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1s, that is, 2.

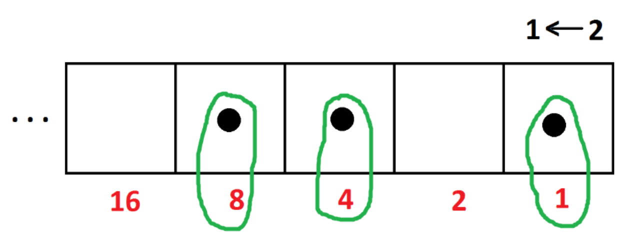
And two dots in this second box are equivalent to one dot, one place to the left. Such a dot must be worth two 2s, that is, worth 4.

And two 4s makes 8 for the value of a dot the next box over.

And two 8s make 16, and two 16s make 32, and two 32s make 64, and so on.



And recall, the code for thirteen is 1101 and we saw this is correct by looking at the values of the dots in this machine.

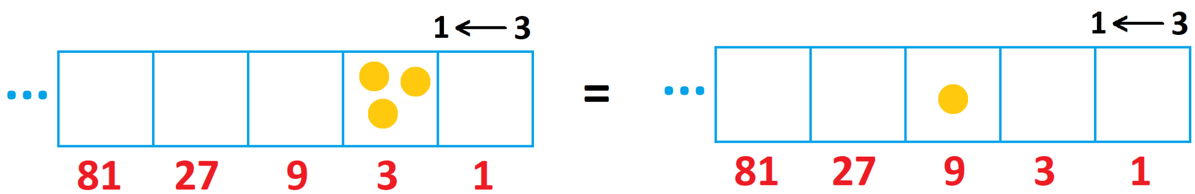


Question 6.1: What number has 1 ← 2 machine code 100101?

Write your answer here.

The same idea must be at play for the 1 ← 3 machine.

Here dots in the rightmost box again are each worth one, but now *three* dots in any one box are equivalent to one dot, one place to the left. We get the dot values in this machine by noting that three 1s is 3, and three 3s is 9, and three 9s is 27, and so on.



Question 6.2:

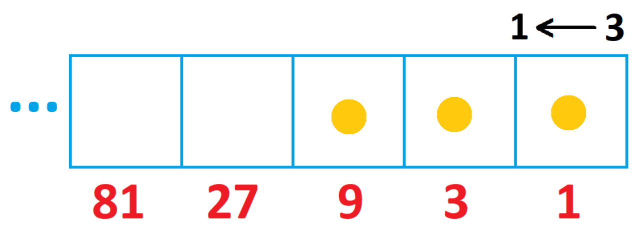
a) Do three 1s make 3?
b) Do three 3s make 9?
c) Do three 9s make 27?
d) Do three 27s make 81?

Write YES four times in this box.

Question 6.3: If one more box is added to the picture of the $1 \leftarrow 3$ machine, what would be the value of a dot in that sixth box?

Write your answer here.

Earlier we tried to work out the $1 \leftarrow 3$ code for the number thirteen. The answer is 111. And now we see that this is absolutely correct: one 9 and one 3 and one 1 does indeed make thirteen.

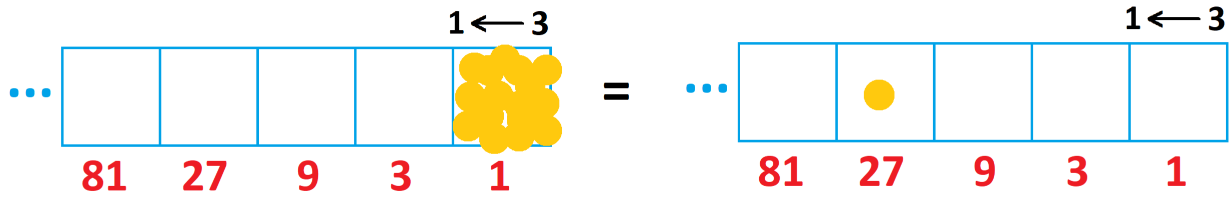


The $1 \leftarrow 3$ machine codes for numbers are called **ternary** or **base three** representations of numbers. Only the three symbols 0, 1, and 2 are ever needed to represent numbers in this system.

> **Question 6.4:** Draw some dots and boxes pictures to show that the $1 \leftarrow 3$ machine code for the number fifteen is 120.
> (And what do one 9 and two 3s add up to?)

Draw your pictures here.

Question 6.5: Put twenty-seven dots in the rightmost box of a $1 \leftarrow 3$ machine code and do all the explosions. You should end up with a single dot in the 27's place. Do you?

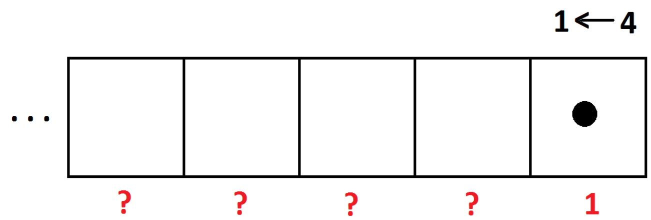


Is it possible to draw some pictures?

Question 6.6 What number has code 2102 in a $1 \leftarrow 3$ machine?

Write your answer here.

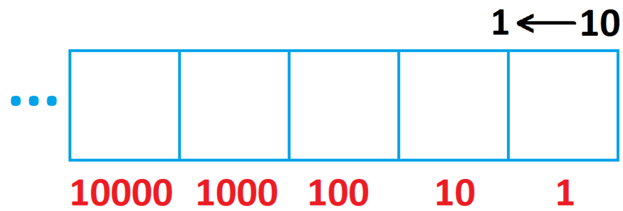
Question 6.7: a) In a $1 \leftarrow 4$ machine, four dots in any one box are equivalent to one dot, one place to their left. What is the value of a dot in each box?



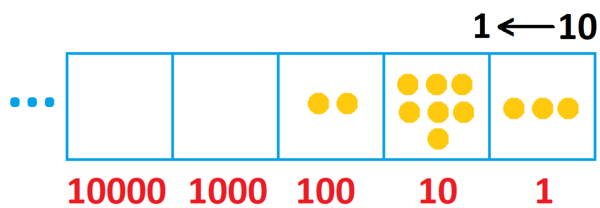
a) What is the $1 \leftarrow 4$ machine code for twenty-nine?
b) What number has 132 as its $1 \leftarrow 4$ machine code?

Write your answers here.

In a 1 ← 10 machine, dots in the rightmost place are worth 1, and we have that ten ones make 10, ten tens make 100, ten one-hundreds make 1000, and so on. A 1 ← 10 has ones, tens, hundreds, thousands, and so on, as dot values.



The code for the number 273 in a 1 ← 10 machine is 273, and this is absolutely correct: 273 is two hundreds, seven tens, and three ones.



In fact, we even speak the language of a 1 ← 10 machine. When we write 273 in words, we write

273 = two hundred seventy three

We literally say, in English at least, two hundreds and seven tens (that "ty" is short for "ten") and three.

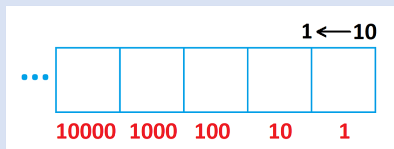
We call the $1 \leftarrow 10$ codes for numbers the **base ten** or **decimal** representation of numbers.

The prefix *dec*- means ten and we have, for example that a decagon is a figure with ten sides and a decade is a period of ten years. Also, December, at one time, was the tenth month of the year. (Care to research the history of how we came up with twelve months for the year and why these months have the names they do?)

SELF-CHECK

The last self-check had us compute the $1 \leftarrow 10$ machine codes of the numbers 24, 382, and 1849 by putting each count of dots in the rightmost box of the machine and conducting (or at least imagine conducting) many many explosions.

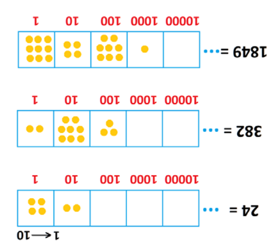
Now we know what a dot in each box of a $1 \leftarrow 10$ machine is worth.



Self-Check 6.1 Quickly draw dots and boxes pictures of the codes for 24, 382, and 1849 without doing any explosions.

ANSWERS

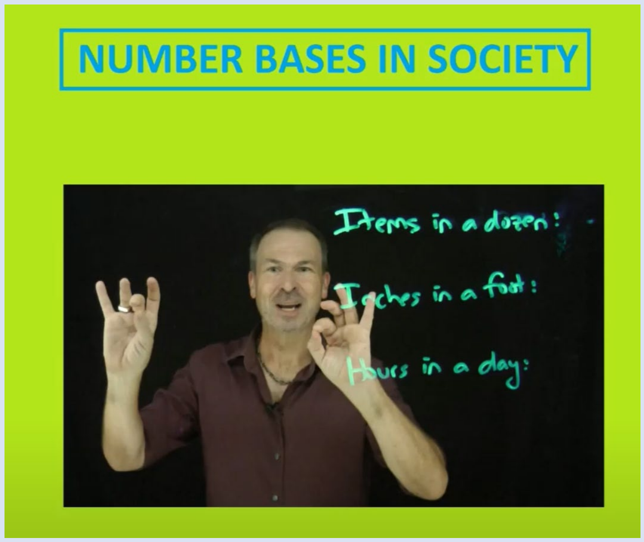
Here are the codes. We can see what they are going to be without putting in dots and doing explosion.



7. Number Bases in Society

We can write codes for numbers in any machine we choose.

Here’s the video for this section.



https://youtu.be/5IRpaOUGEC4

Optional Question 7.1: Write the codes for the number thirty-one in each of the following machines.

a) A $1 \leftarrow 10$ machine.
b) A $1 \leftarrow 2$ machine.
c) A $1 \leftarrow 3$ machine.
d) A $1 \leftarrow 4$ machine.
e) A $1 \leftarrow 5$ machine.
f) A $1 \leftarrow 6$ machine.
g) A $1 \leftarrow 31$ machine.

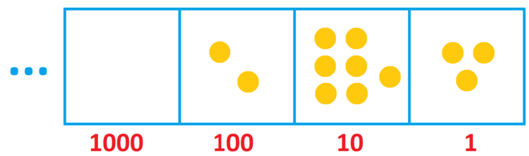
Write your answers here.

These codes are called **number bases:**

The $1 \leftarrow 2$ machine codes are base two (binary) codes,
The $1 \leftarrow 3$ machine codes are base three (ternary) codes,
The $1 \leftarrow 10$ machine codes are base ten codes,

and so on.

And we noticed that our society likes to speak the language of the base ten machine.



<u>273</u> = two <u>hundred</u> seven<u>ty</u> three

digit digit digit

Why do you think we humans chose the 1 ← 10 machine to play with?
Why do we like the number ten so much for arithmetic and counting?

One answer could be because of our human anatomy: we are born with ten digits on our hands. (Thumbs and fingers—and toes too—are called **digits**.) Many historians believe this could well be the reason why we humans have favored base ten. We even call the individual symbols in a long number its “digits.” It is not a coincidence we are using the same word.

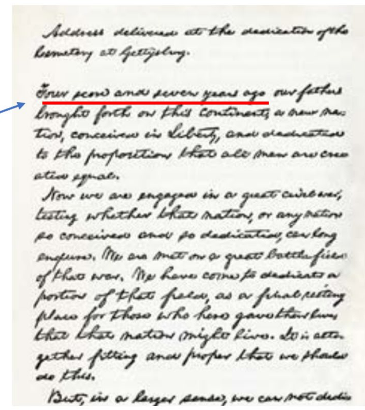
Question 7.2: There are some cultures on this planet that used base twenty for their number systems. Why might they have chosen the number 20 do you think?

Write your guess here.

There is some base-twenty thinking in societies of today. Early American president Abraham Lincoln gave a very famous speech called the Gettysburg Address that begins: “Four score and seven years ago.” The word **score** is an old word for “twenty” and so Lincoln was saying: “four-twenties and seven years ago.” That’s 87 years.

Gettysburg Address, 1863

“Four score and seven years ago …”

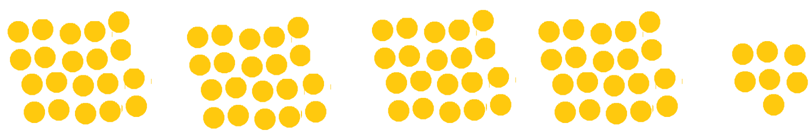

Address delivered at the dedication of the Cemetery at Gettysburg.

Four score and seven years ago our fathers brought forth on this continent, a new nation, conceived in Liberty, and dedicated to the proposition that all men are created equal.

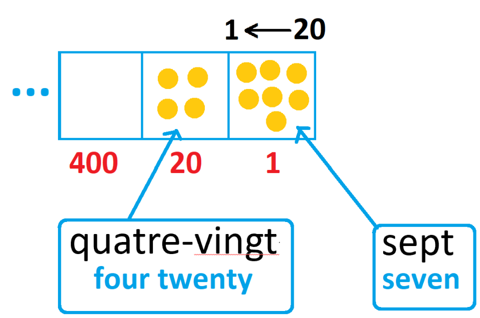
Now we are engaged in a great civil war, testing whether that nation, or any nation so conceived and so dedicated, can long endure. We are met on a great battle-field of that war. We have come to dedicate a portion of that field, as a final resting place for those who here gave their lives that that nation might live. It is altogether fitting and proper that we should do this.

But, in a larger sense, we can not dedi

87 =

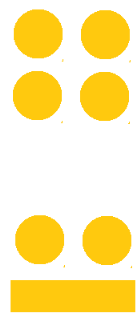


And, in French, the number 87 is said just this way too: *quatre-vingt-sept* translates, word for word, as “four twenties seven.”



Ibo, a Nigerian language of the south-east region, says 87 as *ogu anon na asaa*, which, word-for-word, translates to “twenty four and seven.” It’s the same idea of thinking of four groups of twenty and seven more.

Question 7.3 The Maya of the Mayan Civilization of Mesoamerica used a base twenty system, but they wrote their numbers vertically. They used dots and bars in their number system. Can you make a guess as to how the picture below is meant to be read as "four twenties and seven"?



Write your thoughts here.

Question 7.4: Do you know another language? How is 87 said in your language: in a base-ten way like English, or in a base-twenty way like French or Ibo or Mayan, or in a different way entirely?

Have you another language to share here?

Question 7.4 Continued:

I recently asked on social media how people say "87" in their own languages and received so many responses! Most languages seem to use a base-ten system, but not all.

Malay *lapan puluh tujuh* = "eight ten(s) seven"
Note: "Ten" in Malay is *sepuluh*, but when used in the construction of another number, the "se" is dropped.

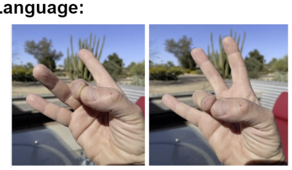
Vietnamese: tám *mươi* bảy = " eight tens seven"

Thai: bet sip jet = "eight tens seven"
(Phonetic, as Thai alphabet is derived from Khmer script)

Mandarin: 八十七 Bāshíqī = "eight ten seven"

Cantonese: eight ten(s) seven

American Sign Language:



="eight seven"

Scottish Gaelic: ochdad 's a seachd = "eighty and seven.

Portuguese: *oitenta e sete* = "eight tens and seven"

Spanish: *ochenta y siete* = "Eighty and seven"

If you would like to see the document, and contribute to it if your language isn't already there, please go to

https://docs.google.com/document/d/1l4ryknunJ7QR9q953PkmTs2NFHvqqe-8n8ajYsEv1bw/edit?usp=sharing

It seems natural for humans to develop base ten and base twenty number systems (think fingers and think toes), but some cultures on Earth actually developed a base-12 number system insead. Wow!

And this could be because there is also a very natural way to count to twelve on one hand. We have four long digits naturally broken into three segments each and a natural pointer to point at them—a thumb!

In fact, in some parts of the world, often in India and south-east Asia, it is common for people to count this way to this day.

> **Question 7.5:** Were your brought up to count to 12 on one hand? Were any of your friends?

Share your answer here.

And there is still "twelveness" in our everyday life.

Question 7.6: How many items are in a dozen?

Write your answer here.

Question 7.7: How many inches are in a foot?

Write your answer here.

Question 7.8: How many hours are in a day? Don't think night, think day!
(The first clocks humans built were sundials, which worked only when there was sunshine, that is, during the day. People in ancient times weren't able to measure time during the night.)

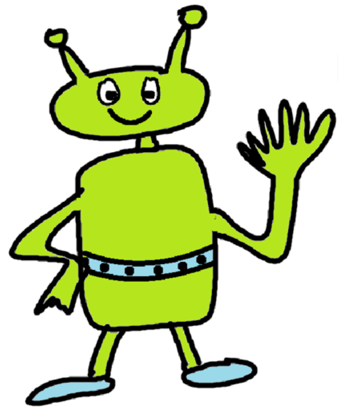
Write your answer here.

Question 7.9: The number twelve is very handy in matters of weights and measures in everyday life. For instance, one might not want to purchase a full unit of some quantity, but perhaps only a half, or a third, or a quarter of that quantity. (These fractions are common in everyday operations.)

a) Eggs are sold in quantities of twelve. How many eggs is half a dozen of them? A third of a dozen? A quarter of a dozen? Are these nice numbers of eggs?
b) If eggs were sold in quantities of ten (to follow base ten thinking), how many eggs is half of that quantity? A quarter of that quantity? A third of that quantity? Do you like the answers you are getting?

Write your answers and thoughts here.

Question 7.10: I happen to know that Martians have six fingers on each of two hands. What base do you think they might use in their society?



My brain thinks there could be more than one possible answer to this question. What is your brain thinking?

The English Language is a Bit of Everything

Even though we do mathematics in a base ten system, English has special words for the first twelve numbers. (Have you noticed that? All the names for the numbers 1 through 12 are their own special words.) Apparently, we still think twelve-ness is important.

one
two
three
four
five
six
seven
eight
nine
ten
eleven
twelve
thirteen
fourteen
fifteen
sixteen
seventeen
eighteen
nineteen
twenty
twenty-one
twenty-two
....

base ten system

After twelve, we fall into a special naming pattern: thirteen, fourteen, fifteen, and so on, up to nineteen, using the special suffix "teen."

But then at twenty, we change to a different pattern. We never use "teen" again and say twenty-one, twenty-two, …, thirty-one, …, sixty-one, …, and so on. We stick with a new pattern for twenty onwards.

So, we follow a base-10 number system, use special words for the first 12 numbers, use a special patten up to 20, and then change to a different pattern for 20 onwards.

English is trying to do it all. That makes English hard and strange!

Question 7.11: (These three sets of questions are from Chris Bolognese.)

A. The following are names of numbers in traditional Welsh.

11: un ar ddeg   13: tri ar ddeg   14: pedwar ar ddeg

a) Can you guess what number is “ddeg”?

b) What numbers are “un”, “tri”, and “pedwar”?

c) What do you think “ar” means?

d) The Welsh for five is “pump”. Make a conjecture about how to say 15 in Welsh. Then do some research to see if you were right.

Write your thoughts here.

B. The following are names of numbers in Finnish.

11: yksitoista   13: kolmetoista   14: neljätoista

a) What do you think “toista” means in Finnish?

b) Make a guess as to how to say one, three, and four in Finnish.

c) The Finnish word “kaksi” means two. How might you say twelve in Finnish?

d) The Finnish word for ten is actually “kymmenen”. How do you think you would say twenty? Do some research to see if you were right.

Write your thoughts here.

C. The following are names of numbers in Mandarin Chinese.

11: Shí yī 13: Shí sān 14: Shí sì

a) What is the word for ten in Chinese?

b) What do you think yī, sān, and sì each mean?

c) The Chinese word for thirty-four is Sān shí sì. Make guesses for the Chinese words for forty, forty-three, and forty-four.

Write your thoughts here.

SELF-CHECK

Self-Check 7.1 In English we say "ninety-three' for the number 93. What number system are we using when we say this?

a) Base ten
b) Base twenty
c) Base twelve
d) Something else

Self-Check 7.2 A dialect of Ibo (a Nigerian language) says *ihule eno na essa* for the number 87. This translates as "twenty into four, and seven." What number system is this dialect using for representing this number?

a) Base ten
b) Base twenty
c) Base twelve
d) Something else

Self-Check 7.3 I happen to know that *Venutians* (beings from the planet Venus) have four fingers on each of two hands. What base do you think they might use in their society?

a) If they focus on one hand, perhaps base four.
b) If they focus on two hands, perhaps base eight.
c) If they focus on fingers and toes (assuming they have two feet and four toes on each foot), perhaps base sixteen.
d) Everything stated in a), b), and c) is reasonable.

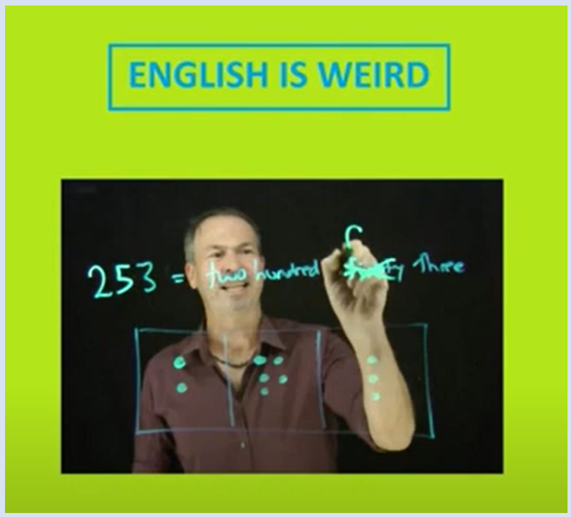
ANSWERS

7.1 a) **7.2** b) **7.3** Everything said is reasonable.

8. English is Weird

We noted in the last section that English is a bit strange in how it names and says numbers. But the truth is that it is actually down and outright weird!

Here's the video for this section.



https://youtu.be/U2JkF3yXEwM

For starters:

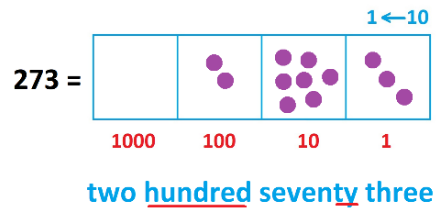
Question 8.1: Have you ever noticed that the spelling of "weird" is weird? What happened to "i before e, except after c"?

Can you think if other English words that break the "i before e, except after c" rule?

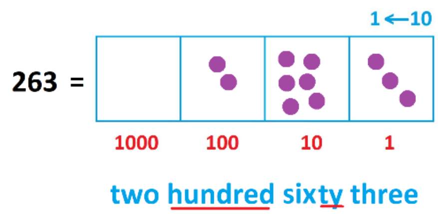
Can you think of one?

The strangeness of English keeps going.

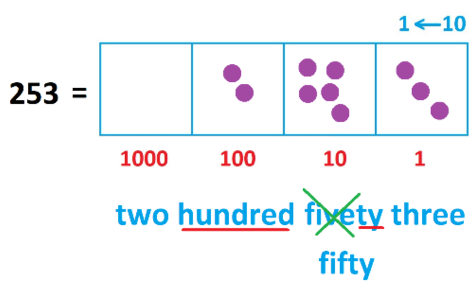
Here's the number 273 in a $1 \leftarrow 10$ machine. It is literally two hundreds, seven tens (*ty* is short for "ten" in English), and three. Nothing too strange there. (Well, "ty" is a bit strange.)



And here is 263. Nothing strange here either.

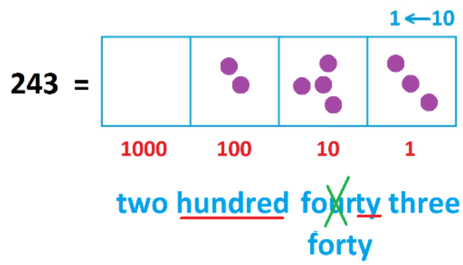


But listen to the number 253. We should say "two hundred five-ty three" but we don't. English has us say "fifty" instead of five-ty.

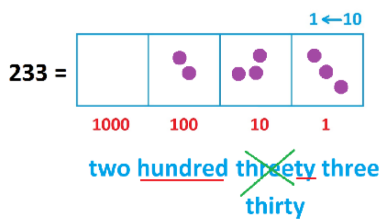


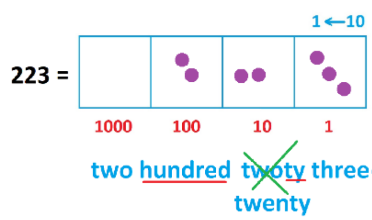
Now saying 243 out loud sounds right, but when writing it out we should write “four-ty,” but English insists we write “forty” instead.

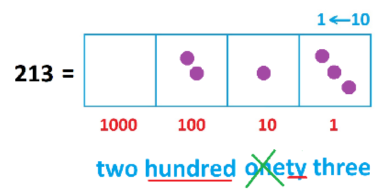
Weird!



And there is 233 and 223 and 213.





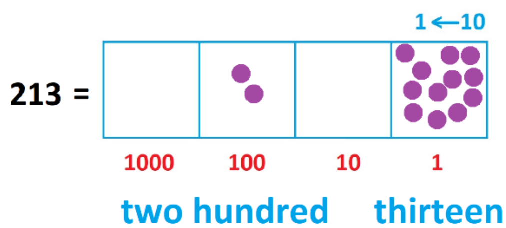


Question 8.2: Something additionally strange happens with the number 213. What do we say instead of “two hundred onety three”? Does what we say make sense?

Write your thoughts here.

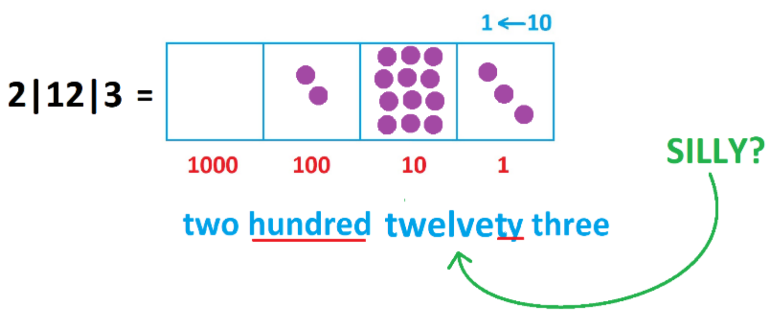
Question 8.3: When Katya thought about this question, she said that "two hundred and thirteen" is really this picture: two hundreds and an extra thirteen dots all in the ones place!

What do you think? Are we allowed to have 13 dots in a single box?

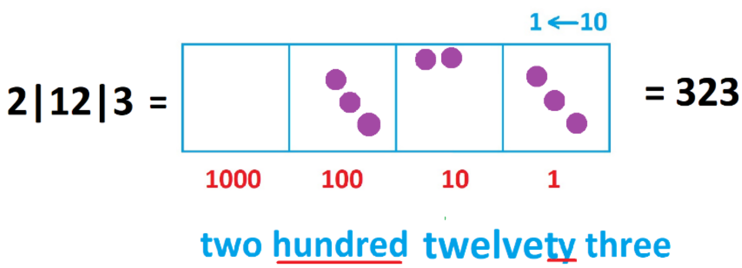


Write your thoughts here.

Most people in society think it is silly to have a large number of dots in any single box. You would never say "two hundred and twelvety three," for example.



But, of course, we know ten dots in any box explode away to produce one dot, one place to their left. So, this number is really 323 in disguise.



Question 8.4: Draw a dots-and-boxes picture of "two-hundred eleventy three." What number is this?

Write your answer here.

Question 8.5: Over a thousand years ago people spoke a version of English we today call "Old English." It included words equivalent to twelvety and eleventy.

What number is twelvety?
What number is eleventy?
(Would saying "thirteen-ty" be just too silly?)

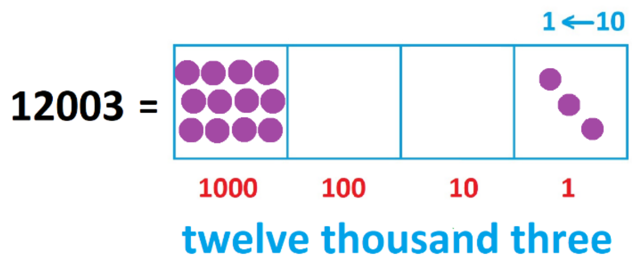
Write your thoughts here.

But society is inconsistent.

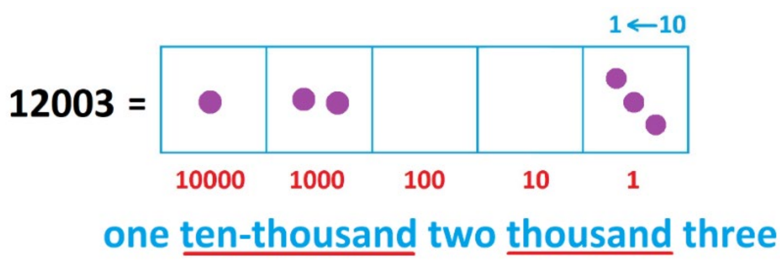
Question 8.6: How do you say the number 12,003 out loud?
Draw a dots and boxes picture of the number you say.

Draw your picture here.

The number 12,003 is pronounced "twelve thousand three" as though we have twelve thousands and three ones. Here we **do** allow twelve dots in a box!



But we don't write what we say. Here's a picture of what we write.



If English and society were consistent, we would say "one ten-thousand two thousand three." But we don't!

What we're learning here is that society and the English language has all sorts of strange demands on how we write and say numbers. The math is always solid and clear. It is just society that makes different, and sometimes strange, choices about speaking the math.

Question 8.7: Do you know another language? How is 12003 said in your language?

Share your answer here.

Question 8.8 Here are four boxes of the $1 \leftarrow 10$ machine. Let's put twelve dots in each box. (We have 12 thousands, 12 hundreds, 12 tens, and 12 ones.)

12	12	12	12

a) How might you say this number out loud?

b) Did you notice as you said this out loud that saying "twelve thousand" sounds acceptable to our ear? And saying "twelve hundred" is okay too? Saying "twelvety" is odd. But saying "and twelve" sounds okay?

That is, did you notice that three-quarters of what you said in part a) sounds acceptable to our ear?

c) What number is 12|12|12|12 really?

d) How do you say your answer to part c) out loud? Do you start with "One ten-thousand, three thousand, ..." ?

Write your answers here.

SELF-CHECK

Self-Check 8.1 Is English weird?

a) Yes
b) No
c) I can't tell. I know English too well and I am too used to it. Anything that is weird to others is normal to me.

Self-Check 8.2 If someone were to say "fifteenty," which is weird, what number would they be communicating?

a) Fifteen tens, which is 150.
b) Some other number.

Self-Check 8.3 What number is "eleven-thousand, eleven-hundred, eleventy, eleven"?

a) 121
b) 1221
c) 12221
d) 122221

ANSWERS

8.1 I wonder how you answered this. **8.2** a) **8.3** c)

9. If English is Weird, we can be Weird Too! ADDITION

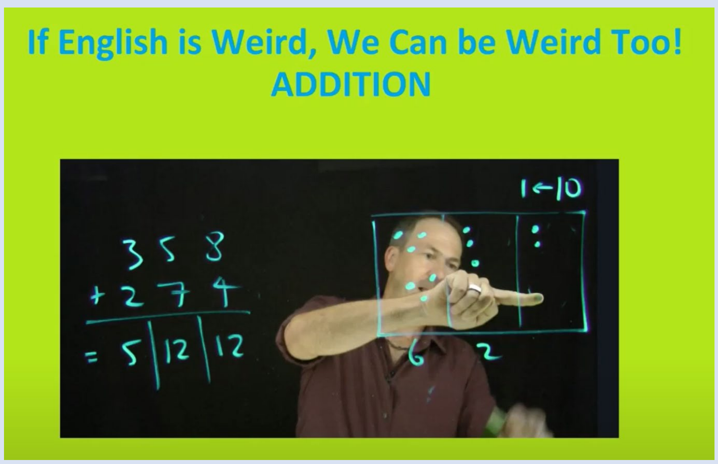
Society has given itself permission to be strange in how it says and writes numbers.

I say … let's go for it! Let's be a little strange too and see what the math then has to say about working with numbers.

Let's next go through the arithmetic we've already learned in school and see how ignoring society actually makes the mathematics so much simpler! We can always fix up what we do to make society happy at any time.

This is going to be fun!

Here's the video for this section.



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We'll focus on the base-ten place-value system, that is, the $1 \leftarrow 10$ machine, the one our society today likes best.

And let's start with an addition problem.

Compute $251 + 124$.

Such a problem is usually set up this way.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline \end{array}$$

And this addition problem is easy to compute:

$2 + 1$ is 3, and $5 + 2$ is 7, and $1 + 4$ is 5.

The answer 375 appears.

$$\begin{array}{r} 251 \\ +\,124 \\ \hline 375 \end{array}$$

But did you notice something curious just then?

I worked from left to right—just as I was taught to read. This is opposite to what most people are taught to do in a mathematics class: go right to left.

Question 9.1: Does it matter? Compute the problem from right to left instead. Do you get the same answer 375?

Why are we taught to work right to left in mathematics classes?

Write your thoughts here.

Many people suggest that the problem we just did is too nice. We should do a more awkward addition problem, one like $358 + 287$.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

Okay. Let’s do it!

Let’s go left to right again:

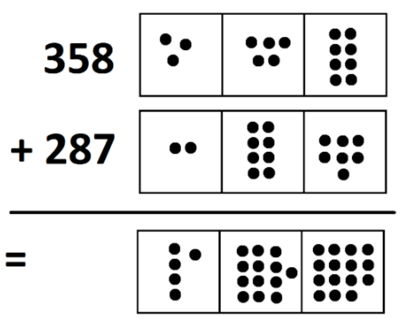
> $3 + 2$ is 5, and $5 + 8$ is 13, and $8 + 7$ is 15.

The answer five-hundred thirteen-ty fifteen appears. (Remember, “ty” is short for *ten*.)

$$\begin{array}{r} 358 \\ +\,287 \\ \hline \end{array}$$

$$5\,|\,13\,|\,15$$

And this answer is absolutely, mathematically correct! You can see it is correct in a $1 \leftarrow 10$ machine. Here are 358 and 287.



5 | 13 | 15

> Adding 3 hundreds and 2 hundreds really does give 5 hundreds.
> Adding 5 tens and 8 tens really does give 13 tens.
> Adding 8 ones and 7 ones really does give 15 ones.

Five-hundred thirteen-ty fifteen is absolutely correct as an answer – and I even said it correctly. We really do have 5 hundreds, 13 tens, and 15 ones. There is nothing mathematically wrong with this answer. It just sounds weird.

But, as we realized in the last section, English is inconsistent about what it thinks is weird and what it thinks is not. I say, let's use weirdness to our mathematical advantage and be weird when it makes matters natural and easy.

> **Question 9.2:** Can we fix up the strange-sounding answer "five-hundred thirteen-ty fifteen" to make society happy?

Any thoughts?

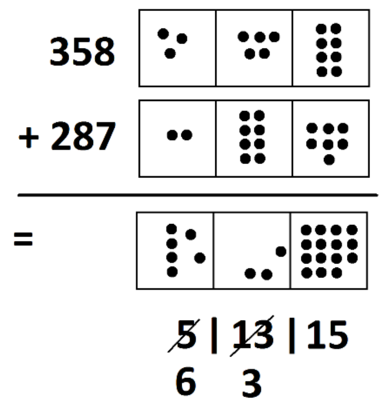
The answer to this question is yes! We can do some explosions since this is a $1 \leftarrow 10$ machine, after all.

Which do you want to explode first: the 13 or the 15?

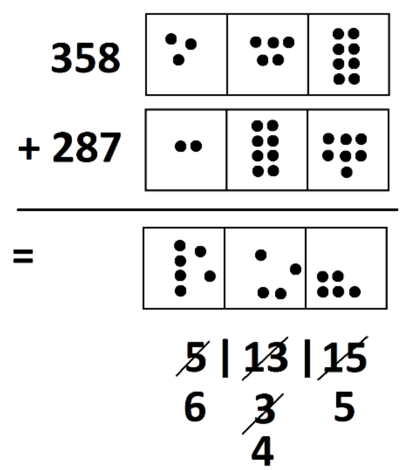
It doesn't matter!

Let's explode from the 13.

Ten dots in the middle box explode to be replaced by one dot, one place to the left.



The answer six hundred three-ty fifteen now appears. This is still a lovely, mathematically correct answer. But society at large might not agree. Let’s do another explosion: ten dots in the rightmost box.



Now we see the answer six hundred four-ty five, which is one that society understands. (Although, in English, “four-ty” is usually spelled *forty*.)

Question 9.3 Write down the answers to the following addition problems working left to right and not worrying about what society thinks! (Actually, do you have to go left to right? Can you go in any order you like?)

For the first four, you I get the answers: 4|6|11, 7|13|8, 4|15|15, and 12|9|6. Do you? What do you get for the last two?

$$\begin{array}{r} 148 \\ +\ 323 \\ \hline = \end{array} \qquad \begin{array}{r} 567 \\ +\ 271 \\ \hline = \end{array} \qquad \begin{array}{r} 377 \\ +\ 188 \\ \hline = \end{array} \qquad \begin{array}{r} 582 \\ +\ 714 \\ \hline = \end{array}$$

$$\begin{array}{r} 310462872 \\ +\ 389107123 \\ \hline = \end{array} \qquad \begin{array}{r} 87263716381 \\ +\ 18778274824 \\ \hline = \end{array}$$

Write your answers underneath each problem. Perhaps use vertical lines to separate numbers—it usually gets hard to read if you don't.

According to a calculator, what is the answer to each of the six problems?

Do you have enough space above to write those answers in too?

Can you now do explosions to convert each weird answer into the same one the calculator says?

(For example, I got that 4|6|11 becomes 4|7|1 after an explosion, which is what my calculator says is the answer to the first problem.)

Here's a lot of space for all this work.

The Traditional Algorithm

How does this dots and boxes approach to addition compare to the standard algorithm most people know?

Let's go back to the example $358 + 287$. Most people are surprised (maybe even upset and angry!) by the left-to-right answer $5 \mid 13 \mid 15$.

$$\begin{array}{r} 358 \\ +\,287 \\ \hline 5 \mid 13 \mid 15 \end{array}$$

This is because the traditional algorithm has us work from right to left, looking at $8 + 7$ first.

8 dots and 7 dots makes 15 dots. But the algorithm doesn't have us write down 15. Instead, we explode ten dots right away and write on paper a 5 in the answer line together with a small 1 tacked on to the middle column. People call this **carrying the one** and it – correctly – corresponds to adding an extra dot in the tens position.

$$\begin{array}{r} 3\overset{1}{5}8 \\ +\,287 \\ \hline 5 \end{array}$$

Now we attend to the middle boxes.

Adding 5 dots and 8 dots and the 1 extra dot from the previous explosion gives us 14 dots. But we perform another explosion right away.

$$\begin{array}{r} {}^{1}\,{}^{1} \\ 358 \\ +\,287 \\ \hline 45 \end{array}$$

On paper we write 4 in the tens position of the answer line, with another little 1 placed in the next column over. This matches the idea of the dots-and-boxes picture precisely.

And now we finish the problem by adding the dots in the hundreds position.

$$\begin{array}{r} {}^{1}\,{}^{1} \\ 358 \\ +\,287 \\ \hline 645 \end{array}$$

The traditional algorithm works right to left and does explosions (*carries*) as one goes along. On paper, it is swift and compact, and this might be why it has been the favored way of doing long addition for centuries.

The dots-and-boxes approach works left to right, just as we are taught to read in English, and leaves all the explosions to the end. It is easy to understand and kind of fun.

Both approaches, of course, are good and correct. It is just a matter of taste and personal style which one you choose to do. (And feel free to come up with your own new, and correct, approach too!)

Question 9.4 Josie doesn't understand what the little 1s at the top of this long addition problem are. Can you explain, in your own way, what they are and what is happening in this addition problem?

$$\begin{array}{cccc} & {}^{1} & {}^{1} & \\ & 7 & 8 & 2 \\ + & 1 & 8 & 9 \\ \hline = & 9 & 7 & 1 \end{array}$$

Some space for you.

Challenge Question 9.5:

a) When adding two numbers the traditional way, why won't we ever "carry a 2"?

b) Give an example of adding a three-digit number and a one-digit number that results in carrying a 1 exactly two times.

c) Give an example of adding a five-digit number and a one-digit number that results in carrying a 1 just one time.

d) Give an example of adding a ten-digit number and a one-digit number that results in carrying a 1 ten times.

Write your answers here.

SELF-CHECK

Self-Check 9.1 If I had 7 tens and 8 tens, then I would have …

a) 15 tens
b) 1 hundred and 5 tens
c) Both of these. (They are the same thing!)

Self-Check 9.2 Which of the following are mathematically correct answers to the following addition problem?

$$\begin{array}{r} 539 \\ +\ 289 \\ \hline = \quad \end{array}$$

a) 7|11|18
b) 8|1|18
c) 8|2|8
d) These are all the same number, really, and all are mathematically correct. (It's just that society doesn't like a) and b).)

Self-Check 9.3 In which direction do we read English sentences?

a) Left to right
b) Right to left if I am standing on head but still holding the book upright.
c) Top to bottom if I am lying sideways on the couch and looking in the mirror at my friend who is standing on her head holding a book upside down between her feet.
d) I am too confused to try to make sense of anything except for answer a), which I know is the correct answer.

ANSWERS:

9.1 All are correct

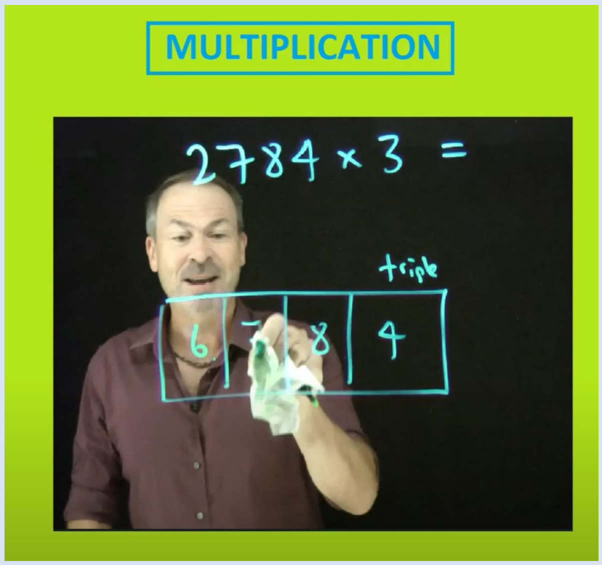
9.2 All are correct

9.3 I am not sure what to answer here. Definitely a). I think d) as well. Maybe b). I can't make sense of c).

10. Multiplication

Let's continue the quirky fun!

Here's the video for this section.



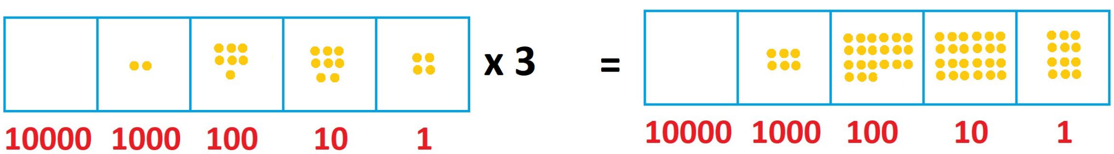
https://youtu.be/ZH0aSjQke2k

Without regard to what society thinks what would be a good—and correct—three-second answer to this multiplication problem? It's asking us to triple a number.

2784 x 3

Can you see that $6\,|\,21\,|\,24\,|\,12$ is the natural answer to this? After all, if we have 2 dots in the thousands place and 7 dots in the hundreds place and 8 dots in the tens place and 4 dots in the ones place, and we triple everything. then we'd have 6 thousands and 21 hundreds and 24 tens and 12 ones as a result. Easy!

(If you think of 2784×3 as $2784+2784+2784$, you can also see the answer $6\,|\,21\,|\,24\,|\,12$ appear.)



The answer is "six thousand, twenty-one hundred, twenty-four-ty, twelve."

2	7	8	4	x 3	=	6	21	24	12

Now, how can we fix up this answer for society? With some explosions of course!

Let's do two explosions from the 24 first, say. (It doesn't matter which explosions we do when). It gives

$$6\,|\,23\,|\,4\,|\,12$$

Now maybe explode from the 12.

$$6\,|\,23\,|\,5\,|\,2$$

Now from the 23.

$$8352$$

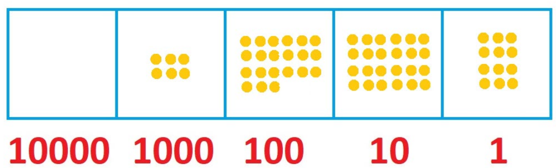
We have that $2784 \times 3 = 8352$.

Question 10.1 Was that too fast?
Did you follow me changing the answer 6|21|24|12 into 8|3|5|2?

Try this.

a) What answer does a calculator give for 2784×3? Is it 8352?

b) Turn 6|21|24|12 into 8|3|5|2 by doing explosions in a different order. Explode from the 12 at the end first. Then explode the explode from the 24 (twice). And then explode from the 21 (again twice). Do you get 8352?



Write your answers here.

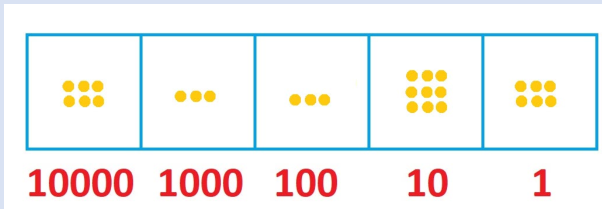
Question 10.2 Compute each of the following without regard to what society thinks. The fix up each of your answers to something that society would accept. Feel free to check your answers with a calculator.

a) 2784×2
b) 2784×4
c) 2784×5
d) 2784×10

Here's some space!

SELF-CHECK

Self-Check 10.1 Which multiplication problem could give this picture as its answer?



a) 21132×3
b) 77777×937
c) The answer is part a)
d) The answer really truly is part a)

Self-Check 10.2 What's 2736×7 ?

a) $14 \mid 49 \mid 21 \mid 35$
b) $14 \mid 49 \mid 21 \mid 42$
c) $14 \mid 49 \mid 21 \mid 49$
d) Something else

Self-Check 10.3 Society doesn't think $14 \mid 49 \mid 21 \mid 42$ is a number. Rewrite this as a number society would accept. (You should get 19152.)

a) Yes. I did this and got 19152.
b) I didn't get 19152 at first. But I kept trying and eventually got it.

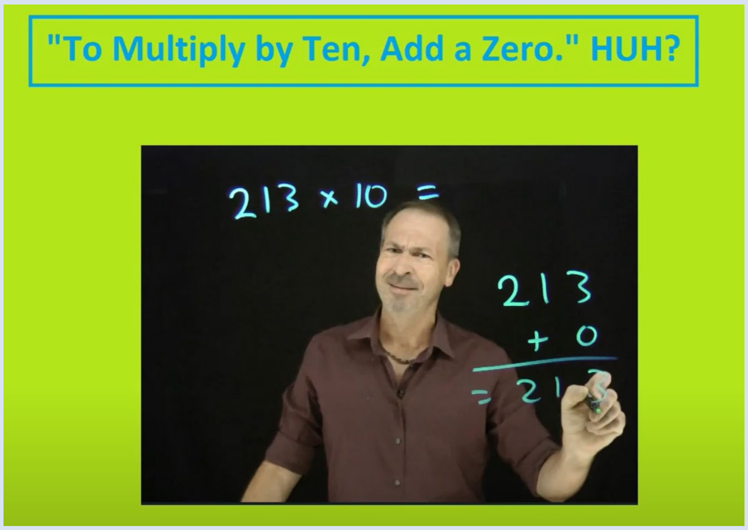
ANSWERS

10.1 a) c) and d) are all correct. **10.2** b) **10.3** Both a) and b) are great!

11. “To Multiply by Ten, Add a Zero.” Huh?

Here’s a true story this time.

And here’s the video for this section.



https://youtu.be/jGPKp9CqUvs

When I was in school, I was told a rule for multiplying by ten: just add a zero.

This rule made no sense to me as stated. To compute 213×10, for instance, you don’t add zero.

$$\begin{array}{r} 2\,1\,3 \\ +\quad\;\; 0 \\ \hline = \qquad\;\; \end{array}$$

Of course, I realized that people meant, "tack a zero to the end of the number."

213 x 10 = 2130

Why does multiplying a number by ten seem to have the effect of appending a zero to the digits of the number? Dots-and-boxes thinking explains why.

Here's the number 213 in a $1 \leftarrow 10$ machine.

	2	1	3

And here is 213×10.

	20	10	30

Now let's perform the explosions, one at a time. We'll need the extra box to the left.

	20	10	30
2	0	10	30
2	1	0	30
2	1	3	0

We have that 2 groups of ten explode in the hundreds place to give 2 dots one place to the left, and 1 group of ten explodes in the tens place to give 1 dot one place to the left, and 3 groups of ten explode in the ones place to give 3 dots one place to the left. The net effect of what we see is all digits of the original number shifting one place to the left to *reveal* zero dots in the ones place.

Indeed, it looks like we just tacked on a zero to the right end of 213. But really it is a whole lot of explosions that pushed each digit one place to the left to leave an empty box at the very end.

Question 11.1 Did you get what happened here?
How would you explain to a friend why 22×10 is 220?
Draw some dots-and-boxes pictures to show what is happening.

A tiny bit of space.

Question 11.2
a) What must be the answer to 476 × 10?
b) What must be the answer to 476 × 100? Why?

Write your answers here.

Question 11.3

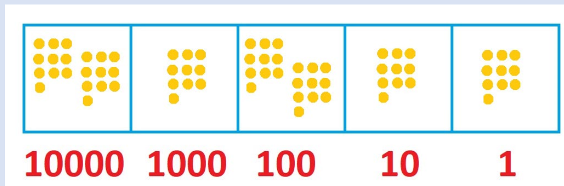
a) What was multiplied by ten to give the answer 3130?

b) What was multiplied by ten to give the answer 978000?

Write your answers here.

SELF-CHECK

Self-Check 11.1 If we do all the explosions in this picture, what final number do we get?



a) 21211
b) 212110
c) 2121100
d) Something else entirely

Self-Check 11.2 Multiplying a number by ten and then multiplying by ten again and then multiplying by ten yet again is the same as just multiplying the number by …

a) One hundred
b) One thousand
c) One gazillion billion million zillion. (That's not a real number.)

Self-Check 11.3 What number was multiplied by ten and then by ten again to give the answer 4500?

a) 45
b) 450
c) 4500

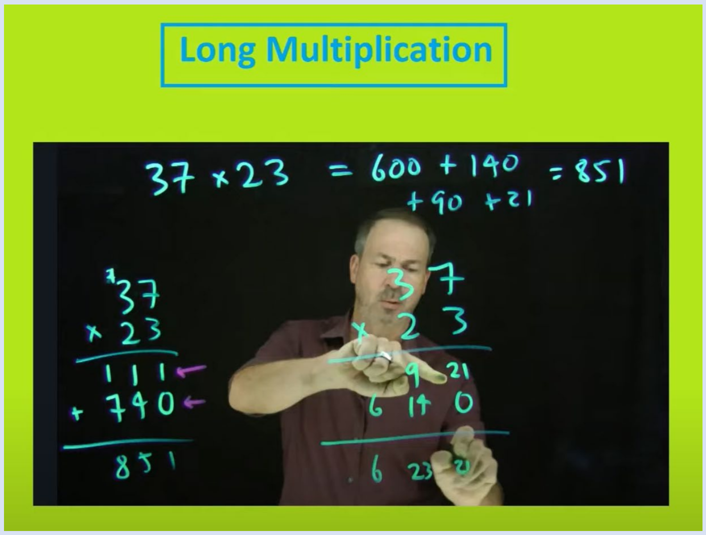
ANSWERS

11.1 b) **11.2** b) **11.3** a)

12. (OPTIONAL) Long Multiplication

Warning: This is a challenging section. Feel free to skip this one.

Here's video for this section.



https://youtu.be/pIR9yw166NE

People wonder if there is a way to conduct long multiplication with dots and boxes. That is, to multiply a number by something more than just a single-digit number.

We've just done an example of this already in multiplying numbers by 10. This is possible because we know our multiples of ten. (Also, the explosions that follow are straightforward.)

x 10

	3	**7**	**=**		**30**	**70**	**=**	**3**	**7**	**0**

But to compute 37×23, say, it looks like we need to know our multiples of 23. That's hard!

x 23

	3	7	=		69	161

After you do some explosions (lots of explosions actually), you'll see that 69|161 is the number 851. We have that $37 \times 23 = 851$.

Question 12.1: Do you want to do the explosions to see that $69 \,|\, 161$ really is the number 851?

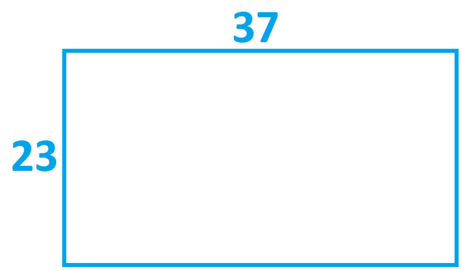
The answer can be NO.

Question 12.2 Fill in the question marks.

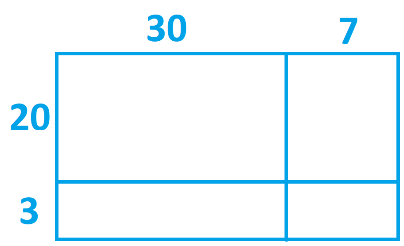
$$37 \times 2 = 6 \,|\, 14 = 74 \qquad 37 \times 11 = 33 \,|\, 77 = ? \qquad 375 \times 11 = 33 \,|\, ? \,|\, ? = ?$$

Write your answers here.

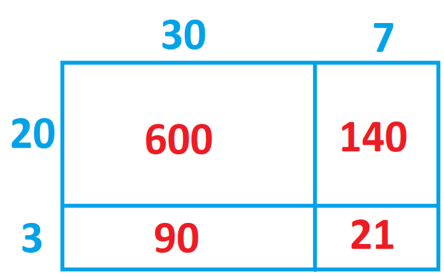
Actually, the most natural way to conduct a long multiplication problem like 37×23 is to think of areas. For instance, if I asked you to find the area of this rectangle, I'd really be asking you to compute 37×23.



But the nice thing about rectangles is that we can chop them up into friendlier pieces



and then work out the areas of the individual pieces.



I see now that $37 \times 23 = 600 + 140 + 90 + 21$. And this is indeed 851. Lovely!

Question 12.3: Use rectangles to compute 31×41 and to compute 134×12.

Draw your pictures and find your answers here.

Let's be very clear what we did in this picture of 37×23.

	30	7
20	600	140
3	90	21

We found a piece of the rectangle of area 600. This came from computing 30×20. This is essentially the same as computing $3 \times 2 = 6$, but our answer is in the hundreds because we are really multiplying 30, which is 3×10, and 20, which is 2×10. (We twice multiply by ten, hence we're seeing two 0 s in our answer.)

We found a piece of the rectangle of area 140. This came from computing 7×20. This is essentially the same as computing $7 \times 2 = 14$, but our answer is in the tens because we are really multiplying 7 and 20, which is 2×10. (We multiply by ten once, hence we're seeing one 0 in our answer.)

We found a piece of the rectangle of area 90. This came from computing 30×3. This is essentially the same as computing $3 \times 3 = 9$, but our answer is in the tens because we are really multiplying 30, which is 3×10, and 3. (We multiply by ten once, hence we're seeing one 0 in our answer.)

And we found a piece of area 21, which comes from computing 7×3.

$$\underline{3 \times 7} = 21$$
$$3 \times 30 = \underline{3 \times 3} \times 10 = 90$$
$$20 \times 7 = \underline{2 \times 7} \times 10 = 140$$
$$20 \times 30 = \underline{2 \times 3} \times 10 \times 10 = 600$$

Phew!

I wasn't taught any of this when I was in school. Instead, I was taught a strange "long multiplication algorithm," which I can't really remember, only half-remember.

But look at this. What do you think of my rectangle/dots-and-boxes/half-remembered long-multiplication mash up here?

$$
\begin{array}{c|c|c|c|}
 & & 3 & 7 \\
\times & & 2 & 3 \\
\hline
 & & 9 & 21 \\
+ & 6 & 14 & \\
\hline
= & 6 & 23 & 21
\end{array}
$$

Do you see the pieces "21" and "90" and "140" and "600"? (This really is doing long multiplication. It's just my chopped up rectangle without the picture of the actual rectangle.)

Do you understand why the answers to 3×7 and 3×3 and 2×7 and 2×3 are written in the positions they are?

Do you see that if we add up the four pieces in this table we get 6 hundreds, and 23 tens, and 21 ones? (That's $600+90+140+21$, so it is the total area of the rectangle.)

Do you see that, after explosions, $6\,|\,23\,|\,21$ is the number 851, the final answer we are meant to get?

Question 12.4: There's a lot to "see" here. Is this making any sense at all?

Some space for some reactions, questions, and/or comments.

Question 12.5: Here's how I computed 23×24 to get the answer 552.

$$\begin{array}{rrrr} & & 2 & 3 \\ \text{x} & & 2 & 4 \\ \hline & & 8 & 12 \\ & 4 & 6 & \\ \hline = & 4 & 14 & 12 \end{array} \qquad = 5|4|12 = 5|5|2$$

a) Draw a rectangle chopped up into four pieces that match what you see in this multiplication problem.
b) Why is the "8" you see really eighty?
c) Why is the "4" you see really four-hundred?
d) Do you "get" what's going on in my work? (Is 23×24 really equal to 552?)

Lots of space.

Question 12.6: My calculator says that 3125×832 is 2600000. Can you get that answer by using my crazy mash-up method?

$$\begin{array}{r} 3\ 1\ 2\ 5 \\ \times \quad 8\ 3\ 2 \\ \hline = \qquad\qquad \end{array}$$

Lots of space.

SELF-CHECK

Self-Check 12.1 This section was challenging and …

a) I found it really hard, and I didn’t really get it.
b) I found it really hard, but I was kinda getting it.
c) Hard? What do you mean?
d) Hard? I wouldn’t know. This section was marked “optional,” so I didn’t read it. (I am not even reading this question.)

ANSWER

12.1 All answers here are valid.

13. NEW NUMBERS: The Opposites of the Counting Numbers

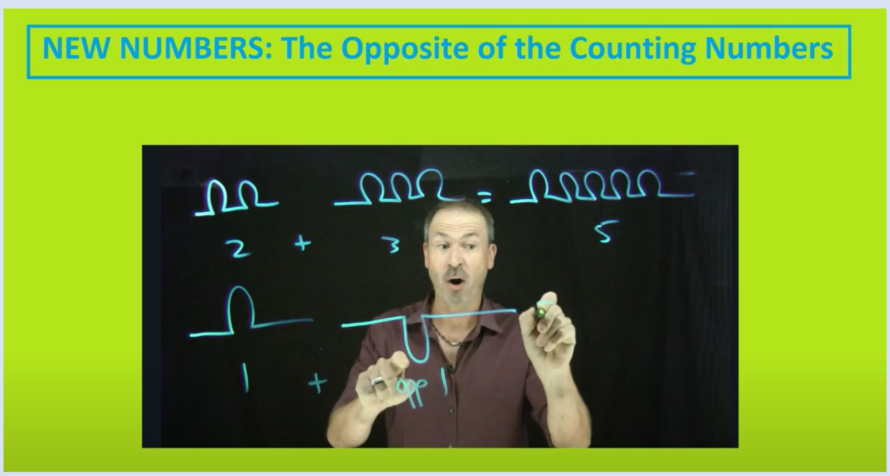
We just did addition and multiplication but skipped over subtraction.

Why?

Because I am someone strange who doesn't believe that subtraction actually exists. To me, subtraction is just addition again: it is the addition of the opposite.

To explain this, I need to tell yet another story that is not true.

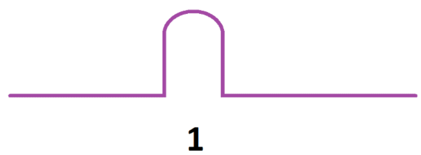
Here's video for this section.



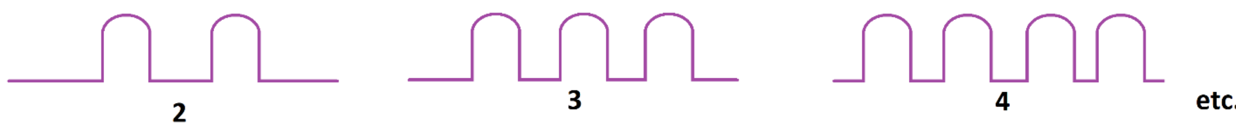
https://youtu.be/0KjCRKCedUs

When I was a young child, I spent my days sitting in a sandbox at the back of my yard. (Not true.) And being a very quiet and contemplative child (not true), I would first take my time in the mornings leveling out the sand in my box to make a perfectly flat horizontal surface. This appealed to my tranquil sensibilities, so much so that I decided to give this level state a name. I called it **zero**.

And I spent many an hour admiring my zero state. (Still not true.) But then, one day, I had a flash of insight. I realized I could reach behind where I was sitting, grab a handful of sand, and make a pile. I called the one pile **1**.



And then I discovered two piles – which I called **2** – and three piles, **3** and so forth.



I had hours of mathematical fun as I created and admired more and more piles of sand.
I had discovered the **counting numbers**.

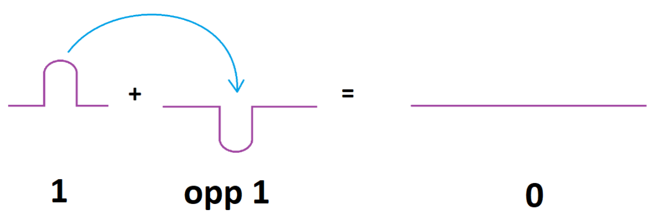
I also discovered addition for the counting numbers simply by lining up piles. For example, I saw that two plus three equals five piles.

Question 13.1: Draw a picture of $3+4+5=12$.

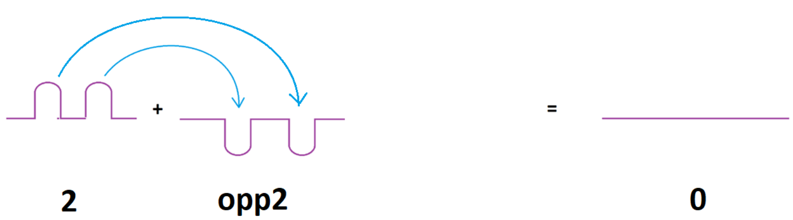
Your answer.

But then, one day I had the most astounding flash of insight of all! Instead of using a handful of sand to make a pile, I realized I take away a handful of sand and make the OPPOSITE of a pile, namely, a hole!

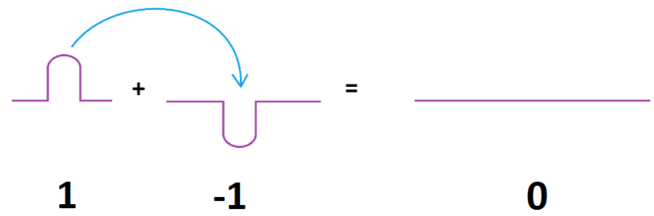
Place a pile and a hole together and they cancel each other out and make the zero state**.** Whoa! For this reason, I called a hole **opp 1**, for the opposite of 1. And I wrote **opp 2** for the two holes, **opp 3** for three holes, and so on.



And I wrote **opp 2** for the two holes, **opp 3** for three holes, and so on.



Later in school I was taught to write "-1" for a hole and say, **negative one**.

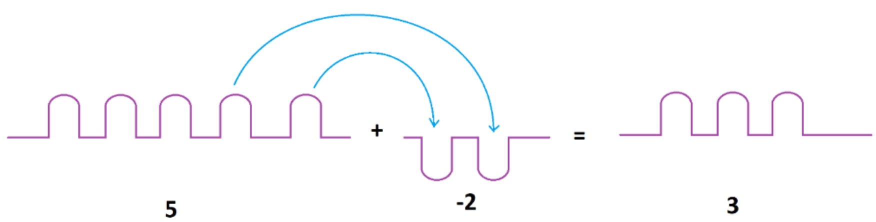


I was told to write "-2" for two holes—**negative two**—and so on. And I was also told to do something called subtraction. But I never really believed in subtraction.

My colleagues would read

$$5-2$$

for example, as **five take away two**. But I was thinking of five piles and the addition of two holes. A picture shows that the answer is three piles.



This, of course, gave the same answer as everyone else: the two holes "took away" two of the piles. But I had an advantage.

For example, my colleagues would say that

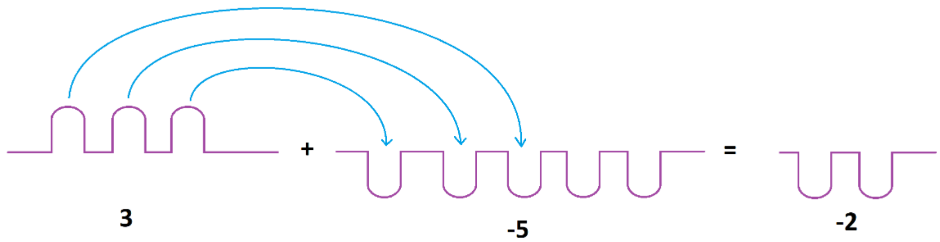
$$3-5$$

has no answer: "You can't take away five things if you only have three to begin with."

But I saw that $3-5$ does have an answer.

$3 - 5$ = three piles and five holes
= two holes
$= -2$

Easy!



Subtraction is just the addition of the opposite!

Question 13.2: Many people would say that $2-6$ has no answer.

But we can think of this as $2 \ + \ -6$.

How many piles are there? How many holes are there? And when we combine them what are we left with?

Your answer.

I came to read a statement like

$$-3+7=4$$

as "three holes and seven piles equals four piles" and I could see it was true by imagining a diagram.

Question 13.3: Draw a piles and holes picture for $-3+7=4$.

Your answer.

Even though my friends would write $5-2$, for instance, for "five take away two," I really would rewrite it as

$$5+-2$$

and think "five piles and two holes".

If a friend wrote $6-9+2-1$, I would rewrite it as

$$6+-9+2+-1$$

and see a lot of piles and holes in my mind.

Question 13.4:

a) A friend wrote $6-7-1$. How do you think I would rewrite it?

b) If I wrote $5+-1+3+-2+-1=4$, what do you think my friends wrote?

c) Can you imagine a picture for each of these two sums? What final value do they each have?

Your answer.

Question 13.5 Some people object to writing something like $6+-9+2+-1$. I admit, it can look confusing. I wish we didn't use a little dash to denote "opposite." Perhaps we can encourage society to start writing

$$6 + opp\,9 + 2 + opp\,1 \text{ ?}$$

Would that be better?

What do you think?

SELF-CHECK

Self-Check 13.1 A picture of $1-1+1-1+1-1+1-1+1$ would have …

a) five piles and four holes, which would combine to leave one pile (1).
b) four piles and five holes, which would combine to leave one hole (-1).
c) five piles and four holes, which would combine to leave one hole (-1).
d) four piles and five holes, which would combine to leave one pile (1).
e) These answers are all very similar and my brain can't keep track of which is saying what!

Self-Check 13.2 The subtraction problem $87-33$ is really the addition problem …

a) $87 \quad + \quad -33$ (That's 87 piles and 33 holes.)

Self-Check 13.3 One-hundred piles and one-hundred-and-one holes would combine to leave …

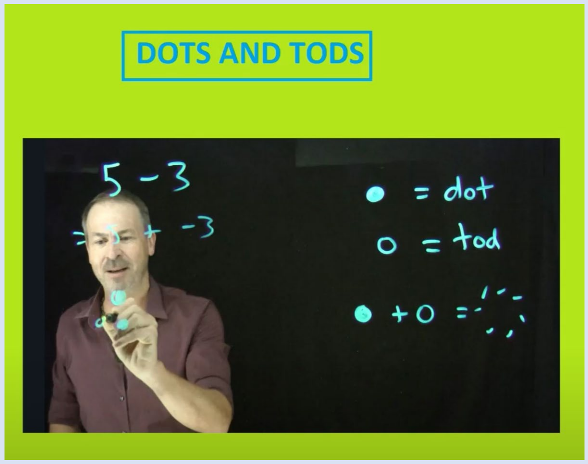
a) One pile
b) One hole
c) A mess of sand

ANSWERS

13.1 The answer is a), but feel free to answer e).
13.2 Umm … the answer is a)
13.3 You'll have b) and probably c) as well.

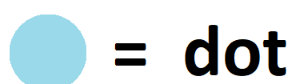
14. DOTS and TODS

Here’s video for this section.



https://youtu.be/ssXFM6kERRw

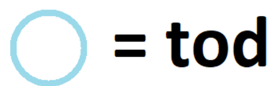
Piles and holes are fine and good. But we began this book with a dot.



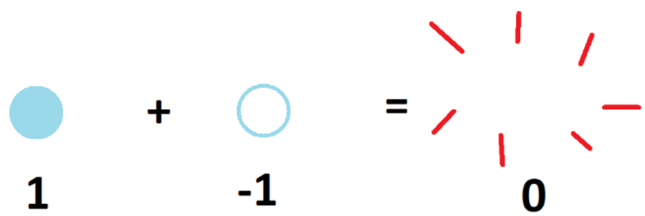
A dot perhaps looks like a pile viewed from above. But if we are going to talk about subtraction in our dots-and-boxes story, we now need the notion of the opposite of a dot. (A hole is the opposite of a pile. What should be the opposite of a dot?)

True Story: When I ask this question to students in countries all over the world, kids always tell me the same one answer.

"Draw an open circle for the opposite of a dot, and call it a **tod**. That's the word *dot*, backwards."

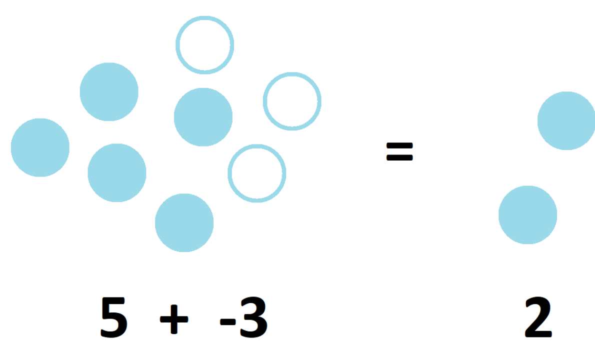


Like a pile and a hole, which each annihilate one another when brought together, a dot and a tod annihilate – POOF! – when brought together, to leave nothing behind.

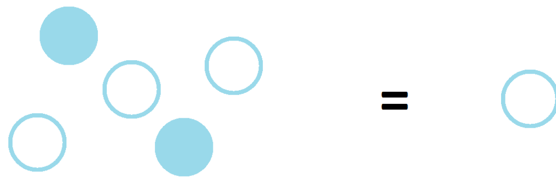


We can conduct arithmetic with dots and tods, just like we did with piles and holes.

For example, for $5+-3$, which is five dots and three tods, there are three annihilations – POOF! POOF! POOF! – to leave behind two dots.

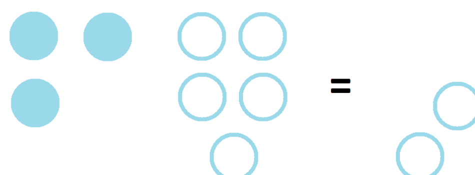


For $2+-3$, which is two dots and three tods, there are two annihilations—POOF! POOF!—to leave one tod.



2 + -3 **-1**

Question 14.1 Here's a picture of $3-5$, giving the answer -2.



3 + -5 = -2

Does this picture make sense to you?

Does it?

Question 14.2 Draw, or just imagine drawing, a dots and tods picture of $1-5$ and find the value of this quantity.

Your answer.

Question 14.3 Imagine drawing a dots and tods picture of $2000 - 1000 + 65$. Can you "see" that the answer will be 1065?

Can you?

SELF-CHECK

Self-Check 14.1 A picture of $1-1+1-1+1-1+1-1+1$ would have …

a) five dots and four tods, which would combine to leave one dot (1).
b) four dots and five tods, which would combine to leave one tod (-1).
c) five dots and four tods, which would combine to leave one tod (-1).
d) four dots and five tods, which would combine to leave one dot (1).
e) I feel like I've done this question before.

Self-Check 14.2 The subtraction problem $87 - 33$ is really the addition problem …

a) $87 \quad + \quad -33$ (That's 87 dots and 33 tods.)
b) I really do feel like I did this question before.

Self-Check 14.3 One-hundred dots and one-hundred-and-one tods would combine to leave …

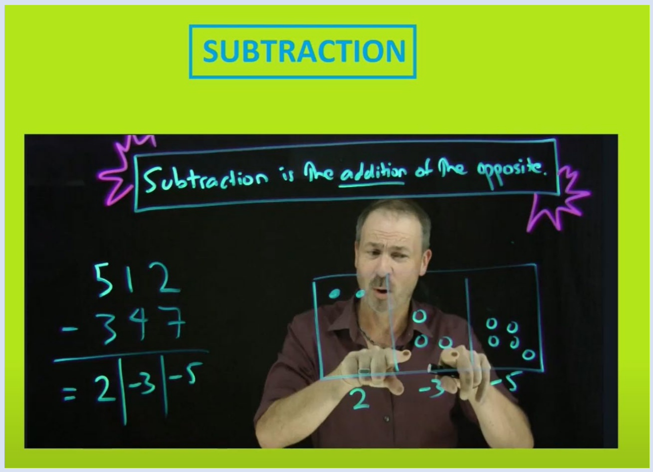
a) One dot.
b) One tod.
c) This is crazy. I've really done all these three questions before.

ANSWERS:

14.1 a) and e)
14.2 a) and b)
14.3 b) and c)
We really have done all these questions before!

15. Subtraction

Here’s video for this section.



https://youtu.be/-j3BJW34BI8

Remember, I do not believe that subtraction exists.

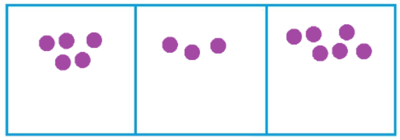
Subtraction is the addition of the opposite.

Let’s play with multidigit subtraction. Consider this subtraction problem.

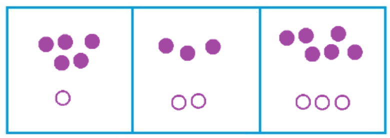
$$\begin{array}{r} 536 \\ -\ \ 123 \\ \hline = \end{array}$$

To me, this is 536 plus the opposite of 123.

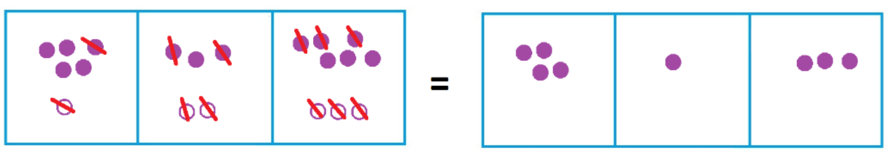
The first number, 536, looks like this in a 1 ← 10 machine: five dots, three dots, six dots.



To this we are adding the opposite of 123. That is, we're adding one anti-hundred (one tod to the hundreds box), two anti-tens (two tods to the tens box), and three anti-ones (three tods to the ones box.



And now there are a lot of annihilations: POOF! and POOF POOF! and POOF POOF POOF!



The answer 413 appears.

And notice, we get this answer as though we just work left to right and saying

5 take away 1 is 4,

3 take away 2 is 1,

and

6 take away 3 is 3.

$$\begin{array}{r} 536 \\ -\ 123 \\ \hline =413 \end{array}$$

Yes. Left to right again!

All right. That example was too nice. How about 512 – 347?

$$\begin{array}{r} 512 \\ -\ 347 \\ \hline = \end{array}$$

Going from left to right, we have

5 take away 3 is 2,
1 take away 4 is -3,

and

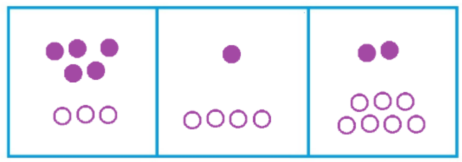
2 take away 7 is -5.

$$\begin{array}{rccc} & 5 & 1 & 2 \\ - & 3 & 4 & 7 \\ \hline = & 2 & |-3 & |-5 \end{array}$$

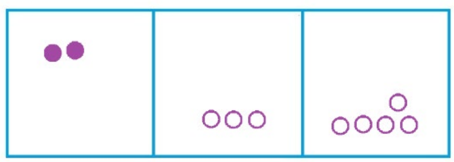
The answer is two-hundred negative-three-ty negative-five.

And this answer is absolutely, mathematically correct!

Here's five hundreds, one ten and two ones (the number 512) in a $1 \leftarrow 10$ machine, together with three anti-hundreds, four anti-tens, and seven anti-ones (the opposite of 347).



And after lots of annihilations we are left with two actual hundreds, three anti-tens, and five anti-ones.



2 **-3** **-5**

The answer really is "two-hundred negative-three-ty negative-five"!

But, of course, this answer is too strange and weird for society to accept. Can we somehow fix up this mathematically correct answer to one that society would accept?

> **Question 15.1** Think about this for a moment. What can we possibly do to fix this answer?

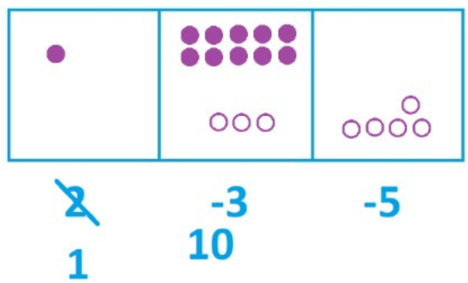
Any ideas?

After a moment it might occur to you to unexplode dots. Any dot in a box to the left must have come from ten dots in the box just to its right, so we can just unexplode it to make ten dots.

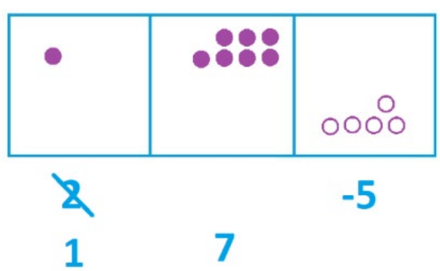
Important Question 15.2: What sound effect should we make for unexploding?

Your Answer.

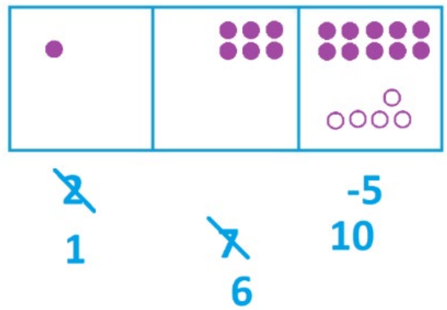
Okay. Let's unexplode one of the two dots we have in the leftmost box. Doing so gives this picture.



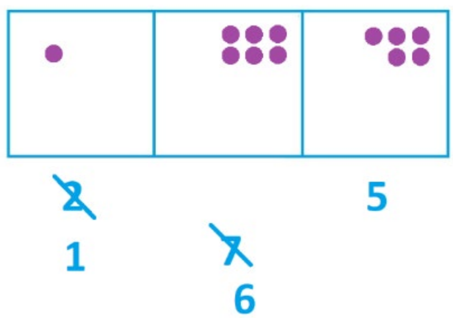
After annihilations, we see we now have the answer one-hundred seventy negative-five. Beautiful!



Let's unexplode again. Let's unexplode a dot from the middle box.



And with some more annihilations we see an answer society can understand: one-hundred sixty-five.



If you want to play with dots and tods in a pre-built $1 \leftarrow 10$ machine go here:

https://www.explodingdots.org/station/OpenMachinesAntidotia

Question 15.3: Can you show that “four-thousand, negative-two-hundred, onety, negative seven” is really 3803?

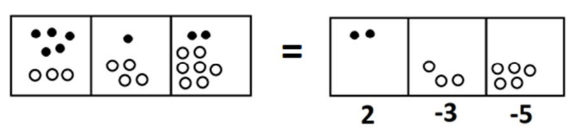
Show me!

Question 15.4: My calculator says that $512 - 347 = 165$.

Working left to right shows me that $512 - 347 = 2\,|\,{-3}\,|\,{-5}$.
Dots-and-boxes show me this too.

Are the two different answers 165 and 2|-3|-5 actually the same?

$$\begin{array}{r} 5\;\;1\;\;2 \\ -\;3\;\;4\;\;7 \\ \hline =2|{-3}|{-5} \end{array}$$



Your thoughts here.

Question 15.5 When Raj saw

$$\begin{array}{rccc} & 5 & 1 & 2 \\ - & 3 & 4 & 7 \\ \hline = & 2 & |-3| & -5 \end{array}$$

he wrote on his paper the following lines.

$$\begin{array}{r} 200 \\ -30 \\ -5 \end{array}$$

He then said that the answer has to be 165.

Can you explain what he is seeing and thinking?

Your thoughts here.

Question 15.6: Compute the following two problems in each of the following ways.

a) Just use a calculator and see what the answer is.
b) Use a method you were taught in school for doing long subtraction.
c) Do each problem left-to-right and fix up the answers to ones that society would accept.

Hopefully the answers for each problem turn out to be the same!

$$\begin{array}{r} 6328 \\ -\ 4469 \\ \hline = \end{array} \qquad\qquad \begin{array}{r} 78390231 \\ -\ 32495846 \\ \hline = \end{array}$$

A question along the way: As you fix up your answers for society, does it seem easier to unexplode from left to right, or from right to left?

Your answers here.

The Traditional Algorithm

I was taught a method in school for doing long subtraction. I want to see now how the dots-and-boxes approach compares to it.

Let's consider again $512-347$.

$$\begin{array}{rccc} & 5 & 1 & 2 \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

The method I was taught has you start at the right to first look at "2 take away 7."
But you can't do $2-7$. (Well, you can: it's -5. But you are not to allowed to write this in this method!)

So, what do you do?

You "borrow one." That is, you take a dot from the tens column and unexplode it to make ten ones. That leaves zero dots in the tens column. We should write ten ones to go with the two in the ones column.

$$\begin{array}{rccc} & & 0 & {\scriptstyle 10} \\ & 5 & \cancel{1} & 2 \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

But we are a bit clever here and just write 12 rather than $10+2$. (That is, we put a 1 in front of the 2 to make it look like twelve.)

$$\begin{array}{cccc} & & 0 & \\ & 5 & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline = & & & \end{array}$$

Then we say, “twelve take seven is five,” and write that answer.

$$\begin{array}{cccc} & & 0 & \\ & 5 & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline = & & & 5 \end{array}$$

The rightmost column is complete. Shift now to the middle column.

We see “zero take away four,” which can’t be done. So, perform another unexplosion, that is, another “borrow,” to see $10-4$ in that column. We write the answer 6.

We then move to the last remaining column where we have $4-3$, which is 1.

$$\begin{array}{cccc} & 4 & {}^{1}0 & \\ & \cancel{5} & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline = & & 6 & 5 \end{array} \qquad\qquad \begin{array}{cccc} & 4 & {}^{1}0 & \\ & \cancel{5} & \cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline = & 1 & 6 & 5 \end{array}$$

That’s complicated!

The truth is that all correct approaches to mathematics are correct. It is just a matter of style as to which approach you like best for subtraction.

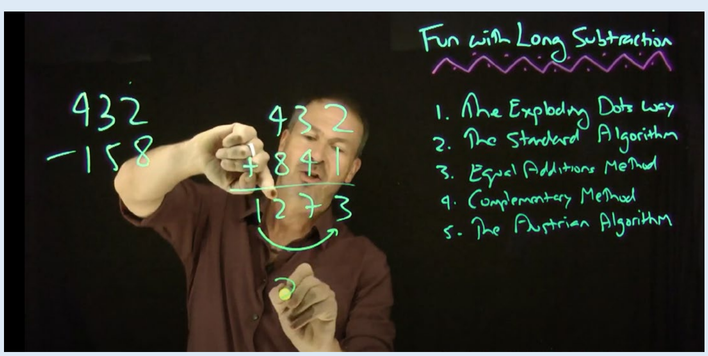
The standard method has you work from right to left and do all the unexplosions as you go along. The dots-and-boxes approach has you “just do it!” and conduct all the unexplosions at the end.

Both methods are fine and correct.

$$\begin{array}{rrrr} & {}^{4}\cancel{5} & {}^{10}\cancel{1} & {}^{1}2 \\ - & 3 & 4 & 7 \\ \hline = & 1 & 6 & 5 \end{array} \qquad\qquad \begin{array}{rrrr} & 5 & 1 & 2 \\ - & 3 & 4 & 7 \\ \hline = & 2| & -3| & -5 \end{array} = 1|7|-5 = 1|6|5$$

OTHER METHODS OF LONG SUBTRACTION

We might think our school method for multi-digit subtraction is “standard,” but at other times and in other cultures alternative approaches were considered standard.
If you are feeling brave, can you make sense of the “Equal Additions Method,” the “Complementary Method,” and the “Austrian Method” I describe in this video?



https://youtu.be/KJb5nfGvXp0

Optional Question 15.7: How might you handle and interpret this subtraction problem?

$$\begin{array}{r} 148 \\ -\ 677 \\ \hline = \quad \end{array}$$

Your thoughts here.

SELF-CHECK

Self-Check 15.1 Which of the following is a subtraction problem with final answer $3 \mid 6 \mid -2$?

a) $784 - 422$
b) $784 - 424$
c) $784 - 426$
d) $2395 - 2037$. (Hang on! There is more than one answer to this problem!)

Self-Check 15.2 The number $3 \mid 6 \mid -2$ is really …

a) Nonsensical and has absolutely no mathematical meaning.
b) It's 7. (It's not, but can you guess why someone might think this?)
c) One unexplosion shows it's 358.
d) It's 300 and 60 and -2, which makes 358.

Self-Check 15.3 Every problem in math must be solved in just one way. Even if you get the right answer and all your thinking is logical and correct, it doesn't matter: there only one way to solve any math problem.

a) This is absolute nonsense for sure!

ANSWERS:

15.1 Both c) and d) are correct.

15.2 Both c) and d) are correct.

15.3 a) is correct.

16. Division

Archeologists often find artifacts from an ancient past and wonder how they were made. They don't see the process that produced the object, just the end of result of the process.

In much the same way, some people like to think of division as the reverse of multiplication. For instance, from

$$3 \times 7 = 21$$

we could focus on the answer 21 and ask about the process:

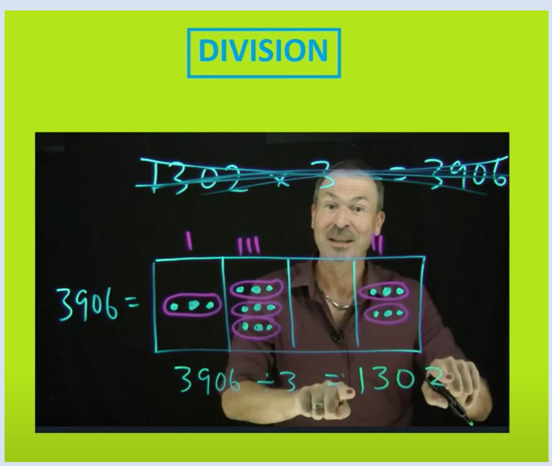
What times 7 made 21?

This would now be considered a division problem and we'd write $21 \div 7 = 3$ after recognizing multiplying 7 seven by 3 is what makes 21.

Let's revisit multiplication for a moment to then see if we can follow it backwards to get to division.

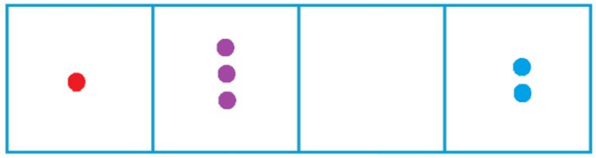
We'll start with a straightforward multiplication problem, say, 1302×3 (with answer 3906).

Here's video for this section.

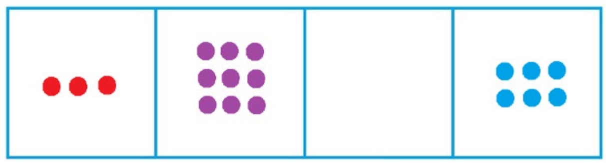


https://youtu.be/fZW9QMwg9u0

Here's what 1302 looks like in a $1 \leftarrow 10$ machine. (I've colored the dots for fun.)

1302 = 

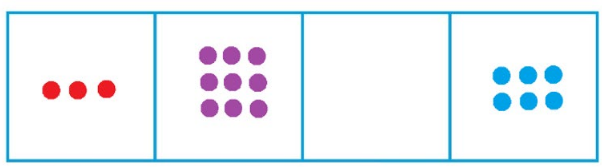
To triple this quantity, we just need to replace each dot in the picture with three dots. We see the answer 3906.

1302 x 3 = 

This is all well and good.

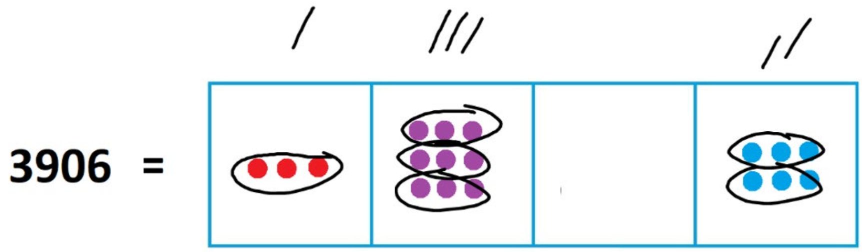
Now let's go backwards and start with the answer to ask:

> Here's a picture of 3906. What times three gives this picture?

3906 = 

We already know the answer is 1302. But what if we didn't know that? What could we do to "undo" this answer?

Well, you we can look for triples of dots that must have come from single dots. And we see plenty of those.



Actually, we see the picture of 3906 is the result of tripling one dot at the thousands level, tripling three dots at the hundreds level, and tripling two dots at the ones level. That is, we see 3906 as the number 1302 tripled.

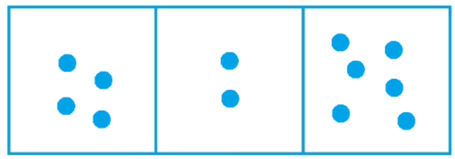
We have just deduced, from the picture, that $3906 \div 3 = 1302$!

So, to divide a number by three, all we need to do is to look for groups of three in the picture of the number. Each group of three corresponds to a dot that must have been tripled. We can just read off the answer to the division problem then by looking at the groups we find!

And we can do the same for any single-digit division problem.

Question 16.1: Here's a picture of 426.

Can you see in the picture that $426 \div 2$ must equal 213? (What was doubled to give this picture?)

426 = 

Your thoughts here.

Question 16.2: Draw a dots-and-boxes picture of 848.
Use you picture to show that $848 \div 4$ must be 212.

How would you explain what is happening to a curious friend?

Your thoughts here.

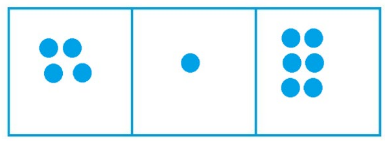
There could be a hiccup in our approach.

Let’s try computing

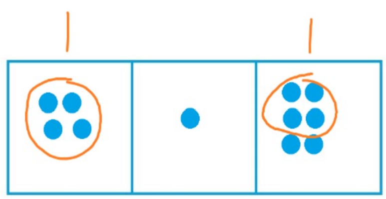
$$416 \div 4 .$$

(The answer is going to be 104.)

We are looking for groups of four—dots that got quadrupled—in a picture of 416.

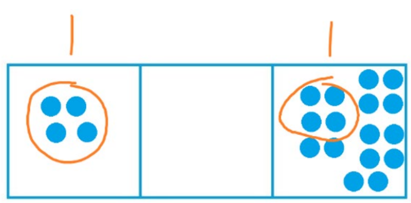
416 = 

We see one group of four at the hundreds level, and one at the ones level. But no more. Hmm. We’re stuck!

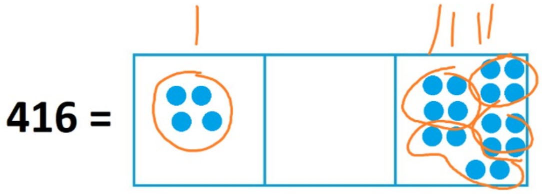
416 = 

But an unexplosion might help!

Let’s unexplode that single dot in the middle box and make it ten dots in the box to its right.

416 = 

This allows to see more groups four.



All dots are now accounted for.

We have one group of four at the hundreds level and four at the ones level. We see that $416 \div 4 = 104$. (That is, 104 was quadrupled to give 416.)

Question 16.3: Try finding $402 \div 3$ with just a dots-and-boxes picture. How can you get to the answer 134?

Some space.

Question 16.4 Use a dots and boxes picture to show that $102 \div 2$ equals 51.

Some more space.

Question 16.5: Use a dots-and-boxes picture to show that $100 \div 4$ equals 25.

Some working space.

Question 16.6 DO THIS ONLY IF YOU DARE

Compute $1000 \div 8$ with a dots-and-boxes picture.
(It might be best, at some point, to just write numbers instead of drawing dots. There are going to be a ridiculous number of dots to draw!)

Some space.

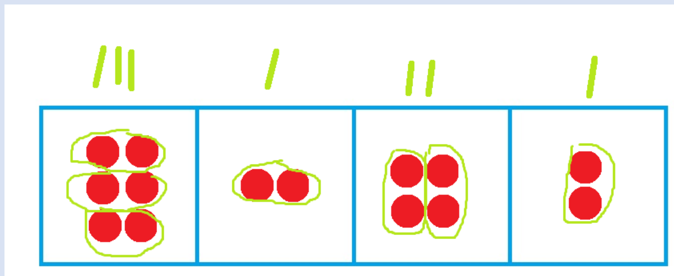
Question 16.7: We saw that $416 \div 4 = 104$.

What's $417 \div 4$? What does a dots-and-boxes picture show?

Some space.

SELF-CHECK

Self-Check 16.1 Genelle did a division problem the dots-and-boxes way and drew this picture. But she forgot what the problem was.



What problem was she solving and what answer did she get for it?

a) She computed $6242 \div 2$ and got the answer 3121.
b) She computed $3121 \div 2$ and got the answer 6242.
c) She computed $2 \div 6242$ and got the answer 3121.
d) She computed $2 \div 3121$ and got the answer 6242.

Self-Check 16.2 Is it possible to compute $10 \div 5$ the dots-and-boxes way to get the answer 2?

a) Yes, it is, and I did it and in my picture I had to do an uexplosion.
b) Yes, it is, but I didn't do it because I didn't feel like doing it.
c) Yes, it is, and I didn't actually do it because I could see in my mind how the picture with an unexplosion would work.

Self-Check 16.3 Is computing $452 \div 1$ the dots-and-boxes way weird?

a) Yes, because anything divided by 1 is just itself. The answer has to be 452. No work!
b) Yes. But you can do it the dots-and-boxes way by circling groups of 1 in the picture. The answer 452 pops out.
c) Same answer as b), except it's cool, not weird, how the dots-and-boxes method works even in an example like this.

ANSWERS:

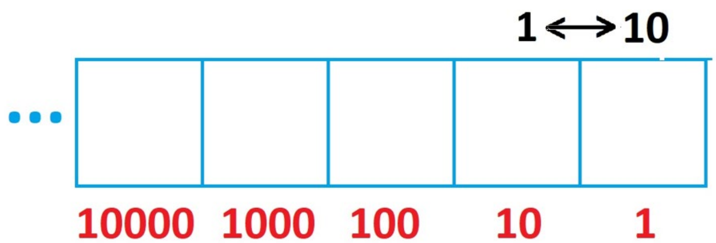
16.1 a)

16.2 a) and c) are the preferred answers. But maybe b) was your answer.

16.3 All three answers are good!

IDLE QUESTION:

We've been playing with a $1 \leftarrow 10$ machine, exploding ten dots to make one, and unexploding one dot to make ten. So, might it be better to use a double arrow notation for the machine? Denote it as a $1 \leftrightarrow 10$ machine?

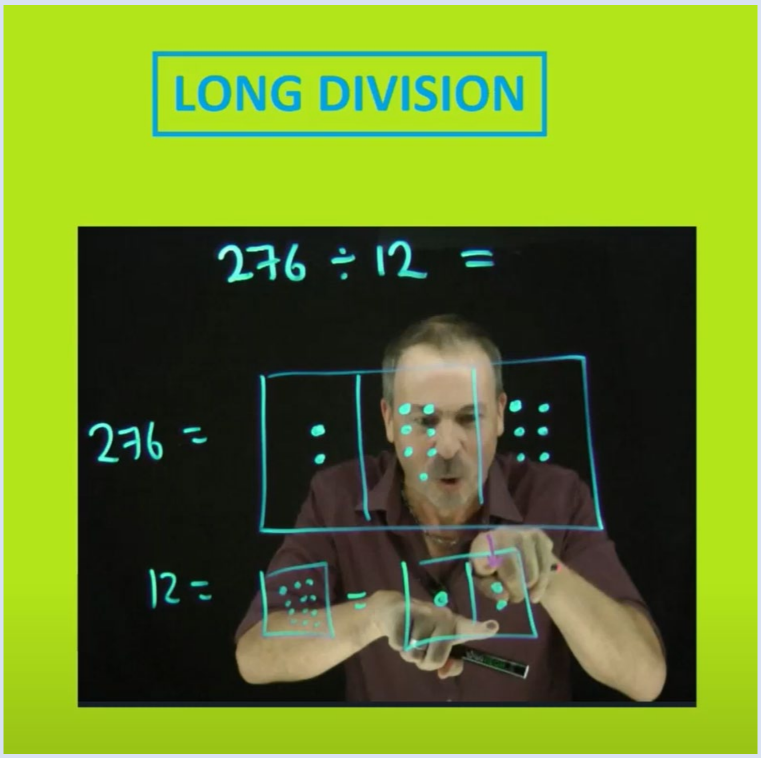


What do you think?

17. Long Division

Division by single-digit numbers is all well and good. What about division by multi-digit numbers? People usually call that **long division**.

Here's video for this section.



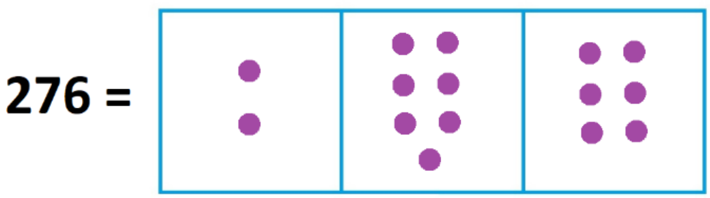
https://youtu.be/4MTmaB4KDBw

Let's consider the problem

$$276 \div 12.$$

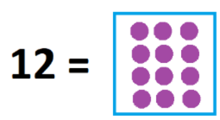
(My calculator says the answer is 23.)

Here is a picture of 276 in a $1 \leftarrow 10$ machine.

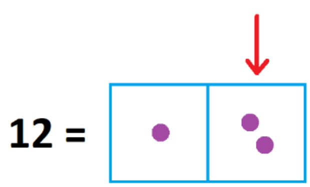


And we are looking for groups of twelve in this picture of 276. That is, we are looking for what got multiplied by twelve to give this picture of 276.

Here's what twelve looks like.



Actually, this is not quite right as there would be an explosion in our $1 \leftarrow 10$ machine. Twelve will look like one dot next to two dots.

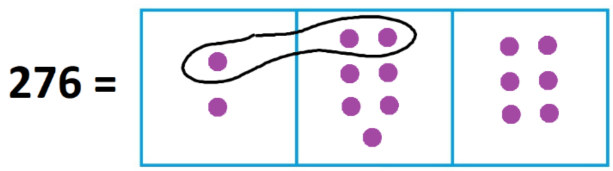


This is going to be confusing. We don't actually see twelve dots, but have to remember there really were twelve dots sitting in the rightmost box. An explosion caused some "spillage."

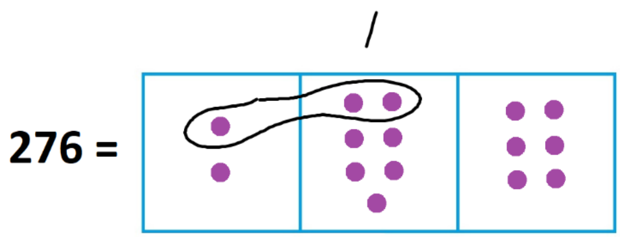
Okay. We are looking for groups of 12 in our picture of 276.

Do we see any "one-dot-next-to-two-dots" in the diagram?

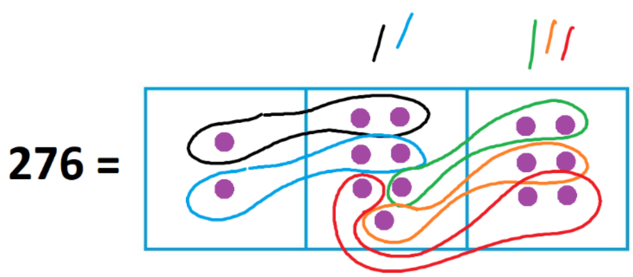
Yes. Here's one.



Within each loop of 12 we find, the twelve dots actually reside in the right part of the loop. (Ten dots must have exploded to spill one dot over to the left.) We have found one group of 12 at the tens level.



And there are more groups of twelve: another one right underneath, and three more one place over. (Remember, the twelve dots in each loop are sitting in the right part of the loop.)



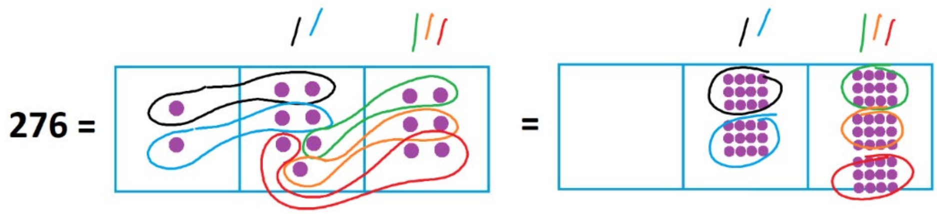
This shows that our picture of 276 is actually a picture of 23 each of whose dots was replaced by 12. We have that 23 was multiplied by 12 to give the answer 276.

$$276 \div 12 = 23 .$$

Question 17.1: Aparna didn't quite understand why the previous picture shows that $276 \div 12$ is 23. She is confused by the statement:

> "Our picture of 276 is actually a picture of 23 each of whose dots was replaced by 12. We have that 23 was multiplied by 12 to give the answer 276."

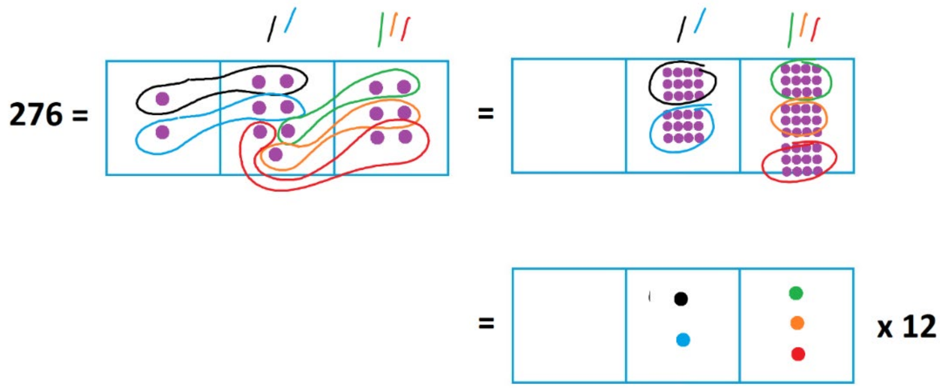
Jad decided to redraw the final figure to help her out. This is what he drew.



a) Do you see what Jad did? Do you think this addition to the picture will help Aparna?

What do you think?

Jad then went further and added another part to the picture.

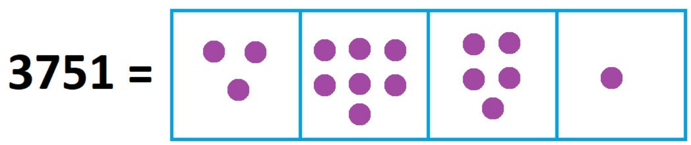


b) Do you think this helps even more or is it just confusing? What do you think he was trying to convey with his final picture?

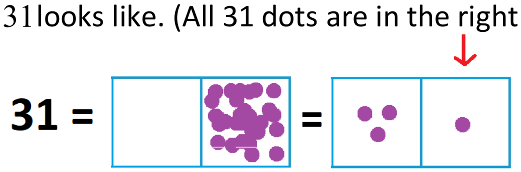
Your thoughts.

Let's compute $3751 \div 31$ this dots-and-boxes way.

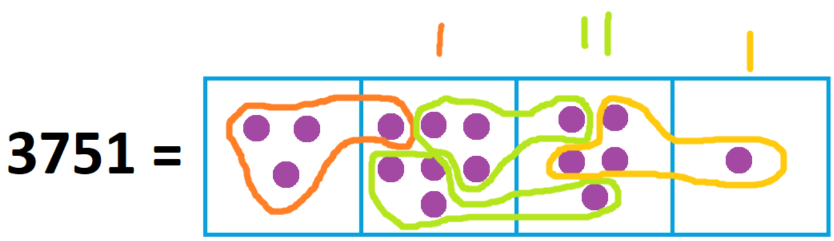
Here's a picture of 3751.



And here's what a group of 31 looks like. (All 31 dots are in the rightmost book.)



And here are all the 31s I can find in the picture of 3751.



I see one at the hundreds level, two at the tens level, and one at the ones level. It must be that 121 multiplies by 31 to give the answer 3751.

$$3751 \div 31 = 121$$

Question 17.2 Is this making any sense to you?
If the answer is no, can you really pin down which part(s) of this page is confusing and try to devise a question you could ask that might help you if answered?

Your response.

Question17.3: See if you can compute

$$2783 \div 23$$

using dots and boxes. Do you get the answer 121?

Your work.

Question 17.4:

a) Compute $4473 \div 21$ with dots and boxes to get the answer 213.

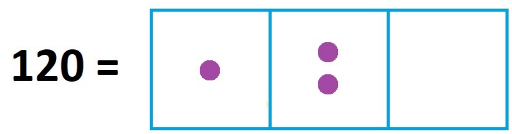
b) Now compute $4473 \div 213$. Do you see the answer 21?

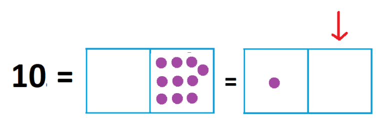
Your work.

Question 17.5: SOMETHING TRICKY

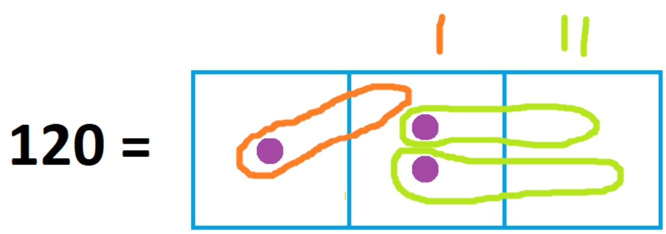
Ricky is wondering about $120 \div 10$. He knows the answer is going to be 12, but he is wondering how the dots-and-boxes approach is going to show this.

He starts by drawing this.





He says he is looking for "one dot next to no dots" and then hunts for them. He thinks he's finding them.



Is he finding them? Is he getting the answer 12 to $120 \div 10$?
Is everything he is doing good, and fabulous, and correct?

What do you think?

Your thoughts.

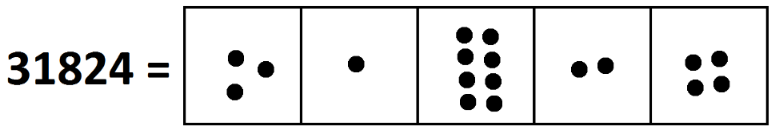
Question 17.6 Care to try this one on your own before I do it on the next page?

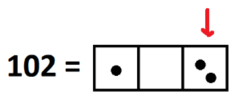
$$31824 \div 102$$

(There's a zero in that "102.")

Your work.

Okay. Let's compute $31824 \div 102$. Here's the picture.

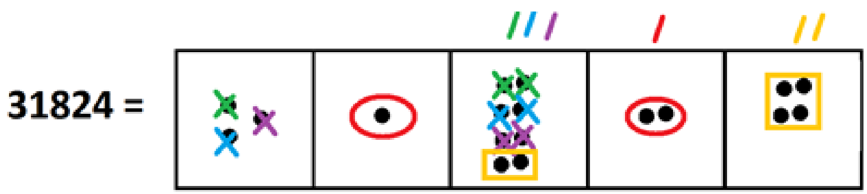




(In the picture of 102, all one-hundred-and-two dots are sitting in the rightmost position. But with many explosions, lots of dots "spilled" over to the left.)

We are looking for groups of one dot–no dots–two dots in our picture of 31824.

And we can spot a number of these groups and the answer 312 to $31824 \div 102$ appears!



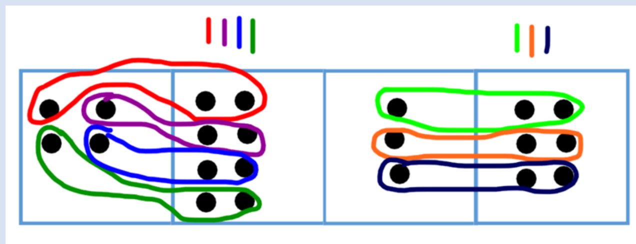
Question 17.7: Can you make sense of my picture here? (I found drawing loops to be messy, so I drew Xs and circles and boxes instead. Is that okay?)

And do you also see how I circled a double group in one hit at the very end?

Your comments.

SELF-CHECK

Self-Check 17.1 Genelle did a division problem the dots-and-boxes way and drew this picture. But she again later forgot what the problem was.



What problem was she solving and what answer did she get for it?

a) She computed $4836 \div 12$ and got the answer 403.
b) She computed $4836 \div 21$ and got the answer 403.
c) She computed something else entirely.

Self-Check 17.2 Compute $4664 \div 22$ the dots and boxes way. What answer do you get?

a) I asked my smartphone the question and it gave me the answer 212. (Why would I do this by hand if I have technology to use?)
b) I did it the dots-and-boxes way and got the answer 212.
c) I did it the dots-and-boxes way and didn't get the answer 212. (But now I think I must have made a slip. I'll try it again.)

Self-Check 17.3 If I am looking for the pattern "three dots—two dots—one dot" when doing a division problem the dots-and-boxes way, I am looking for, in my picture, groups of …

a) 321
b) Yes. Groups of 321.
c) Really yes. I must be dividing by three-hundred twenty-one and so am looking for groups of three-hundred twenty-one.

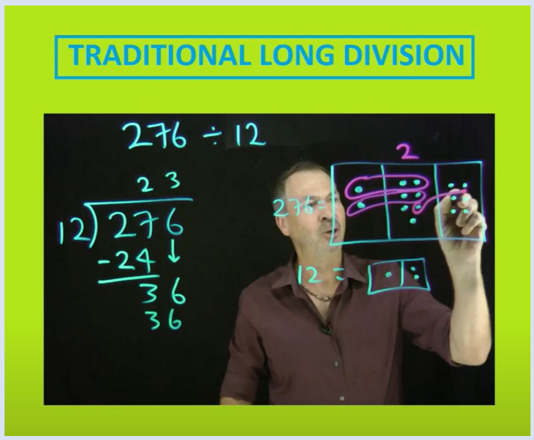
ANSWERS:

17.1 a) **17.2** All three answers are acceptable. **17.3** All three answers are correct.

18. (OPTIONAL) TRADITIONAL LONG DIVISION

This section is completely optional.

Here's video for this section.



https://youtu.be/SwV9VywxEH0

I remember as a young lad being taught an algorithm for conducting long division.

It looked like this.

Example: Compute $276 \div 12$.

Answer: We see $276 \div 12 = 23$.

$$\begin{array}{r} 23 \\ 12\overline{)276} \\ -24\downarrow \\ \hline 36 \\ -36 \\ \hline 0 \end{array}$$

And I remember as a young lad being thoroughly perplexed by this algorithm!

Did you learn an algorithm like this one too?

Can see for yourself what this algorithm is doing?

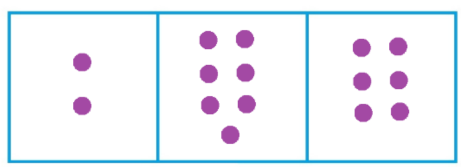
Can you explain why it works?

I couldn't then! But we can now.

Here's what we did to compute $276 \div 12$.

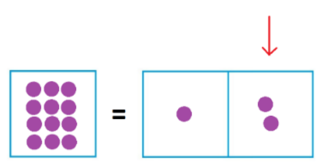
We drew a picture of the number 276 in a $1 \leftarrow 10$ machine.

276 =



And we drew a picture of twelve. It appears as one dot next to two dots, but it is really twelve dots in the rightmost box.

12 =



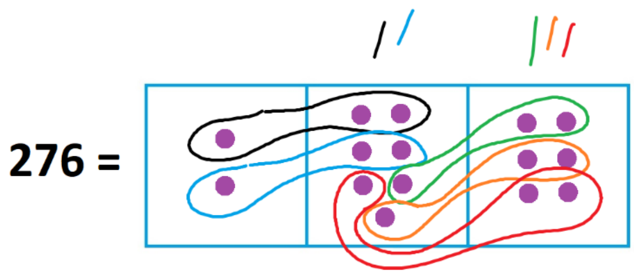
We looked for groups of twelve in the picture of 276.

We first saw two groups among the left two boxes. That is, we first found 2 groups of twelve at the tens level. This accounts for two dots in the left most box, and four of the dots in the middle box.

That left three dots still to consider in the middle box and the six in the rightmost box.

Then we found 3 more groups of twelve at the ones level.

This accounts now for all the dots in the picture. Zero dots are left over.

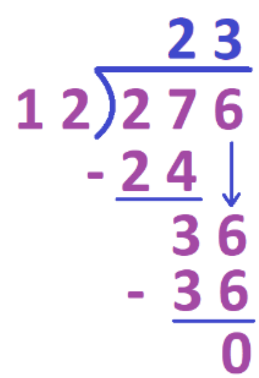


Now think through the school algorithm. Can you see now that it is actually the same process—just without the pictures?

The algorithm first identifies 2 groups of twelve at the tens level just by looking at the first and second boxes.

Then the subtraction simply observes that there are 3 dots in the middle box unaccounted for after doing this.

The algorithm then draws our attention to the second and third boxes, where we can next see 3 groups of twelve at the ones level with no dots are left unaccounted for.

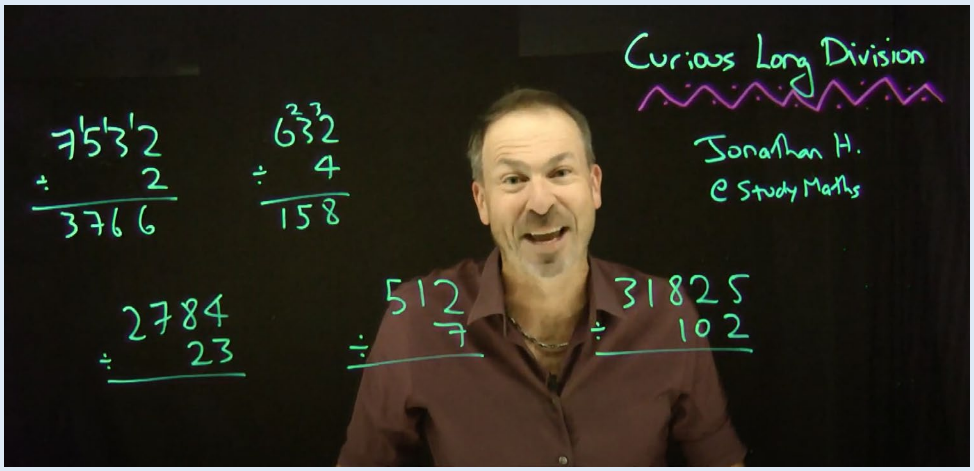


Question 18.1: Compute $2756 \div 13$ the traditional way and the dots-and-boxes way, at the same time, side by side. Can you see they both doing the same thing?

Your work.

SPEED DIVISION

Here’s a video that shows again that fun alternative pencil-and-paper way to conduct division. I give the explanation of it here too.



https://youtu.be/sWrjGjM2Vjc

SELF-CHECK

Self-Check 18.1 I tried this section and …

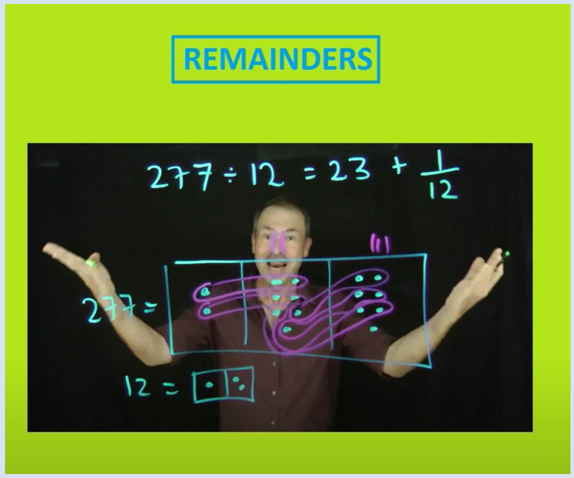
a) I kinda got it.
b) I kinda really didn't get it. (Perhaps I can talk to someone about it.)
c) This section was labeled "optional," so I didn't actually try it. I am not even reading this question.

ANSWER:

18.1 All answers here are acceptable.

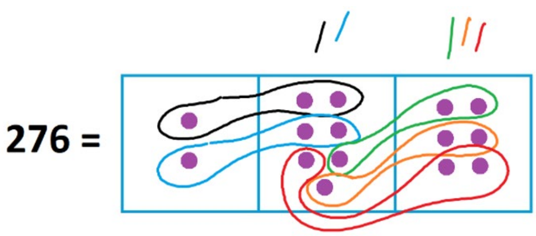
19. Remainders

Here's video for this section.



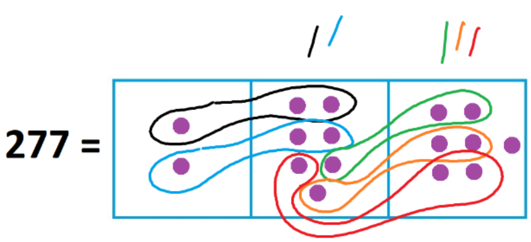
https://youtu.be/6znZCnqgknI

We saw that 276 ÷ 12 equals 23.



Suppose we tried to compute 277 ÷ 12 instead. What picture would we get? How should we interpret the picture?

Well, we'd see the same picture as before except for the appearance of one extra dot, which we fail to include in a group of twelve.



This shows that 277 ÷ 12 equals 23 with a remainder of 1.

You might write this as

$$277 \div 12 = 23 \ R\ 1$$

or with some equivalent notation for remainders. (People use different notations for remainders in different countries.)

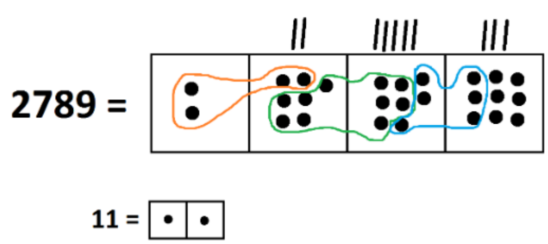
Or you might be a bit more mathematically precise and say that 277 ÷ 12 equals 23 with one more dot still to be divided by twelve.

$$277 \div 12 = 23 + \frac{1}{12}.$$

(And some people might like to think of that single dot as one-twelfth of a group of twelve. All interpretations are good!)

Question 19.1 Compute 2789 ÷ 11 the dots-and-boxes way.

Do you get a picture like this?



How do you interpret this picture?

Your work.

Question 19.2 Use dots and boxes to show that $4366 \div 14$ equals 311 with a remainder of 12.

Your work.

Question 19.3 Use dots and boxes to show that $5481 \div 131$ equals 41 with a remainder of 110.

Your work.

Question 19.4 Here are too many division problems you might or might not want to try. Just do a few of them, whichever seem interesting to you.

a) Compute $4840 \div 4$.

b) Compute $721 \div 7$.

c) Compute $126 \div 6$.

d) Compute $126 \div 3$.

e) Compute $126 \div 2$.

f) Compute $126 \div 1$.

g) Compute $3641 \div 11$.

h) Compute $3642 \div 11$.

i) Compute $3649 \div 11$.

j) Compute $3900 \div 12$.

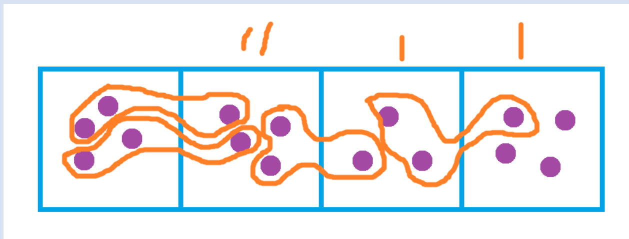
k) Compute $100 \div 9$.

l) Compute $1000000000 \div 9$.

Lots of space!

SELF-CHECK

Self-Check 19.1 This picture shows that $4434 \div 21$ equals 211 with a remainder of …

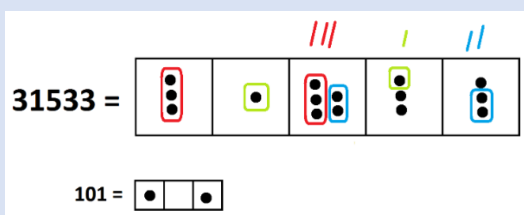


a) 3
b) 211
c) There is no remainder.

Self-Check 19.2 It turns out that $444 \div 3$ equals 148. What then is $446 \div 3$?

a) It's still 148.
b) It's 148 with a remainder of 1.
c) It's 148 with a remainder of 2.
d) It's 148 with a remainder of a gazillion.

Self-Check 19.3 What is this picture showing?



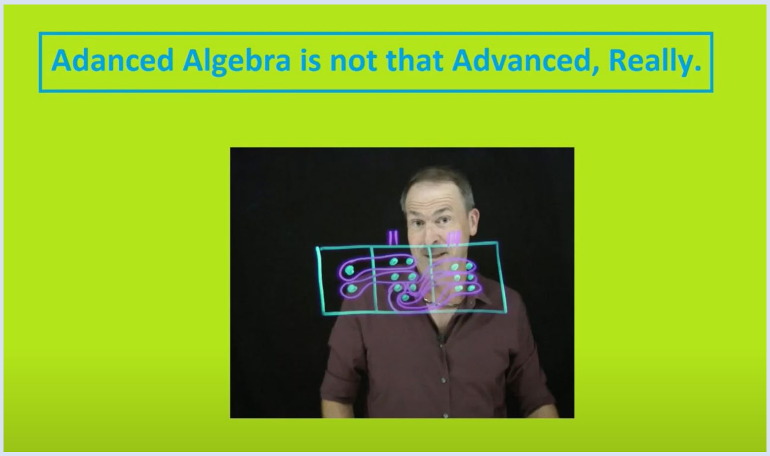
a) A happy pony leaping through a field of peonies.
b) It is impossible to interpret this picture. Who knows what the artist was thinking?
c) It shows that $31533 \div 101 = 312$ with a reminder of 21.

ANSWERS

19.1 a) **19.2** c) **19.3** c)

20. (OPTIONAL) Advanced Algebra is not that Advanced, Really.

And here’s the video for this section.



https://youtu.be/righkhHrpfg

Everything we do in current society is based on the number 10. We humans like that number because we were born with ten digits on our hands to count with.

But there is absolutely nothing special about the number ten. We could do of arithmetic in base 6 or base 12 (Martian), or base 4 or base 8 (Venution), or base 2 (computer), or in any other number machine we care to work in. The mathematics is exactly the same!

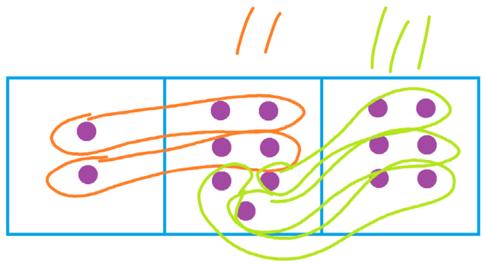
Algebra is about doing mathematics in any system whatsoever and not being locked into our humanness. (Did you read the introduction to this book? I really do believe that mathematics is universal.) Moreover, not only can we can do addition and subtraction and multiplication and division in any base in any base we like, we can do arithmetic even before we’ve decided which base we want to work in!

The way to do that is to use a symbol to represent a base number. But there is a snag. People seem to use the same symbol over and over again when they think “any old number.” They use the letter x. And now that letter has been so overused over the decades that people are now actually scared of that one symbol. What a shame. (I personally like to use the symbol n for “number” or maybe for us, b for “base.”)

Take a deep breath and look at this picture. (It has the symbol x.) Can you see that it is showing that division in base 10, as we've been doing, and division in algebra look exactly the same?

Arithmetic

$$276 \div 12$$
$$= 23$$



Algebra

$$\left(2x^2 + 7x + 6\right) \div (x+2)$$
$$= 2x + 3$$

IT'S THE SAME!

And remember, the symbol x here just means "any base you like." We can go back to being human and choose x to be 10 if we want, and this takes us back to the familiar calculation $276 \div 12 = 23$. But it is also fun to play with other possibilities as to what x could be. We can be human or Martian or Venutian or a computer, or something else entirely, with ease.

Algebra is about just having fun not being locked into our humanness.

SELF-CHECK

Self-Check 20.1 What do you think of the picture on the previous page?

a) It completely freaks me out!
b) I am intrigued by it. I do see a connection between what is written on the left and what is written on the right.

ANSWER:

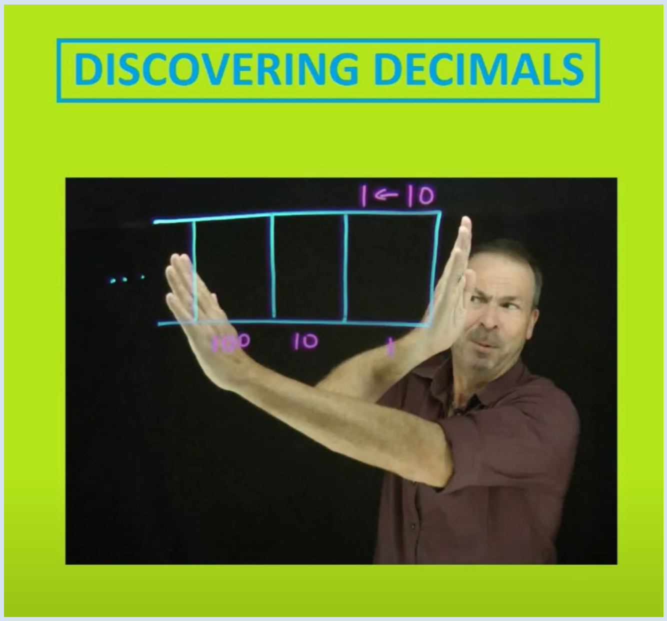
20.1 I am genuinely curious as to how you answer this question.

If you want to start playing with some high school algebra right now, go to the sections at the end of this book.

21. Discovering Decimals

Let me go back to thinking about the basic dots-and-boxes machines. Something about them has been bothering me all this time.

Here's video for this section.



https://youtu.be/4GpmPgQS6Kc

Recall, no matter the machine, we had boxes going to the left as far as we pleased.

But that seems awfully lopsided! Why can't we have boxes going infinitely far to the right as well?

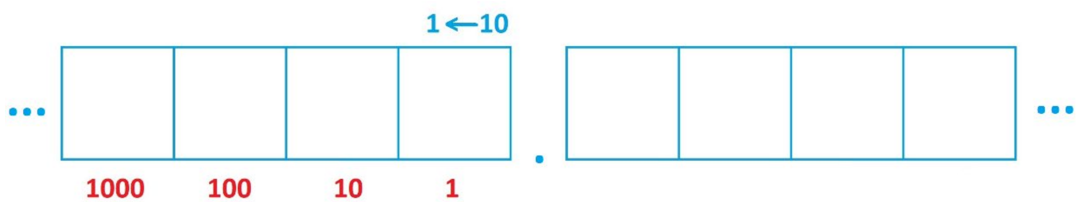
Mathematicians like symmetry and so let's follow suit and now make all our machines symmetrical. Let's have boxes going to the left and to the right.

But the challenge now is to figure out what those boxes to the right mean.

Focusing on the $1 \leftarrow 10$ machine.

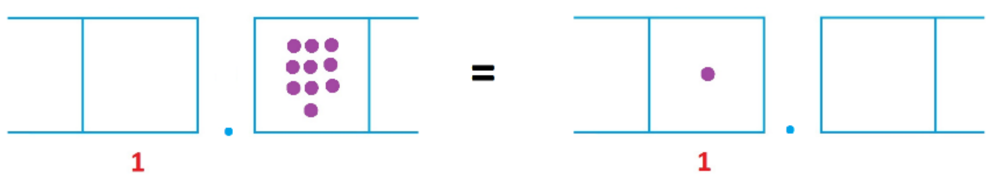
Let's focus on $1 \leftarrow 10$ machine and see what boxes to the right could mean for that machine.

To keep the left and right boxes visibly distinct, we'll separate them with a point. Society calls this point—for base ten, at least—a **decimal point**. (There's the prefix *dec-* again!)



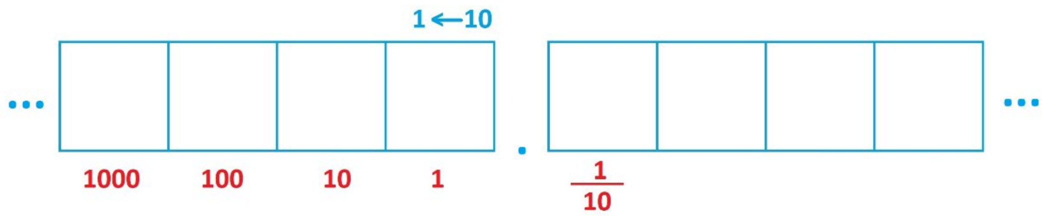
So, what does it mean to have dots in the right boxes? What are the values of dots in those boxes?

Since this is a $1 \leftarrow 10$ machine, we do know that ten dots in any one box explode to make one dot one place to the left.

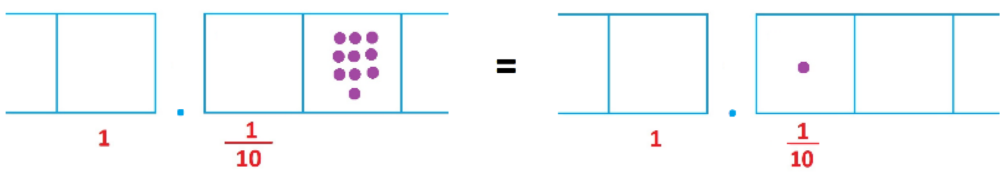


So, ten dots in the box just to the right of the decimal point are equivalent to one dot in the 1s box. Each dot in that box must be worth one-tenth.

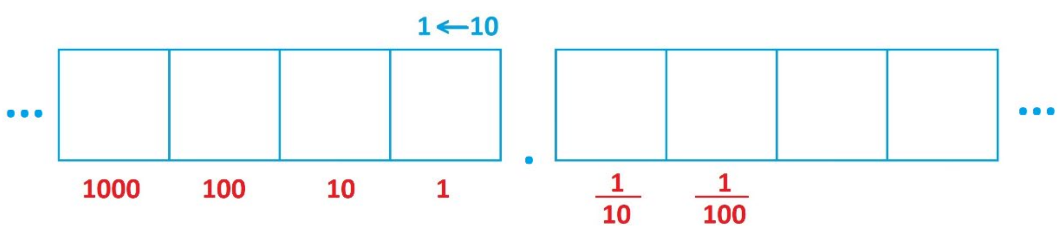
We have our first place-value to the right of the decimal point.



In the same way, ten dots in the next box over are worth one dot in the one-tenth place.

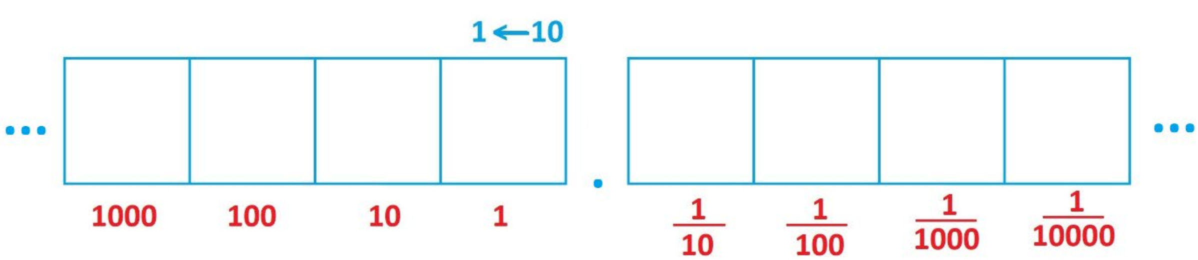


And so, each dot in that next box over must be worth one-hundredth.



We now have two place values to the right of the decimal point.

And ten one-thousandths make a hundredth, and ten ten-thousandths make a thousandth, and so on.



We see that the boxes to the left of the decimal point represent place values given by tens multiplied together and boxes to the right of the decimal point represent place values given by tenths multiplied together.

We have just discovered decimals!

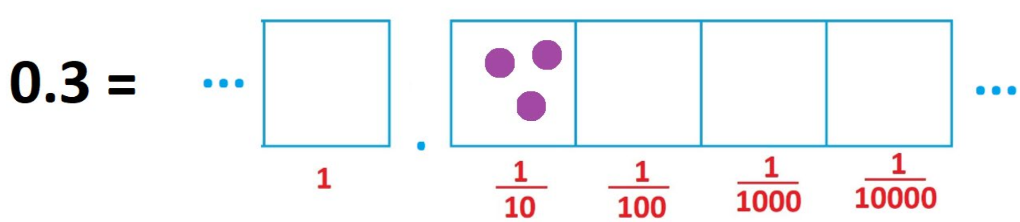
> **Question 21.1:** If the point that separates left and right boxes in a $1 \leftarrow 10$ machine is a called **decimal point**, what might we call the point that separates left and right boxes for a $1 \leftarrow 2$ machine?

Your answer.

> **Question 21.2:** Do all cultures use a point to separate boxes?

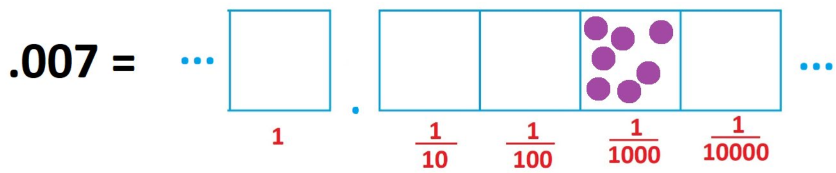
Your answer.

When people write 0.3, for example, in base ten, they mean the value of placing three dots in the first box after the decimal point.



We see that 0.3 equals three tenths: $0.3 = \frac{3}{10}$.

Seven dots in the third box after the decimal point is seven thousandths: $0.007 = \frac{7}{1000}$.



Some people might leave off the beginning zero and just write $.007$ rather than 0.007. This is just a matter of personal taste. (You'll see I've already used both styles of presentation.)

Question 21.3: Draw a dots-and-boxes picture of two one-hundredths.

Your answer.

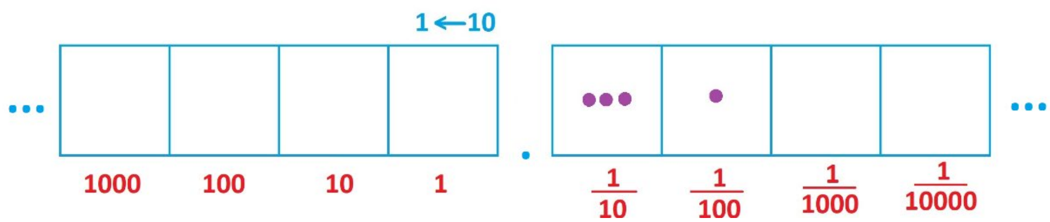
Question 21.4: What fraction is 0.6?

Your answer.

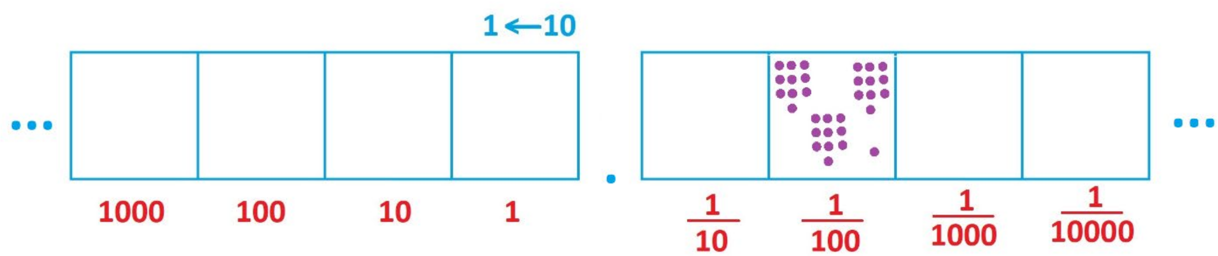
Question 21.5: Some people read 0.6 out loud as "point six" and others read it out loud as "six tenths." Which is more helpful for understanding what the number really is?

Your answer.

There is a possible source of confusion with a decimal such as 0.31. This is technically three tenths and one hundredth: $0.31 = \frac{3}{10} + \frac{1}{100}$.



But some people read 0.31 out loud as "thirty-one hundredths," which looks like this.



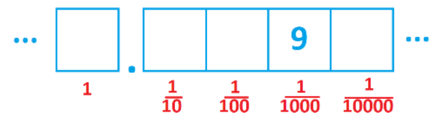
Are these the same thing?

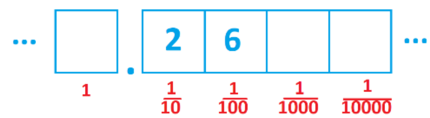
Well, yes! With three explosions we see that thirty-one hundredths becomes three tenths and one hundredth. And with three unexplosions, we can turn three tenths and one hundredth back into thirty-one hundredths.

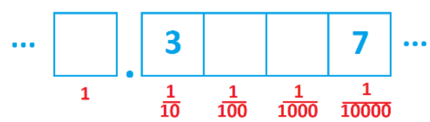
Question 21.6: Do you see this?

Your answer.

Question 21.7 For each picture, write the decimal the picture represents and the fraction that that decimal equals. (For example, the anwer to part a is 0.009, which is $\frac{9}{1000}$.)

a) 

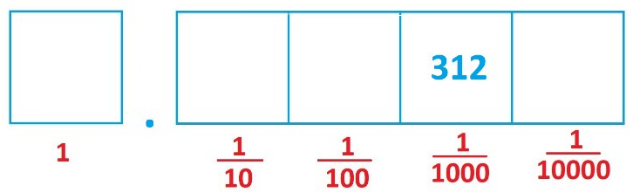
b) 

c) 

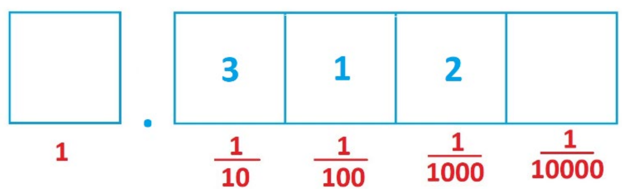
Your answers.

Question 21.8 A schoolteacher asked his students to each draw a $1 \leftarrow 10$ machine picture of the fraction $\frac{312}{1000}$.

JinJin drew:



Subra drew:



The teacher marked both students as correct.

Are both of these responses indeed valid? Explain your thinking.

Your answers.

SELF-CHECK

Self-Check 21.1 The decimal 0.06 corresponds to which fraction?

a) $\frac{6}{10}$

b) $\frac{6}{100}$

c) $\frac{6}{1000}$

d) What's a decimal?

Self-Check 21.2 The decimal $.023$ corresponds to which fraction?
(Perhaps drawing a dots-and-boxes picture helps?)

a) $\frac{23}{10\ 000}$

b) $\frac{23}{1\ 000}$

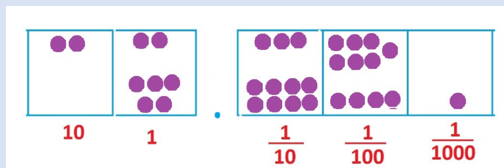
c) $\frac{23}{100}$

d) $\frac{23}{10}$

Self-Check 21.3 Aparna was asked to compute $22.37+5.841$. She wrote this answer for her teacher.

$$\begin{array}{r} 22.3\;7\;\;\; \\ +\;\;\;5.8\;4\;1 \\ \hline 2|7.11|11|1 \end{array}$$

Her teacher was confused, so she added this picture to her page.



Her teacher was still a bit puzzled, but she had an idea now as to what Aparna might be thinking. She said "I was expecting to see the answer 28.211. Does your work lead to that answer?"

What could Aparna do next to show her teacher that $2\,|\,7\;\;.11\,|\,11\,|\,1$ is indeed the number 28.211 in disguise?

a) She can do some explosions! After all, ten dots in a box are the same as one dot, one place to the left. And I checked: $2\,|\,7\;\;.11\,|\,11\,|\,1$ does indeed become 28.211.

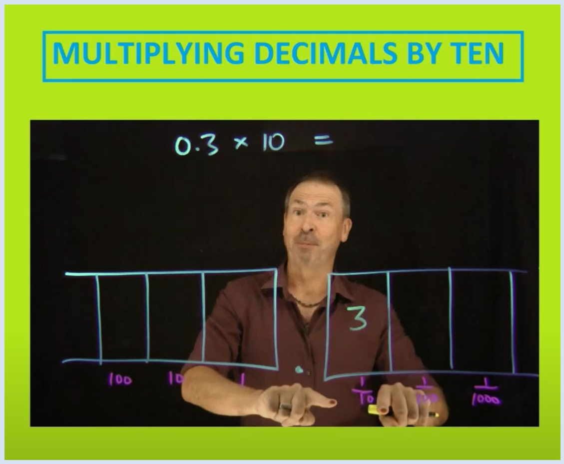
b) She can do some explosions! After all, ten dots in a box are the same as one dot, one place to the left. (I didn't check that $2\,|\,7\;\;.11\,|\,11\,|\,1$ becomes 28.211, but I will.)

ANSWERS

21.1 b) **21.2** b) **21.3** Both a) and b) are fine.

22. Multiplying Decimals by Ten

Here's video for this section.



https://youtu.be/79nb69OB0Cw

The decimal 0.6 is the fraction $\frac{6}{10}$, and so multiplying it by 10 should give the answer 6.

We have $10\times\frac{6}{10}=1$.

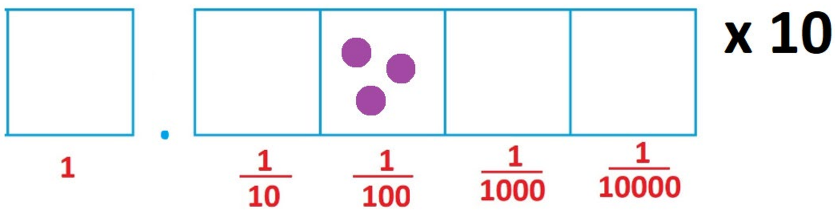
This means $10\times 0.6=6$.

Question 22.1: Can you explain why 10×0.03 equals 0.3?

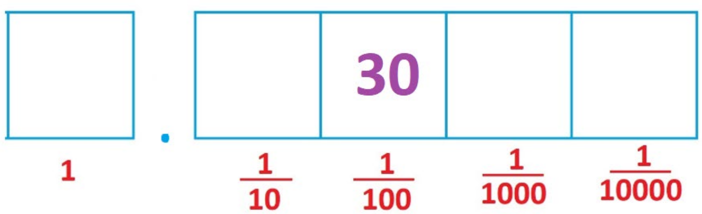
This technically a YES/NO question. But it would be lovely if you explain it.

Here's a dots-and-boxes way to explain the previous question.

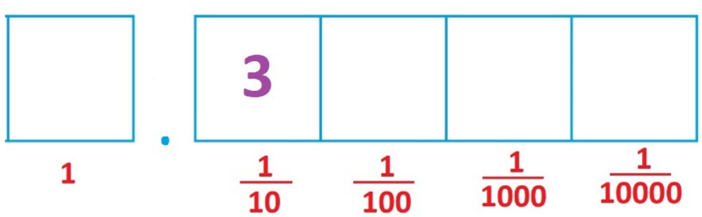
A picture of 0.03 has three dots in the hundredths place. We want to multiply this quantity by ten.



This will give 30 dots in the hundredths place.



Three explosions then produce 3 dots in the tenths place instead.



And this is the number 0.3.

We have

$$10 \times 0.03 = 0.3 .$$

Question 22.2: Use a dots-and-boxes picture to show that 22.37×10 equals 223.7?

Some space.

Question 22.3: Can you explain why 22.37×100 equals 2237?

Some space.

SELF-CHECK

Self-Check 22.1 What is 0.07 times 10?

a) It's really $\frac{7}{100} \times 10$, which is $\frac{7}{10}$, which is 0.7.

b) A dots-and-boxes picture of 0.07 has 7 dots in the hundredths place. Multiplying by ten gives 70 dots in the hundredths place. Explosions then produce 7 dots in the tenths place. We have the answer 0.7.

c) It is impossible to know what the answer is.

Self-Check 22.2 When we multiplied 22.37 by 10 to get the answer 223.7, it looks like the decimal point shifted to the one spot to the right. Did that really happen?

a) NO! The decimal point did not move. Explosions, instead, make it look like the digits of the number are moving one place to the left.

Self-Check 22.3 Should we memorize a rule about how a decimal point "moves" when multiplying a decimal by ten?

a) NO! Just visualize the explosions.

ANSWERS:

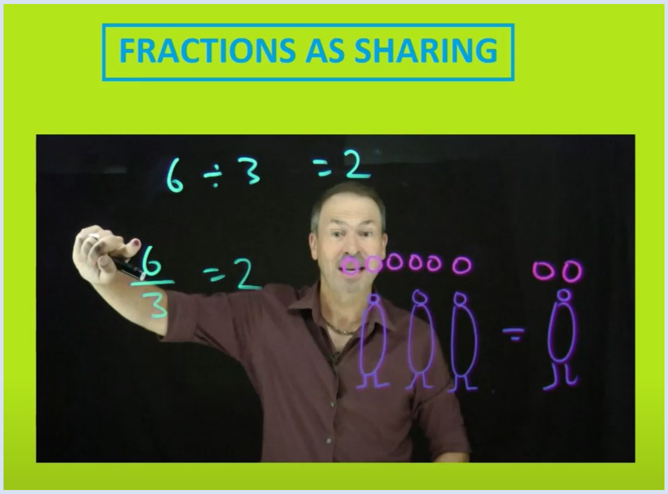
22.1 a) and b) are both correct.
22.2 a)
22.3 a)

If you want to see the full mathematics of working with decimals, see chapter 6 here.

https://gdaymath.com/lessons/gmp/9-1-chapter-content/

23. Fractions as Sharing

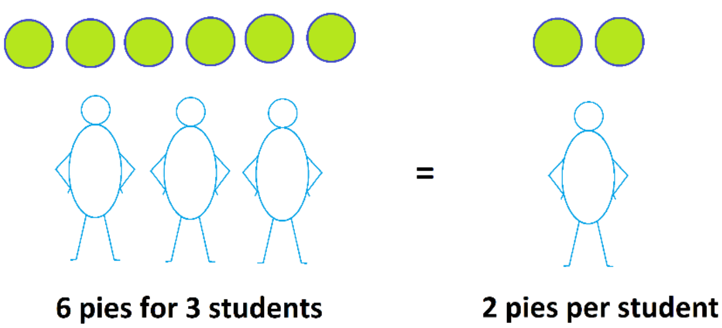
Here's video for this section.



https://youtu.be/vrgpT68qAnI

Some people like to think of division as the act of sharing.

For example, if I share 6 pies equally among 3 students, each student will receive 2 pies.



We often write this as

$$6 \div 3 = 2 .$$

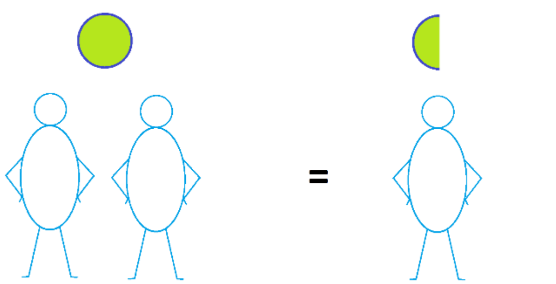
But we could also write this using the notation of fractions.

number of pies

$$\frac{6}{3} = 2 \text{ pie per student}$$

number of students

This type of thinking is meaningful. For example, in this setting, the fraction $\frac{1}{2}$ represents the act of sharing 1 pie equally among 2 students. And we see that each student really does get half a pie.



1 pie for 2 students **Half a pie per student**

number of pies

$$\frac{1}{2} = \textbf{half} \text{ pie per student}$$

number of students

Question 23.1: Here's a sharing problem: $\frac{20}{5}$

How many pies are there? How many students?
How much pie does each student receive?
(Do you want to draw a picture of this sharing problem?)

Your answers.

Question 23.2: In early grades, we call the fraction $\frac{1}{3}$ a **third**.

Draw a pie-and-students picture of $\frac{1}{3}$ and show how much an individual student receives. Do we indeed call what you drew "a third"?

Your picture and answer.

Challenge Question 23.3: Draw a picture that interprets the fraction $\frac{2}{3}$ as a sharing problem. How many pies are there? How many students? Draw a picture of how much pie each student gets.

Your picture.

SELF-CHECK

Self-Check 23.1 Sharing 60 pies equally among 3 students gives 20 pies per student. We normally write this statement as $60 \div 3 = 20$. But we could also write it as …

a) $\frac{3}{20} = 60$ b) $\frac{60}{20} = 3$ c) $\frac{20}{60} = 3$ d) $\frac{60}{3} = 20$

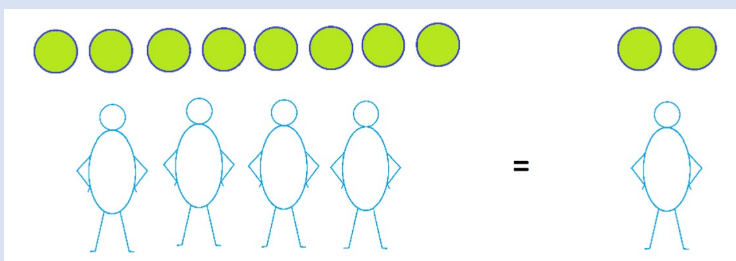
Self-Check 23.2 The fraction $\frac{1}{4}$ corresponds to the answer of the division problem

a) $1 \div 4$

b) Distributing one pie equally among four students.

c) Someone eating a whole pie for themselves and sharing nothing.

Self-Check 23.3 What's a correct description of this picture?



a) $\frac{8}{4} = 2$ b) $8 \div 4 = 2$ c) $\frac{8}{4} = 8 \div 4 = 2$ d) The pies are green. (Odd!)

ANSWERS

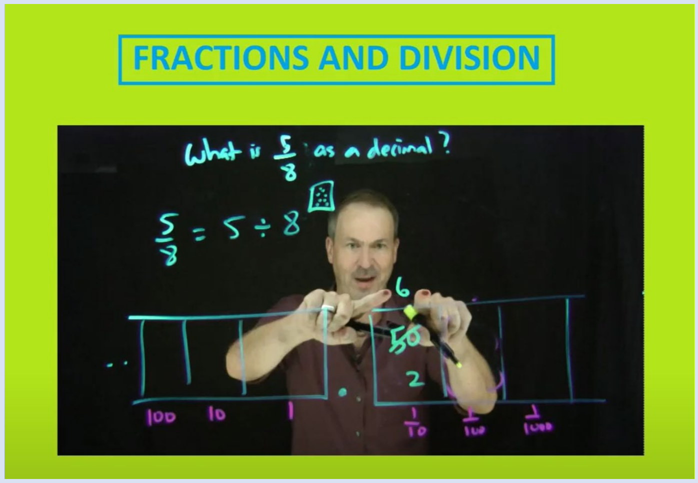
23.1 d) **23.2** a) and b) are both correct. **23.3** All options are correct.

If you want to see the full pie-per-student story for fractions, see chapter 5 here.

https://gdaymath.com/lessons/gmp/9-1-chapter-content/

24. Fractions and Division

Here's video for this section.



https://youtu.be/SrzRVLVkXAM

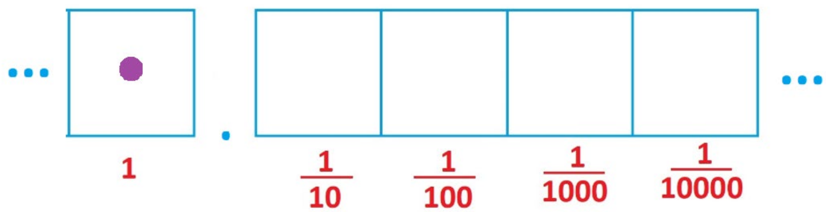
We just set up matters so that a fraction is a number that is an answer to a division problem.

$\frac{2}{3}$ is the answer to $2 \div 3$.

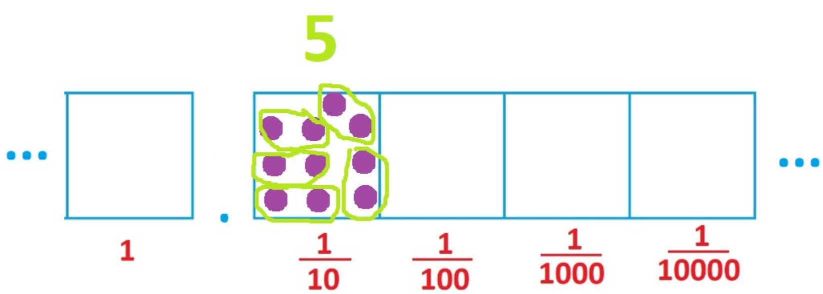
$\frac{1}{2}$ is the answer to $1 \div 2$.

Moreover, we can compute $1 \div 2$ in a $1 \leftarrow 10$ machine by making use of decimals. The method is exactly the same as for division without decimals.

For $1 \div 2$ we seek groups of two in the following picture.



We don't see any groups of two. So, let's unexplode. Then we see five groups of two at the tenths level.



We have that $\frac{1}{2}$ is 0.5 as a decimal. (And as a check, $\frac{5}{10}$ does indeed equal $\frac{1}{2}$.)

Question 24.1: Compute $\frac{1}{5} = 1 \div 5$ as a decimal to get 0.2.

Some space.

Question 24.2: Compute $\frac{1}{4}$ as a decimal by computing $1 \div 4$. Do you get 0.25?

Some space.

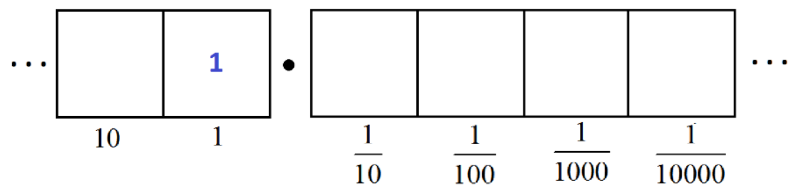
Question 24.3: Compute $\frac{1}{10} = 1 \div 10$ as a decimal.

(Why should you get the answer 0.1?)

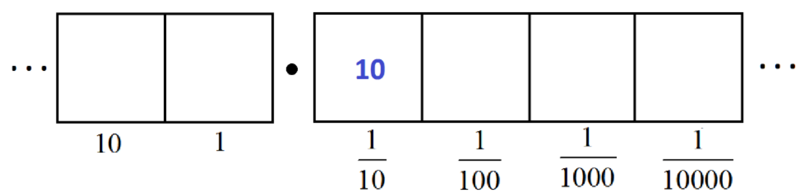
Some space.

Another example: Let's write $\frac{1}{8}$ as a decimal. We need to compute $1 \div 8$ in a $1 \leftarrow 10$ machine.

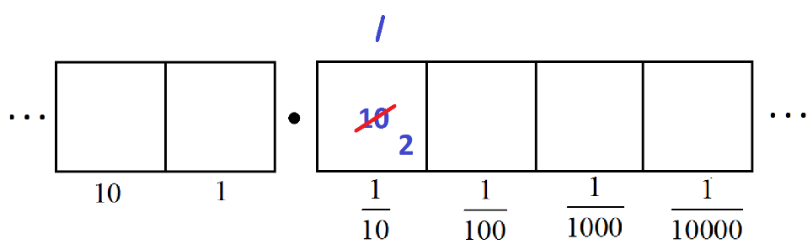
We seek groups of eight in the following picture. (I won't draw dots this time and just write numbers.)



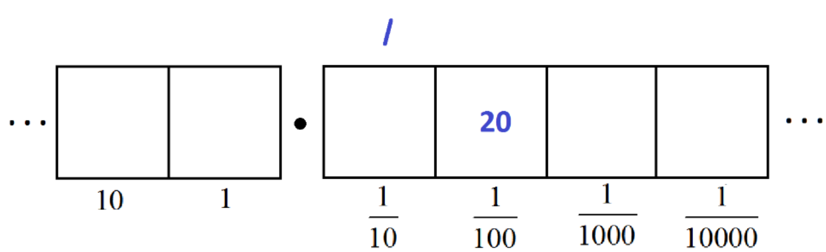
None are to be found right away, so let's unexplode.



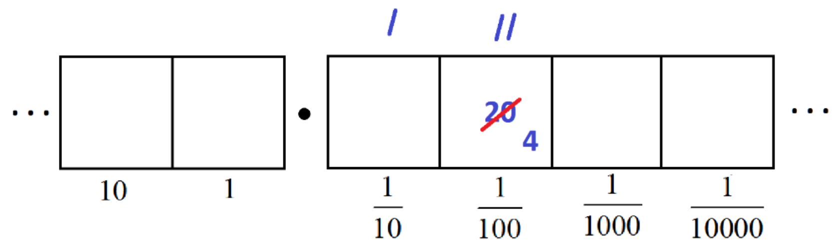
We have one group of 8, leaving two behind.



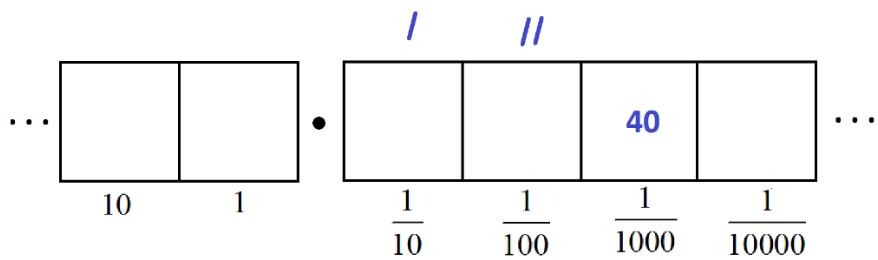
Two more unexplosions.



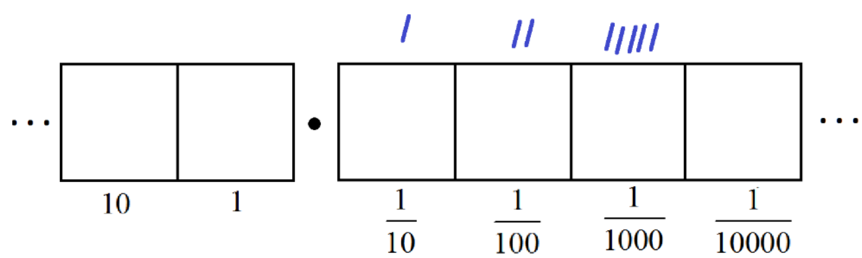
This gives two more groups of 8 leaving four behind.



Unexploding again



reveals five more groups of 8 leaving no remainders.



We see that, as a decimal, $\frac{1}{8}$ turns out to be 0.125.

This is what a calculator shows too. Super!

Question 24.4: The decimal 0.125 is the fraction $\frac{125}{1000}$. Is this really the fraction $\frac{1}{8}$?

Some space.

Question 24.5: My calculator says that $\frac{5}{8}$ is 0.625.

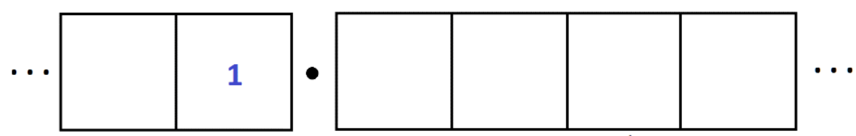
Can you show this the dots-and-boxes way too?

Lots of space!

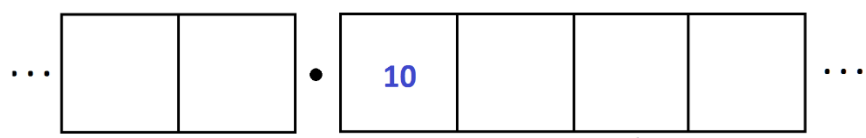
Not all fractions lead to simple decimal representations.

For example, consider the fraction $\frac{1}{3}$.

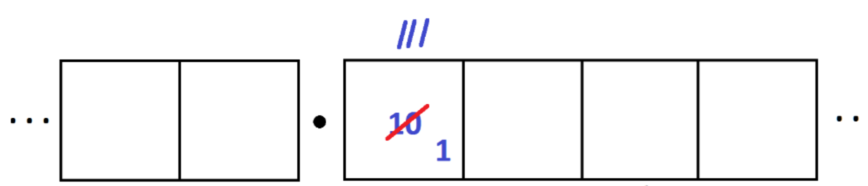
To compute it, we seek groups of three in the following picture.



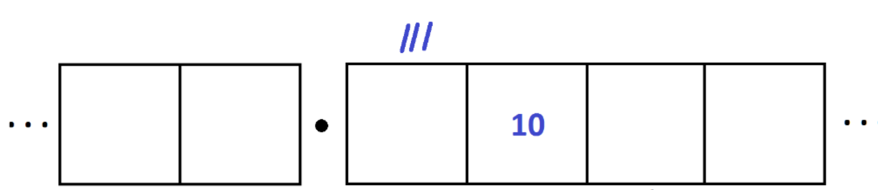
Let's unexplode.



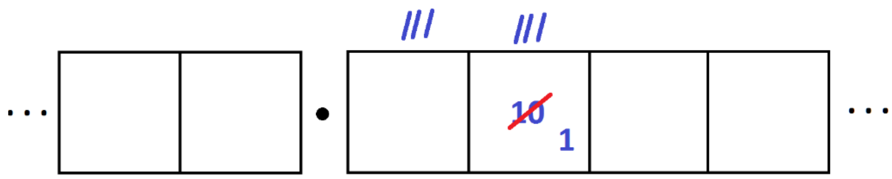
We see three groups of 3 leaving one behind.



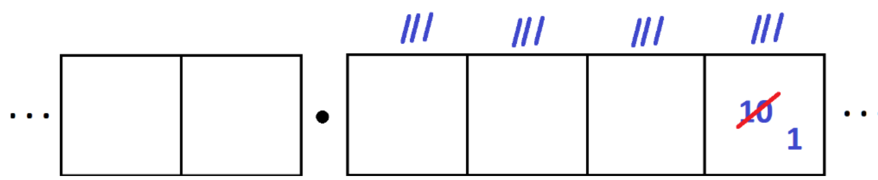
Unexploding gives another ten dots to examine.



We find another three groups of 3 leaving one behind.



And so on. We are caught in an infinitely repeating cycle.



This puts us in a pickle! As human beings we cannot keep doing something forever.

But it seems very tempting to write

$$\frac{1}{3} = 0.33333333\cdots$$

with the dot-dot-dot at the end (this symbol of three dots is called an **ellipsis**) representing the instruction "keep going with this pattern forever."

In our minds we can almost imagine what this means. But as a practical human being it is beyond our abilities: one cannot actually write down those infinitely many 3s represented by the ellipses.

Nonetheless, many people choose not to worry about this and just carry on and say that some decimals are infinitely long. The fraction $\frac{1}{3}$ is one of those fractions whose decimal representation goes on forever.

Question 24.6: Are you worried about not being able to do a mathematical process forever? Or is it kinda fun to imagine having an infinite number of 3s as part of an answer to a math problem?

Your thoughts?

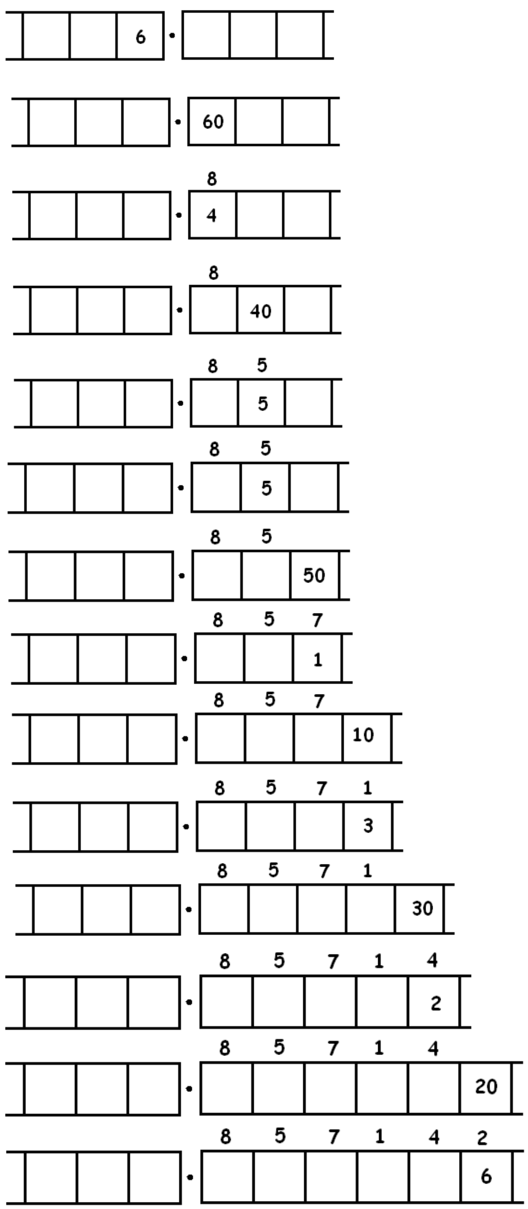
Question 24.7: Write $\frac{1}{9} = 1 \div 9$ as a decimal. Does it also fall into an infinitely repeating pattern?

Some space.

Are you ready for a complicated example?

Let's convert the fraction $\frac{6}{7}$ to a decimal. It will turn out to be an infinitely long repeating decimal.

Here's my work. Can you make sense of it? (Perhaps just do the problem $6 \div 7$ on your own and don't look at my work.)



Do you see, with this 6 in the final rightmost box, that we have returned to the very beginning of the problem? This means that we shall simply repeat the work we have done and obtain the same sequence 857142 of answers, and then again, and then again.

We have $\frac{6}{7} = 0.857142\ 857142\ 857142\ 857142\cdots$.
(I think my calculator shows this too.)

Question 24.8: Write $\frac{5}{6} = 5 \div 6$ as a decimal. Does it fall into an infinitely repeating pattern?

Some space.

Question 24.9: Show that $\frac{1}{11} = 1 \div 11$ as a decimal is $0.09090909....$.

Some space.

Optional Challenge 24.10: Which of the following fractions give infinitely long decimal representations? (We've done some of these already.)

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10} \quad \frac{1}{11} \quad \frac{1}{12}$$

Probably not enough space.

SELF-CHECK

Self-Check 24.1 This section was a lot of work!

a) I agree!

b) I thought it was fine.

Self-Check 24.2 We know $\frac{1}{3} = 0.33333\cdots$. What's $\frac{2}{3}$?

a) If you compute $2 \div 3$ with dots and boxes you get $0.66666\cdots$.

b) It must be $0.66666\cdots$ because $\frac{2}{3}$ is double $\frac{1}{3}$, and double $0.33333\cdots$ is $0.66666\cdots$.

c) Both a) and b) are correct. (And I think answer b was a clever way to avoid work!)

d) Actually, both a) and b) are incorrect and $\frac{2}{3}$ as a decimal is something else.

Self-Check 24.3 $\frac{5}{7}$ as a decimal is

a) $0.712854\ 712854\ 712854\cdots$
b) $0.714285\ 714285\ 714285\cdots$
c) $0.714258\ 714258\ 714258\cdots$
d) I can't believe you are making me do this!

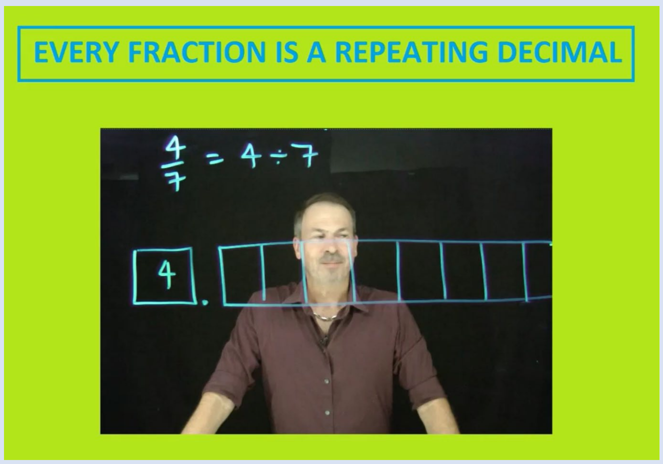
ANSWERS

24.1 Either answer could be true.
24.2 a) b) and c) are all correct.
24.3 b) and possibly d).

25. Every Fraction is a Repeating Decimal

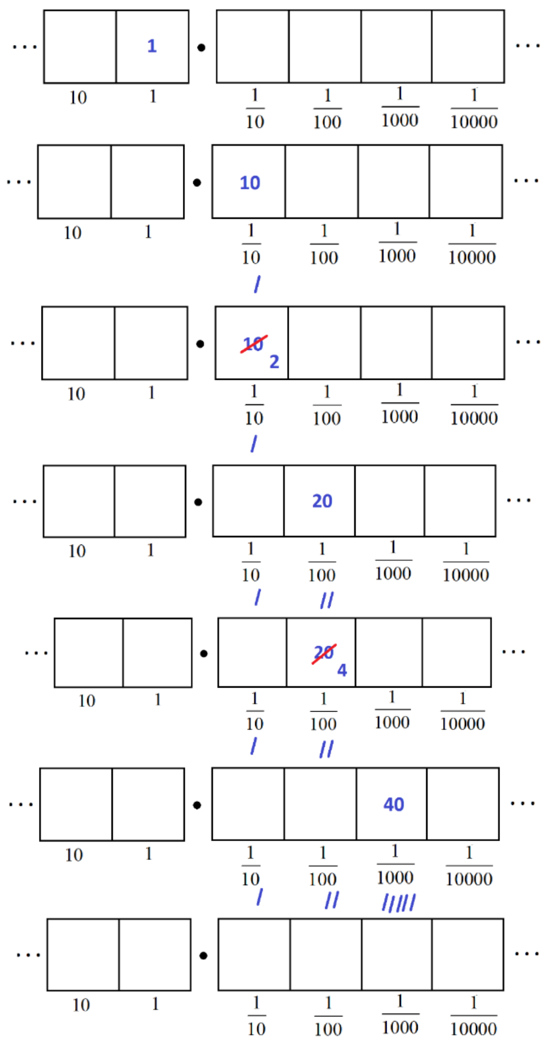
The title of this section reveals what's coming next.

Here's video for this section.



https://youtu.be/w3Ormnijr_0

We used division in a $1 \leftarrow 10$ machine to write fractions as decimals. For example, we saw that $\frac{1}{8}$, computed as $1 \div 8$, has decimal representation 0.125.



Other fractions have infinitely long decimal representations.

For example, we computed $\frac{1}{3}$ as $1 \div 3$ and got

$$\frac{1}{3} = 0.33333\cdots.$$

And if you did the last self-check you might have computed

$$\frac{5}{7} = 0.714285\ 714285\ 714285\cdots.$$

People often use a horizontal bar (it's called a **vinculum**) to denote a block of digits that is repeated infinitely often in a decimal. For example, we write

$$\frac{5}{7} = 0.\overline{714285}$$

and

$$\frac{1}{3} = 0.\overline{3}.$$

Also, $0.38\overline{142}$ means repeat the group 142 forever after the beginning 38 "hiccup."

$$0.38\overline{142} = 0.38142142142142\cdots.$$

Question 25.1:

a) Write $41.\overline{37}$ as an infinitely long decimal.
b) Write $534.6012012012012012012\cdots$ with vinculum notation.

Your answers.

Question 25.2: Not everyone in the world uses a vinculum to represent a repeated block of digits in a decimal representation. Some people use an arc, or parentheses, or dots. Can you guess which infinitely long decimal each of these quantities represents?

a) $0.4\overparen{9305}$

b) $2.35(517)$

c) $719.320\dot{6}578\dot{3}$

Your answers.

Optional Challenge Question 25.3: What's $0.\overline{125} + 0.1\overline{38}$?

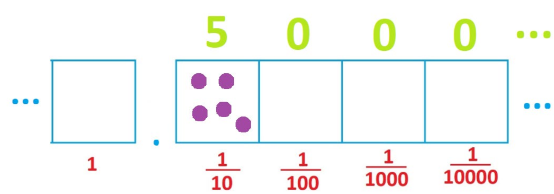
Some space.

All the examples of fractions with infinitely long decimal representations we've seen so far fall into a repeating pattern. This is curious.

We can even say this is true too for our finite decimal examples: they fall into a repeating pattern of zeros after an initial start.

$$\frac{1}{8} = 0.12500000.... = 0.125\overline{0}$$
$$\frac{1}{2} = 0.50000.... = 0.5\overline{0}$$

Question 25.4 It seems silly to write $\frac{1}{2}$ as $0.50000000\cdots$. But could we if we wanted to? Is it correct to say that there are five tenths, and zero hundredths, and zero thousandths, and zero ten-thousands, and so on?



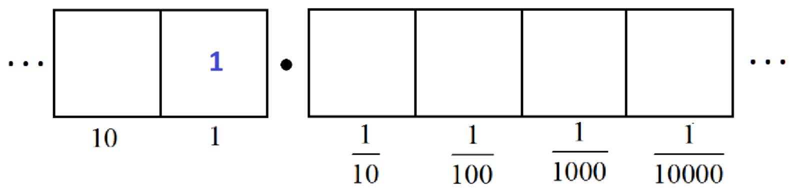
What do you think?

This leads to a big question.

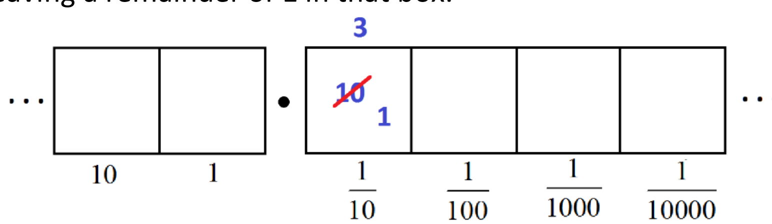
Does every fraction have a decimal representation that eventually repeats?

The answer to this question is YES and our method of division explains why.

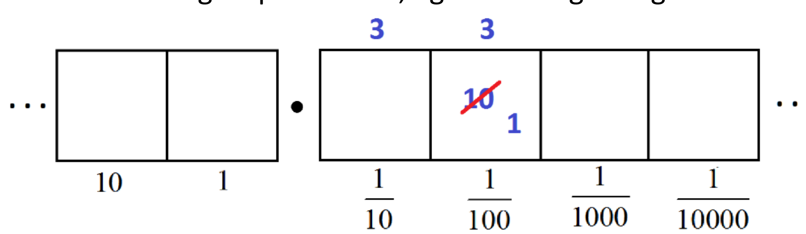
Let’s go through the division process again—slowly—first with a familiar example. Let’s compute the decimal representation of $\frac{1}{3}$ again in a $1 \leftarrow 10$ machine. We think of $\frac{1}{3}$ as the answer to the division problem $1 \div 3$, and so we need to find groups of three within a diagram of one dot.



We unexplode the single dot to make ten dots in the tenths position. There we find three groups of three leaving a remainder of 1 in that box.



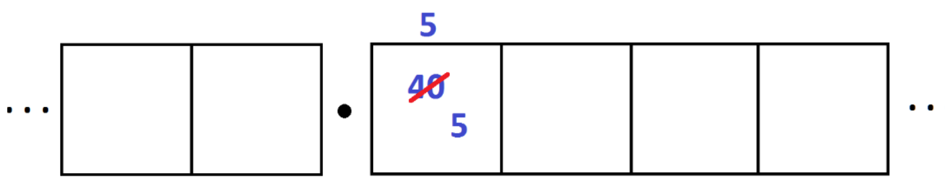
Now we can unexploded that single dot in the tenths box and write ten dots in the hundredths box. There we find three more groups of three, again leaving a single dot behind.



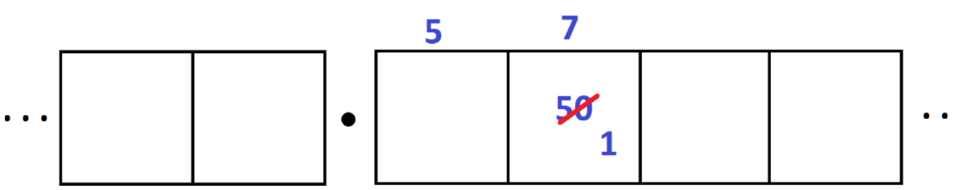
And so on. We are caught in a cycle of having the same remainder of one dot from cell to cell, meaning that the same pattern repeats. Thus, we conclude $\frac{1}{3} = 0.333\ldots$. The key point is that the same remainder of a single dot kept appearing.

Let's compute the decimal representation of $\frac{4}{7}$ in the $1 \leftarrow 10$ machine. That is, let's compute $4 \div 7$ and be sure to take note of the remainders that occur.

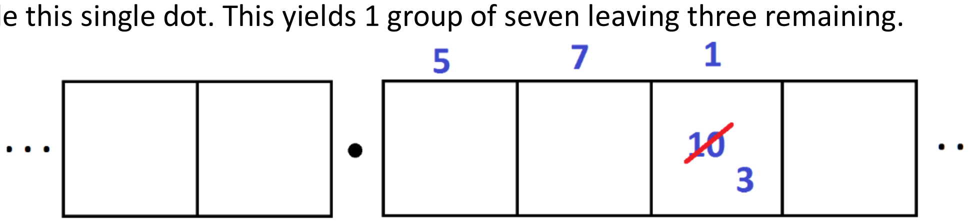
We start by unexploding the four dots to give 40 dots in the tenths cell. There we find 5 groups of seven, leaving five dots over.



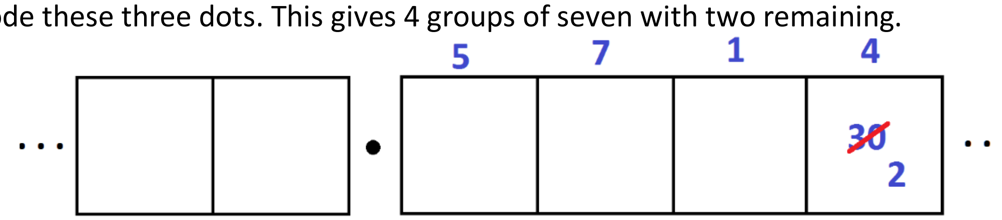
Now unexplode those five dots to make 50 dots in the hundredths position. There we find 7 groups of seven, leaving one dot over.



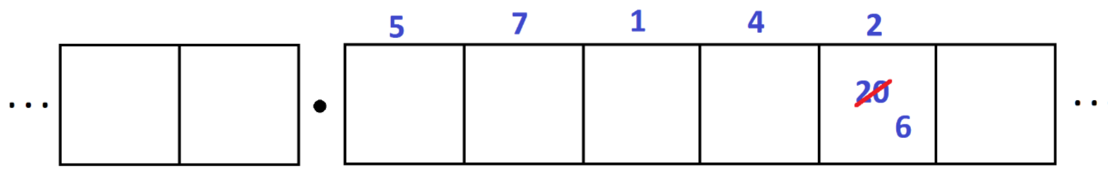
Unexplode this single dot. This yields 1 group of seven leaving three remaining.



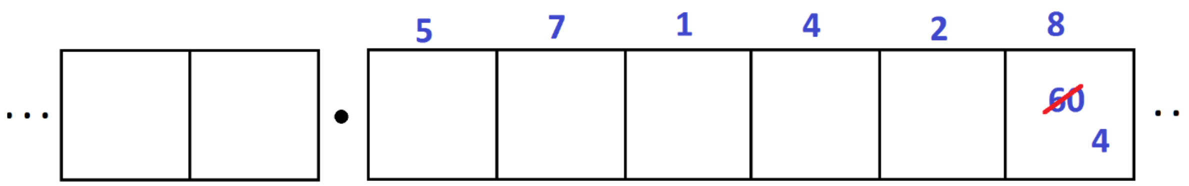
Unexplode these three dots. This gives 4 groups of seven with two remaining.



Unexplode the two dots. This gives 2 groups of seven with six remaining.



Unexplode the six dots. This gives 8 groups of seven with four remaining.



But this is the predicament we started with: four dots in a box!

So now we are going to repeat the pattern and produce a cycle in the decimal representation. We have

$$\frac{4}{7} = 0.571428\ \ 571428\ \ 571428\ \cdots.$$

Question 25.5 A couple of times now you have been invited to compute the decimal representation of $\frac{5}{7}$. (Feel free to do it now, if you haven't done it already.)
Did you get a remainder that repeated?
What was the value of that first repeated remainder?

Some space.

Let's step back and think about things for a moment.

In computing $\frac{4}{7}$ (and $\frac{5}{7}$) we were looking for groups of 7 in our dots-and-boxes picture.

There could be remainders. But there are only the following options for what the remainders could be.

Remainder = 0 or 1 or 2 or 3 or 4 or 5 or 6.

Question 25.6 When looking for groups of seven, why won't you ever see a remainder of 7? Why won't you ever see a remainder of 8?

Your thoughts.

So, in computing $\frac{4}{7}$ there are only seven possible remainders—0,1,2,3,4,5,6—which means we can't keep doing our work and keep getting different remainders. We are going to have to eventually repeat one of them. And as soon as we do, we're in a cycle!

The fraction $\frac{4}{7}$ simply had to go into a repeating pattern for its decimal representation.

In the same way, the decimal representation of $\frac{18}{37}$ must also cycle.

Imagine a dots-and-boxes picture of $18 \div 37$.

We'd be looking groups of 37 dots.

There are likely to be remainders.

There are only thirty-seven possible remainders: 0, 1, 2, …, 35, 36.

We can't keep getting different remainders: we must repeat at least one of them.

As soon as we do, we're in a cycle.

Question 25.7 How would you explain that $\frac{21}{345}$ must have a decimal representation that falls into a repeating pattern?

Your thoughts.

Question 25.8 Can you see that when we compute $1 \div 4$ with dots-and-boxes, we fall into a repeated remainder of 0? (We get that $\frac{1}{4} = 0.25\overline{0}$.)

Some space.

We have just established something amazing!

Every fraction has a decimal representation that falls into a repeating pattern.
(A pattern of repeating zeros is allowed.)

Question 25.9 Find the decimal representation of $\frac{23}{45}$.

(This question isn't as bad as it looks. After a "hiccup," its decimal representation repeats just one digit over and over again.)

Some space.

SELF-CHECK

Self-Check 25.1 The dots-and-boxes approach shows that if you represent a fraction as a decimal, that decimal representation is sure to have a repeating pattern.

a) It sure does!

Self-Check 25.2 Hang on! I just worked out $\frac{3}{20}$ and got 0.15 for its decimal representation. There's no repeating pattern in that.

a) Oh no! All of mathematics has just fallen apart!

b) There is a repeating pattern. It's a repeating pattern of zeros. Mathematics is safe!

$$0.15 = 0.15000000\cdots = 0.15\overline{0}.$$

Self-Check 25.3 Could the fraction $\frac{8}{13}$ have a decimal representation with a block of fifteen digits that repeat?

a) I don't know. And I don't want to compute $8 \div 13$. It's hard!

b) If I imagine computing $8 \div 13$, I can see that there are only thirteen possible remainders. I'd have to start repeating remainders by the fourteenth place, if not before. I don't have fifteen different remainders to get a block of fifteen digits that repeat.

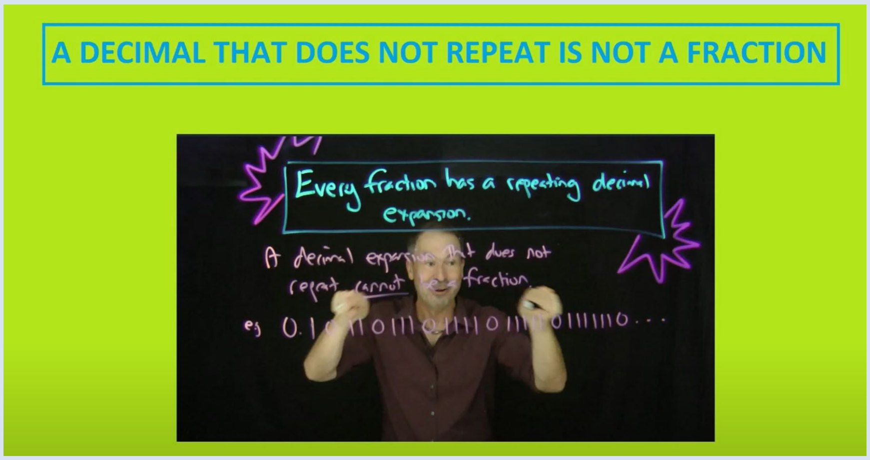
The answer to this question is: NO.

ANSWERS

25.1 a) **25.2** b) **25.3** Both a) and b) are acceptable.

26. A Decimal that Does Not Repeat is not a Fraction

Here's video for this section.



https://youtu.be/ERCMjdUlw2M

We established in the previous section

> Every fraction has a decimal representation that falls into a repeating pattern.

Remember, some fractions have a decimal representation that fall into a repeating pattern of zeros. People usually don't bother writing these repeating zeros and call them **finite decimals** instead.

For example, $\frac{1}{2} = 0.5$ is the infinite decimal $0.50000\cdots = 0.5\overline{0}$.

And 0.03 is the infinite decimal $0.030000\cdots = 0.03\overline{0}$.

Question 26.1 Is the whole number 2 a fraction?
Does it have a decimal representation?
Does it have a decimal representation with a repeating pattern?

Your thoughts.

That every fraction has a repeating decimal representation now opens up a curious idea.

A decimal representation that does not repeat corresponds to a quantity that is not a fraction!

Pause! Do you get the logic here?

Question 26.2:

a) Jozsef believes that "Every crow is a black."
You show Jozsef a bird that is not black. Would he say that the bird is a crow?

b) Lana believes that "Every birthday cake has icing."
You give her a cake with no icing. Would she deem it a birthday cake?

c) We know that all whole-number multiples of ten end with 0.
I see a whole number and doesn't end with 0. Can it be a multiple of ten?

d) Every fraction has decimal representation that repeats.
I see the decimal representation of a number and it doesn't repeat. Can that number be a fraction?

Your thoughts.

For example, consider this quantity.

$$0.10\;1100\;111000\;11110000\;1111100000\cdots$$

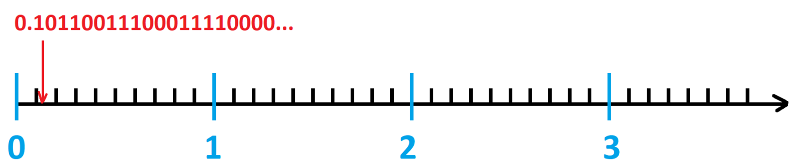
It has a pattern in its decimal representation: one 1, one 0, two 1s, two 0s, three 1s, three 0s, and so on. This means we can know any digit of this number by just following the pattern long enough.

But the pattern we have is NOT a repeating pattern.

This means that this number is not a fraction!

Question 26.3: Whoa! Pause again! Take this in.

The quantity $0.10\;1100\;111000\;11110000\cdots$ is a bit bigger than 0.1, which is $\frac{1}{10}$, and so is just to the right of one-tenth mark on the number line.



Yet it cannot be a fraction: it doesn't have a repeating decimal representation.

There are numbers on the number line that are not fractions!

Do you find this freaky?

Your reaction?

We can invent all sorts of numbers that can't be fractions!

For example

$$0.\,8\,0\,8\,00\,8\,000\,8\,0000\,8\,00000\,8\,000000\,8\cdots$$

and

$$0.\,1\,2\,3\,4\,5\,6\,7\,8\,9\;\;10\;\;11\;\;12\;\;13\;\;14...98\;\;99\;\;100\;\;101\;\;102...999\;\;1000\;\;1001\;\;1002\;\;1003....$$

are numbers that are not fractions.

People call a number on the number line that is not a fraction an **irrational number**.

Question 26.4: Lucinda says that

$$0.34334333433334333334333333 4\cdots$$

is an irrational number slightly bigger than a third.

Is she right?

Your answer.

Question 26.5: Create two numbers that you personally know are not fractions.

Your answers.

SELF-CHECK

I learnt in school that between any two fractions lies another fraction.

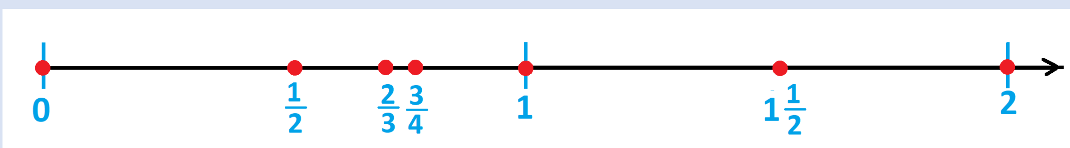
For example:

Between 0 and 1, there's $\frac{1}{2}$.

Between 1 and 2, there's $1\frac{1}{2}$.

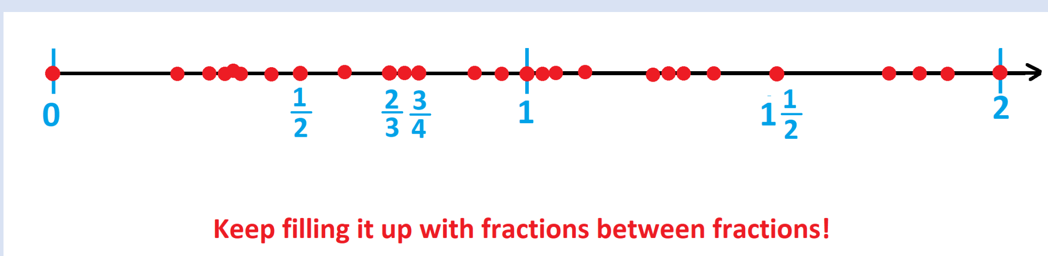
Between $\frac{1}{2}$ and 1, there's $\frac{3}{4}$.

And $\frac{2}{3}$ fits between $\frac{1}{2}$ and $\frac{3}{4}$.



You can keep finding fractions between fractions. (In fact, the average of any two fractions is a fraction right between the two.)

It looks like we can just fill up the entire number line with fractions.



Let me say that again:

It looks like the entire number line fills up with fractions.

But we've just proved it doesn't!

There are numbers on the number line that aren't fractions –lots of them, in fact!

FACT: The fractions <u>don't</u> fill up the entire number line!

Whoa!

Self-Check 26.1 Does your brain now hurt?

a) Yes
b) No
c) My brain hurts too much to be able to answer this question.

I said at the very beginning of this book that I feel like math goes "beyond my humanness."

As a human, I feel I know what the counting numbers are. And I think I have a feel for fractions. But now we've just learned there are even more numbers than just the fractions. And, on top of that, I don't at all have a feel for these new numbers, what they are and how they work.

Self-Check 26.2 I said that I find math "powerful" and "inspiring" and "thrilling," and even "comforting" (though I am not sure if I am feeling comforted right now.)

Having gone on this crazy math journey with me playing with counting and arithmetic and dots-and-boxes, what are some words that describe how you feel about math right now?

Final question:

Self-Check 26.3 How awesome are you for having come to the end of this book?

a) Exceptionally incredibly fabulously awesome!

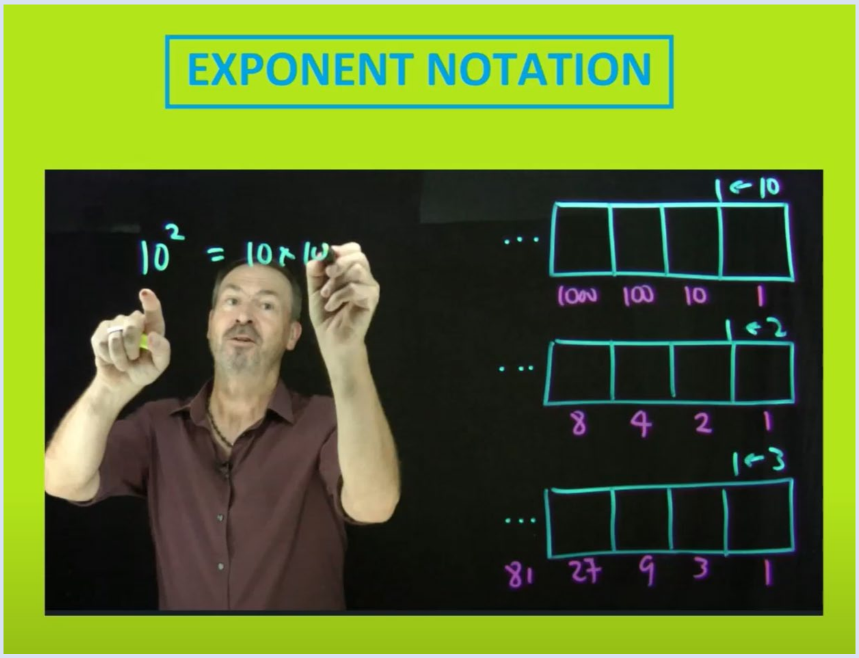
ANSWERS

Thank you for being your fabulous genuine human self and for being a great mathematician!

Some
HIGHSCHOOL ALGEBRA

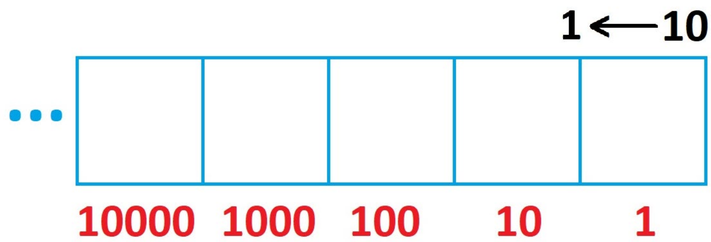
27. Exponent Notation

Here's video for this section.

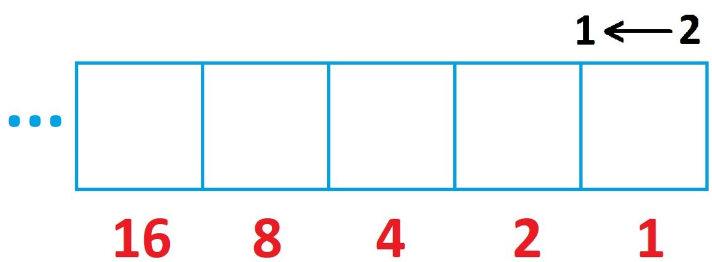


https://youtu.be/l5c7javvqDE

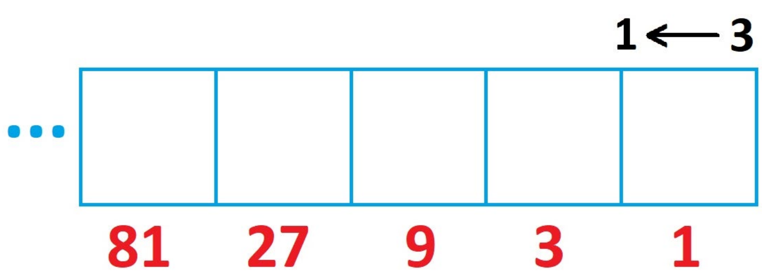
We’ve been playing almost exclusively with the $1 \leftarrow 10$ machine, with dots in boxes worth 1 or 10 (ten) or 100 (ten tens) or 1000 (ten groups of ten tens!), and so on.



But we could just as well have been playing in a $1 \leftarrow 2$ machine, with dots in boxes worth 1 or 2 (two) or 4 (two twos) or 8 (two fours), etc.



Or with a $1 \leftarrow 3$ machine with dots in boxes worth 1 or 3 (three) or 9 (three threes) or 27 (three nines), etc.



Or with any other machine.

It helps to have a quick notation for the types of numbers we see as dot values in these machines. And, luckily, mathematicians have one. They use superscripts to denote repeated multiplication and they call the notation *exponent notation*.

For example, 10^2 means "a product of two tens" and 10^5 means "a product of five tens."

$10^2 = 10 \times 10$ (This is one hundred.)

$10^5 = 10 \times 10 \times 10 \times 10 \times 10$. (This equals $100\,000$.)

Similarly, we have

$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$, a product of seven twos. (This equals 128.)

$3^4 = 3 \times 3 \times 3 \times 3$, a product of four threes. (This equals 81.)

Question 27.1

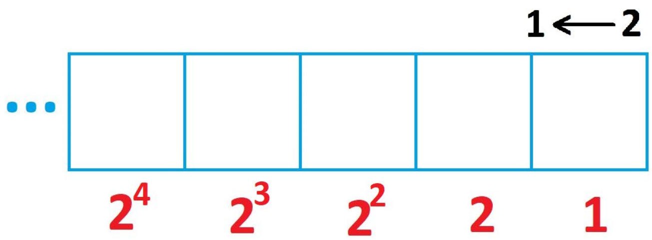
a) What is the value of 5^3?
b) What is the value of 3^2?
c) What is the value of $1^{3338289836382}$?

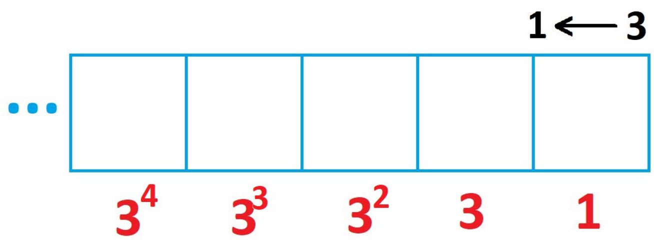
Your answers.

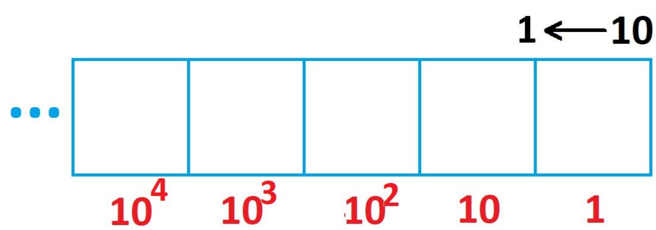
Question 27.2 What are the values of 2^2 and 2^3 and 2^4 and 2^5 and 2^6 and 2^7 and 2^8 and 2^9 and 2^{10}?

Your answers.

This notation allows us to write the place values of our machines a little more neatly and consistently.



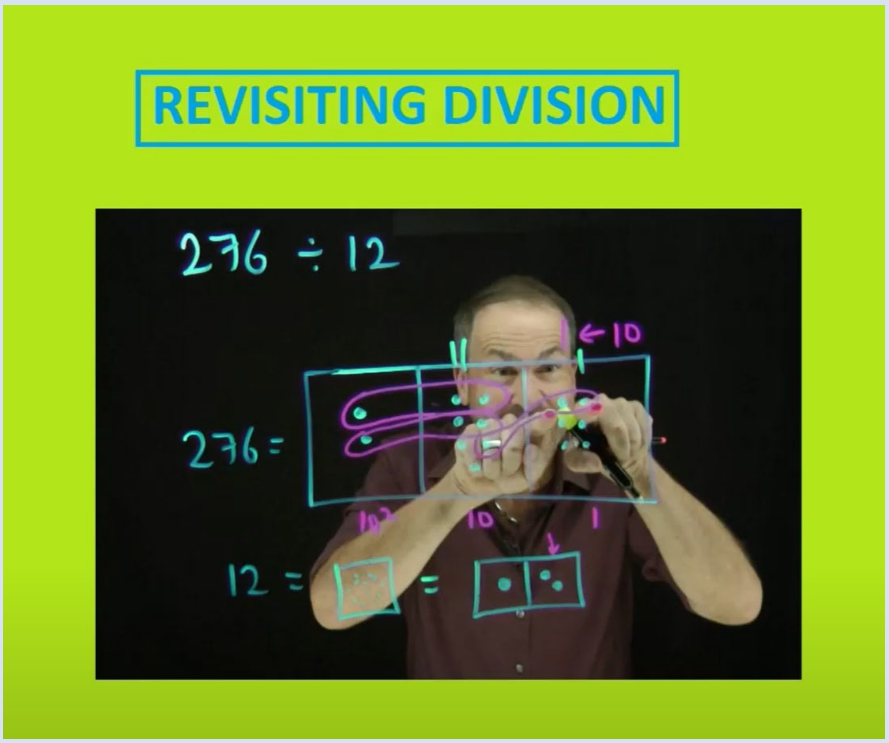




28. Revisiting Division

Let's revisit an example from the past.

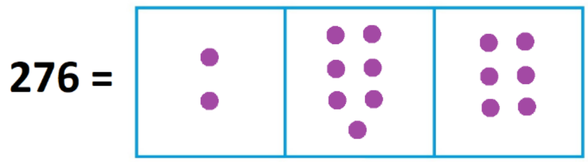
Here's video for this section.



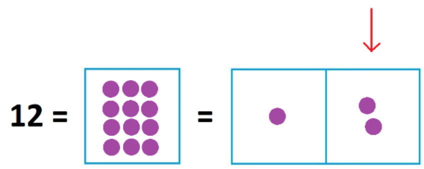
https://youtu.be/Fj8UoOs-ZEI

Back in section 17 we computed $276 \div 12$.

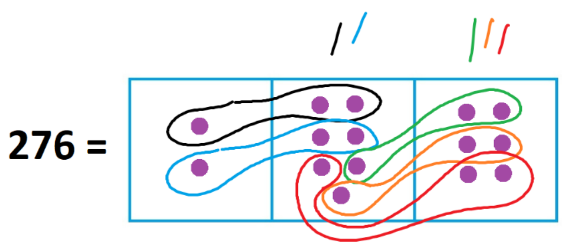
We started by drawing a picture of the number 276 in a $1 \leftarrow 10$ machine.



And we drew a picture of the number 12. It appears as one dot next to two dots, but it is really twelve dots in the rightmost box.



Then we identified groups of 12 in the picture of 276. And there are plenty of them: two at the tens level and three at the ones level. (Remember, the twelve dots appear in the right part of each loop: an explosion must have "spilled" one dot over to the left.)



This shows that our picture of 276 is actually a picture of 23, each of whose dots was replaced by 12. That is, we see that 276 as the number 23 multiplied by 12, and so

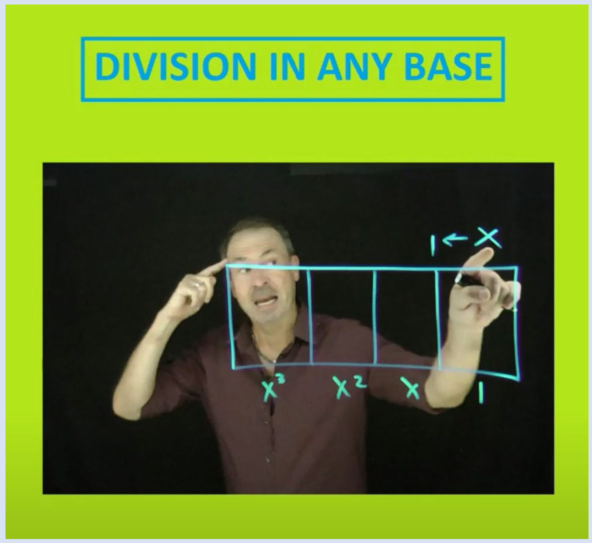
$$276 \div 12 = 23.$$

Question 28.1: Do you recall this process? Would you like to practice it by showing that $27999 \div 132$ equals 212 with remainder of 15?

Some space.

29. Division in Any Base

Here's video for this section.



https://youtu.be/Yx0XQQW1hF4

Let's now live up to the promise of Section 20. Let's see that we're really doing advanced algebra right now.

The only thing we have to realize is that there is nothing special about a $1 \leftarrow 10$ machine. We could do all our arithmetic in a $1 \leftarrow 2$ machine if we desired, or a $1 \leftarrow 5$ machine, or even a $1 \leftarrow 37$ machine. The math doesn't care in which machine we conduct it. It is only us humans with a predilection for the number ten that draws us to the $1 \leftarrow 10$ machine.

So, let's now be bold and do our work in <u>all</u> possible machines, all at once! That sounds crazy, but it is surprisingly straightforward.

What I am going to do is draw the picture of a machine, but I am not going to tell you which machine it is. It could be a $1 \leftarrow 10$ machine again, I am just not going to say. Maybe it will be a $1 \leftarrow 2$ machine, or a $1 \leftarrow 4$ machine or a $1 \leftarrow 13$ machine. You just won't know as I am not telling. It's the mood I am in!

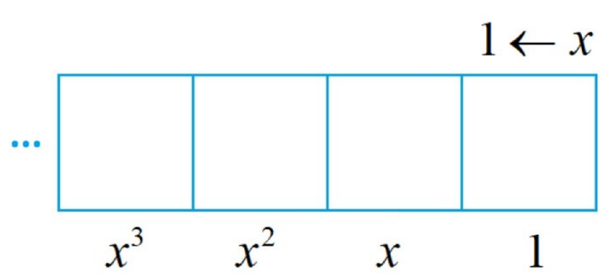
Now, in school algebra there seems to be a favorite letter of the alphabet to use for a quantity whose value you do not know. It's the letter x. Always the letter x. (It's a weird obsession.)

> **Question 29.1:** What can you find on the internet about why the letter *x* is the favored letter to represent an unknown quantity in mathematics? (Watch out! There are multiple thoughts, theories, and combinations of details. Don't fully believe the first explanation you encounter.)

What did you find?

So, let's work with an $1 \leftarrow x$ machine with the letter x representing some number whose actual value we do not know.

Following section 27, the place values of boxes in an $1 \leftarrow x$ machine will be 1 and x and x^2 and x^3, and so on.



As a check, if I do tell you that x actually is 10 in my mind, then the values $1, x, x^2, x^3$, ... match the numbers 1, 10, 100, 1000, …, which is correct for a $1 \leftarrow 10$ machine.

If, instead, I tell you that x is really 2 in my mind, then the values 1, x, x^2, x^3, ... match the numbers 1, 2, 4, 8, …, which is correct for a $1 \leftarrow 2$ machine.

This $1 \leftarrow x$ machine really is representing all machines, all at once!

Okay. Out of the blue! Here's an advanced algebra problem.

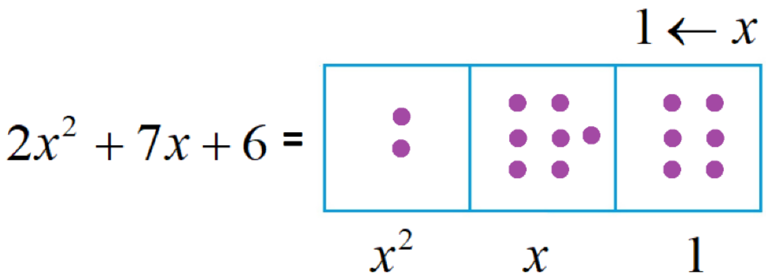
Compute $\left(2x^2 + 7x + 6\right) \div \left(x + 2\right)$.

Whoa!

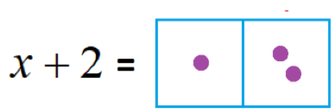
STEP 1: **Deep breath!** This does look scary, so take a deep breath.
STEP 2: **Do Something!**

We can at least draw a picture for the challenge using dots and boxes.

Here's what $2x^2 + 7x + 6$ looks like in an $1 \leftarrow x$ machine. It's two x^2s, seven xs, and six ones.

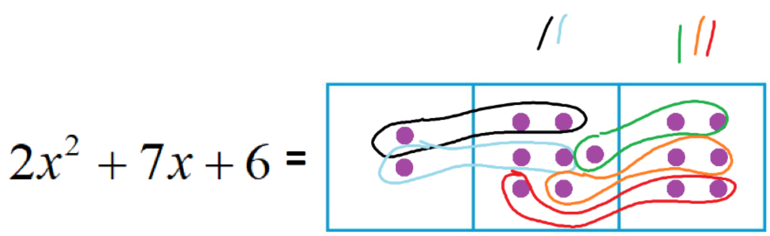


And here's what $x + 2$ looks like.



The division problem $(2x^2 + 7x + 6) \div (x + 2)$ is asking us to find copies of $x + 2$ in the picture of $2x^2 + 7x + 6$.

And we can find them!

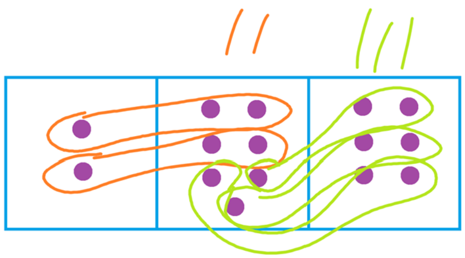


I see two copies of $x + 2$ at the x level and three copies at the 1 level. The answer is $2x + 3$.

And we have been here before!

Arithmetic

$$276 \div 12$$
$$= 23$$



Algebra

$$\left(2x^2+7x+6\right)\div(x+2)$$
$$=2x+3$$

IT'S THE SAME!

What's going on?

Suppose I told you that x really was 10 in my head all along.

Then $2x^2+7x+6$ is the number $2\times100+7\times10+6$, which is 276.

And x + 2 is the number 10 + 2, that is, 12.

So, we just computed 276 ÷ 12 and got the answer 2x + 3, which is $2\times10\ +\ 3=\ 23$.

That is, if I do announce that x was 10, then we just repeated a school arithmetic problem.

But if I tell you that x is actually, instead, 2 in my mind, then

$2x^2+7x+6=2\times4+7\times2+6$, which is 28,

$x+2=2+2$, which is 4,

and

$2x+3=2\times2+3$, which is 7,

and we've just ascertained that 28 ÷ 4 = 7, which is correct!

Doing division in an 1 ← x machine is really doing an infinite number of division problems all in one hit.

Whoa!

Question 29.2 What is $(2x^2+7x+6)\div(x+2)=2x+3$ saying if x is 5?

Your answer.

Question 29.3: Compute

$$(2x^3+5x^2+5x+6)\div(x+2)$$

in an $1 \leftarrow x$ machine. (You should get the answer $2x^2+x+3$.)

If I tell you x is 10 in my mind, can you see that this matches 2256 ÷ 12 = 213?

Your answer.

Quantities expressed in an $1 \leftarrow x$ machine are called **polynomials**. They are just like numbers expressed in base 10, except now they are "numbers" expressed in base x. (And if someone tells you x is actually 10, then they really are base 10 numbers!) Work in polynomial algebra is just a repeat of school arithmetic.

Question 29.4 Draw $x^3 + 2x^2 + 3x + 1$ in an $1 \leftarrow x$ machine.
Add $3x^3 + 7x^2 + 4x + 1$ to the picture.
What, then, is $\left(x^3 + 2x^2 + 3x + 1\right) + \left(3x^3 + 7x^2 + 4x + 1\right)$?

If x is the number 10 throuout all this, what ordinary arithemetic problem have you conducted?

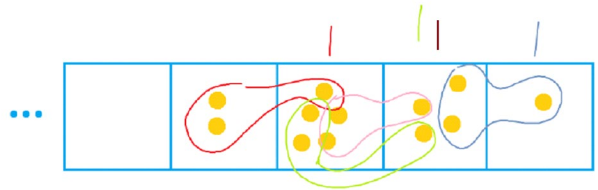
Your answer.

Question 29.5 What do you think $\left(5x^2 + 4x + 9\right) - \left(2x^2 + x + 6\right)$ equals?

What does this translate to if x is 10?

Your answer.

Question 29.6 Kennedy drew this picture. She was working on a division problem.



a) If this is a picture of a division problem in a $1 \leftarrow 10$ machine, what division problem is she conducting and what is its answer?
b) If this is a picture of a division problem in an $1 \leftarrow x$ machine, what division problem is she conducting and what is its answer?

Your answer.

Question 29.7

a) Compute $(2x^4 + 3x^3 + 5x^2 + 4x + 1) \div (2x + 1)$.

b) Compute $(x^4 + 3x^3 + 6x^2 + 5x + 3) \div (x^2 + x + 1)$.

If x is 10 in both these problems what two division problems in ordinary arithmetic have you just computed?

Your answer.

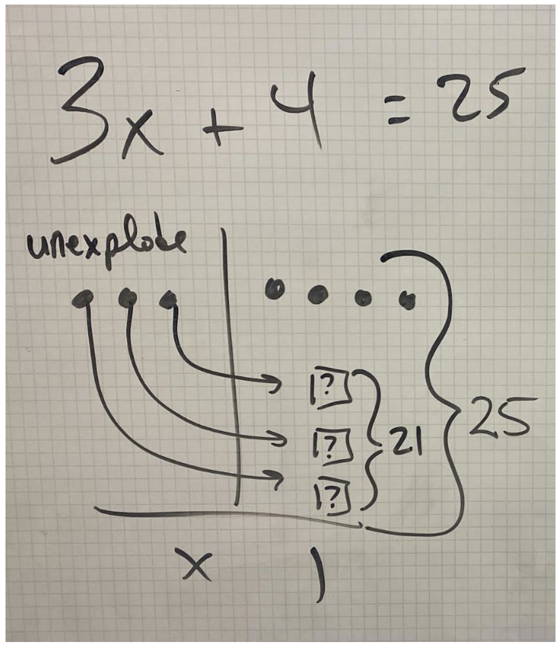
Question 29.8

a) Show that $(x^4 + 4x^3 + 6x^2 + 4x + 1) \div (x + 1)$ equals $x^3 + 3x^2 + 3x + 1$.
b) What is this saying for $x = 10$?
c) What is this saying for $x = 2$?
d) What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11?

Your answer.

Challenge Question 29.9 Washington state middle-school teacher David Buitenveld shared with me an idea for solving some equations using an $1 \leftarrow x$ machine.

Can you make sense of his approach from this picture he shared?



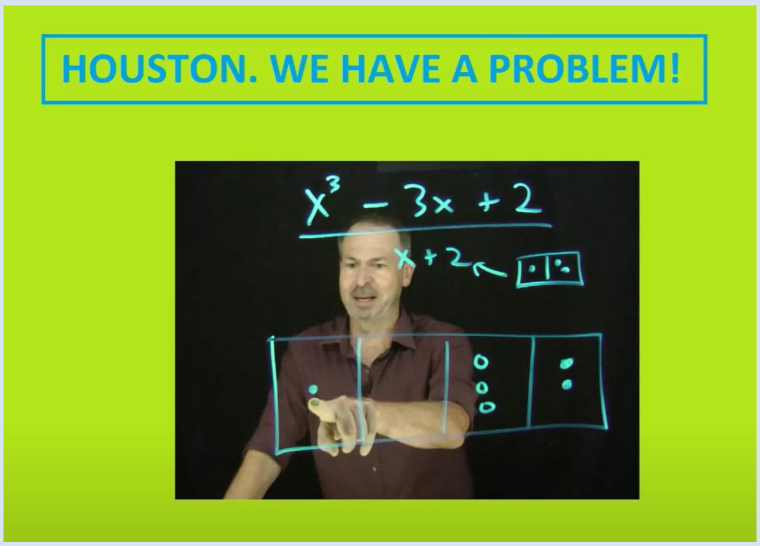
Your thoughts.

30. A Problem!

Okay. Now that we are feeling really good about doing advanced algebra, I have a confession to make. I've been hiding a problem. A serious problem!

I've been choosing examples that only use dots. What about tods?

Here's video for this section.

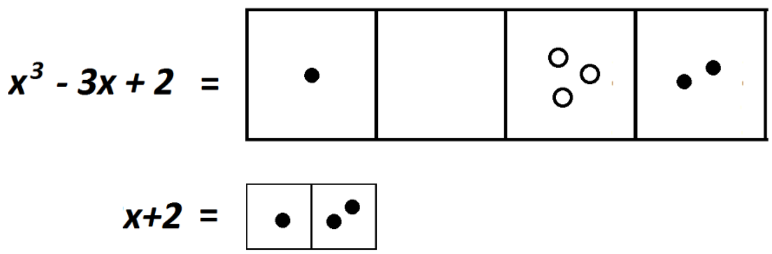


https://youtu.be/1PI5YH794Xg

Consider, for example,

$$\left(x^3 - 3x + 2\right) \div \left(x + 2\right).$$

Here's what we have in an $1 \leftarrow x$ machine.



We are looking for one dot next to two dots in the picture of $x^3 - 3x + 2$. But we don't see any!

What can we do besides weep a little?

Do you have any ideas?

Maybe you are thinking to unexplode one of the leftmost dots. That is a great idea! It really is. But it has a snag: we don't know the value of x so we don't know how many dots to draw when we unexplode. Bother!

We need some amazing flash of insight for something clever to do. Or maybe polynomial problems with negative numbers just can't be solved with this dots-and-boxes method

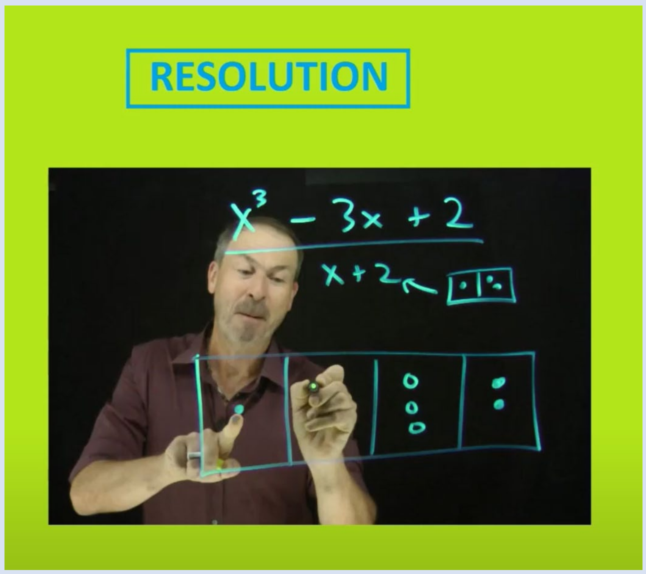
Question 30.1 What do you think? Have you a flash of insight?

Thoughts?

31. Resolution

Let’s see if we can bring resolution to matters.

Here’s video for this section.

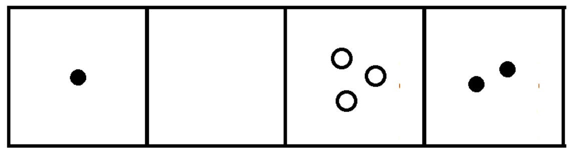


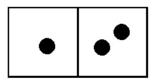
https://youtu.be/Cb2vqDU_6Bg

We are stuck on computing the following.

$$\left(x^3 - 3x + 2\right) \div \left(x + 2\right)$$

We have this $1 \leftarrow x$ machine picture.

$x^3 - 3x + 2 =$ 

$x+2 =$ 

We seek copies of $x + 2$—one dot next to two dots—anywhere in the picture of $x^3 - 3x + 2$. But we don't see any!

And we can't unexplode dots to help us out as we don't know the value of x. (We don't know how many dots to draw when we unexplode.)

The situation seems hopeless at present.

But I have a piece of advice for you, a general life lesson in fact. It's this:

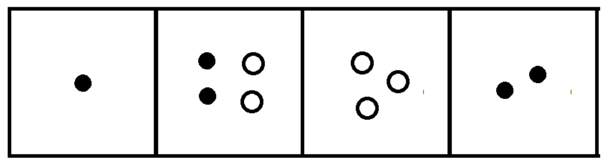
IF THERE IS SOMETHING IN LIFE YOU WANT … MAKE IT HAPPEN! (And deal with the consequences.)

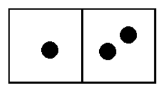
Right now, is there anything in life we want?

Look at that single dot way out to the left. Wouldn't it be nice if we had two dots in the box next to it, to make a copy of $x + 2$?

So, let's just put two dots into that empty box! That's what I want, so let's make it happen!

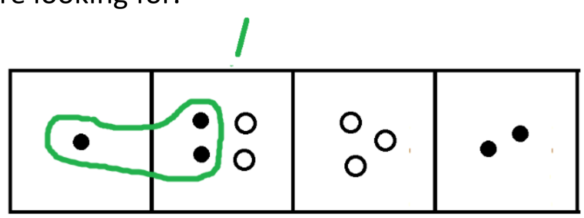
But there are consequences: that box is meant to be empty. And in order to keep it empty, we can put in two tods as well!

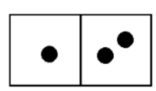
$x^3 - 3x + 2 =$ 

$x+2 =$ 

Brilliant!

We now have one copy of what we're looking for.

$x^3 - 3x + 2 =$ 

$x+2 =$ 

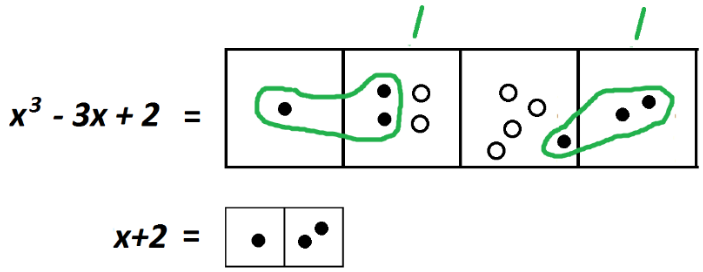
But there is still the question: Is this brilliant idea actually helpful?

Hmm.

Well. Is there anything else in life you want right now? Can you create another copy of $x + 2$ anywhere?

I'd personally like a dot to the left of the pair dots in the rightmost box. I am going to make it happen!

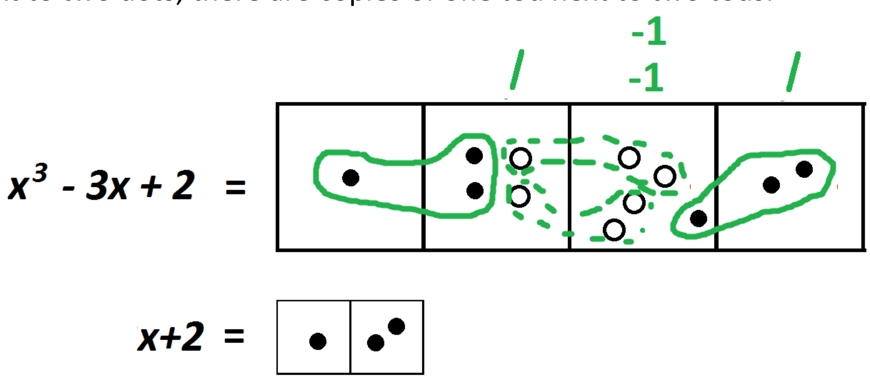
I am going to insert a dot and tod pair. Doing so finds me another copy of $x + 2$.



This is all well and good, but are we now stuck? Maybe this brilliant idea really just isn't helpful.

Ooh! But stare at this picture for a while. Do you notice anything?

Look closely and we start to see copies of the exact opposite of what we're looking for! Instead of one dot next to two dots, there are copies of one tod next to two tods.



Whoa!

And how do we read the answer? We see that $(x^3 - 3x + 2) \div (x + 2)$ is $x^2 - 2x + 1$.

Fabulous!

So actually, we *can* do all polynomial division problems with this dots-and-boxes method, even ones with negative numbers.

Question 31.1 Try computing $(x^4-1)\div(x-1)$. Can you get the answer x^3+x^2+x+1?

Some space.

Question 31.2 Show that $(x^6+1)\div(x^2+1)$ equals x^4-x^2+1.

Some space.

Question 31.3: Play with $\left(4x^4 - 7x^3 + 10x^2 - 4x + 2\right) \div \left(x^2 - x + 1\right)$ to see that it equals $4x^2 - 3x + 3$ with a remainder of $2x - 1$ yet to be divided by $x^2 - x + 1$.

People typically write this answer as follows, using a fraction notation for division.

$$\frac{4x^4 - 7x^3 + 10x^2 - 4x + 2}{x^2 - x + 1} = 4x^2 - 3x + 3 + \frac{2x - 1}{x^2 - x + 1}.$$

Some space.

Question 34.4 If you can do this problem—written with fraction notation for division—you can probably do any polynomial division problem!

$$\frac{4x^5 - 2x^4 + x^3 + x^2 + x - 2}{x^2 - x + 1}$$

(You should get the answer $4x^3 + 2x^2 - x - 2$.)

Some space.

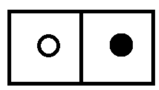
Optional Challenge Question 31.5 SOMETHING WILD!

Use an $1 \leftarrow x$ machine to compute

$$\frac{1}{1-x}.$$

Here we are being asking to divide the simple polynomial 1, which is just a single dot

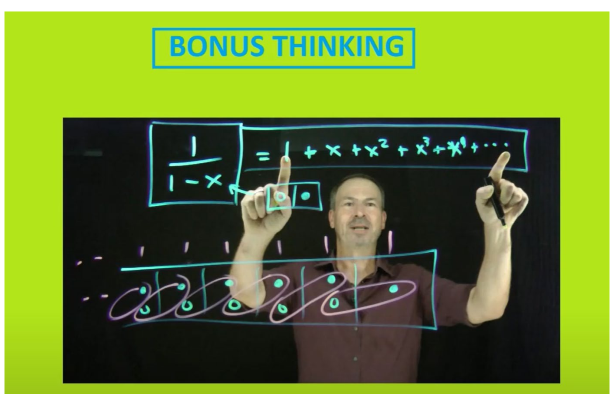
by $1-x$ which looks like this, one tod and one dot.



Is it possible to do this division problem?

EXTRA: Computing $\frac{1}{1-x-x^2}$ leads to something mighty curious. What famous sequence of numbers do you see appear in your work?

Here's a video of me going through these two questions.



https://youtu.be/xDjg9Rx-2U4

Some space for you.

SOLUTIONS

SOLUTIONS

1.1 My notation "$1 \leftarrow 2$" is trying to show that two dots in a box change into one dot, one place to the left. Do you think my notation is okay?

1.2 You do get the same code no matter when you choose to do particular explosions.

1.3 It really is 1101. Put thirteen dots in the rightmost box of a machine and try to be neat and careful about erasing pairs of dots, replacing each pair with a dot one place to their left. (It is easy to get messy and lose track of what is happening!)

1.4 They are:

11: 1011	**16: 10000**
12: 1100	**17: 10001**
13: 1101	**18: 10010**
14: 1110	**19: 10011**
15: 1111	**20: 10100**

1.5 10011 is the number nineteen.

2.1 It's 10001.

2.2 $30 = 16 + 8 + 4 + 2$ and it's code is 11110.

2.3 It is the number $16 + 8 + 1$, which is twenty-five.

2.4 We have $50 = 32 + 16 + 2$ and it's code is 110010. (This is the same as what you get in taking Lela's picture and doing all the explosions!)

2.5 11111 corresponds to the number $16 + 8 + 4 + 2 + 1$, which is 31.

2.6 Ari's reasoning is good and the smallest five-digit code is 10000. This the code for sixteen.

2.7 10011 is the number $16 + 2 + 1 = 19$.

2.8 $15 = 8 + 4 + 2 + 1$ and has code 1111.

$30 = 16 + 8 + 4 + 2$ and code 11110.

2.9 A bicycle is a vehicle with two wheels; Binoculars are viewing devices with two lenses; To bisect something is to cut it into two equal parts; A biped is a creature that walks on two legs; A bivalve is a mollusk with shells composed of two parts.

3.1 128

3.2 $32+8+4+1=45$

3.3 $64+32+4=100$

3.4 It is 11001000.

3.5 $1,048,576$

3.6 It's 1023.

3.7 There is! The first one is the forty-seventh number in the list of doubling numbers:

$$70,368,744,177,664.$$

3.8

500 has code: 111110100.

999 has code 1111100111.

The biggest number you can make is 1111111111 which corresponds to 1023.

4.1 My codes are all correct. (Phew!)

4.2 Thirteen is 111.

4.3 There are three dots among the four in 20142 that would explode to make the code 20212. (What number is this?)

4.4 a) Amit is correct, that there are zero dots in the fourth box to the right when you work out the code for the number sixteen. Writing 0121 is technically correct. So too is writing 00121 and 00000121. It's just that most people don't bother writing zeros in the front of the codes. They are not helpful.

b) Alas, Shania is incorrect. The code for sixteen really does have 1 dot in the rightmost box, 2 dots the next second-right box, and 1 dot in the third rightmost box. The code can't be 1210.

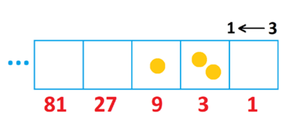
4.5 It's twenty-six.

6.1 It's thirty-seven.

6.2 Yes Yes Yes Yes

6.3 243

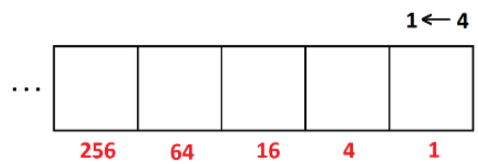
6.4 You really do get 120.



6.5 You do indeed end up with one dot in the 27's place. A dot there really is worth twenty-seven 1s.

6.6 It's sixty-five.

6.7 a) Boxes have the following values:



b) The number twenty-nine has code **131** in a 1 ← 4 machine.

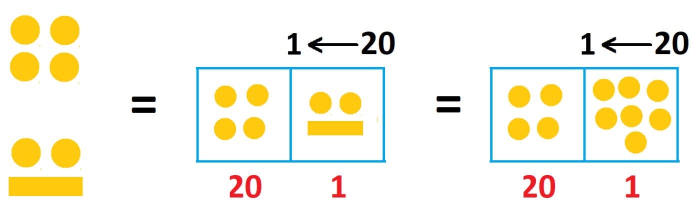
c) Thirty. (This is one more than the code for twenty-nine!)

7.1 **.** a) **31** b) **11111** c) **1011** d) **133** e) **111** f) **51** g) **10**

Did you notice that thirty-one has codes that are just a string of ones in base 2 and base 5? (It's also a string on ones in base 30. Do you see why?) Can you find a number which has codes a string of ones in four different bases?

7.2 They were likely thinking of fingers AND toes!

7.3 In the Mayan system a dot represents 1 and a bar represents 5. So, we see in the picture four dots (4) on top of two dots and a bar (7). This is four twenties and seven.



7.4 Please do look at the document and contribute to it if your language is missing.

7.5 Do you count to twelve this way?

7.6 12

7.7 12

7.8 The first clocks humans used were indeed sundials, which, of course, work only during daylight hours. The ancient Egyptians divided the day into ten main hours and two extra twilight hours—early morning and early evening. With the invention of water clocks and mechanical clocks people could start measuring time during the night as well. Since the day was divided into 12 hours, they divided the night into 12 hours as well. This is why we, today, say “There are 24 hours in a day.” But this is really coming from “There are 12 hours in, literally, a day.”

7.9 A half of 12 and a third of 12 and a quarter of 12 are all nice whole numbers. (6,4, and 3). But a half, third, and a quarter of 10 are getting into very unfriendly numbers! (Well, a “half of 10” is okay—it’s five—but the other two aren’t so nice!)

7.10 If Martians focus on one hand, they might go with base six. But, if like humans who focus on two hands, they might go base twelve. And if they are thinking fingers and toes (do they have six toes on each of two feet), they might go base twenty-four.

7.11

A a) *ddeg* is 10 b) 1, 3, 4
c) You might think *ar* means “and,” but it translates as “on.”
d) 15 is actually Pymtheg, not “Pump ar ddeg” like you’d expect.

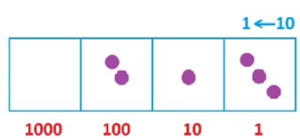
B a) *toista* possibly means “and ten.” b) 1 = yksi, 3 = kolme, 4 = neljä
c) 12 = kaksitoista
d) 20 is kaksikymmentä, which is sort of like “2 tens.”

C a) Shí means ten. b) 1 = yī, 3 = sān, 4 = sì
c) 40 = Sì shí [4 10
43 = Sì shí sān [4 10 3]
44 = Sì shí sì [4 10 4]

8.1 I thought of “foreign.”

8.2 We say “two hundred and thirteen” or “two hundred thirteen.” We do say that there are two hundreds, we don’t say that there is one ten, but we do say there is an extra thirteen to go with the two-hundred. It’s a bit confusing!

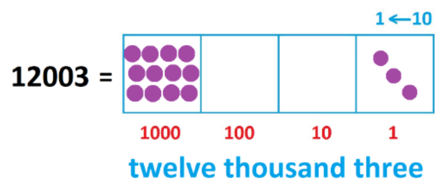
8.3 Katya is right. What we are saying is her picture. But then, we normally expect ten dots to explode to become one dot, one place to the left. So, what we say is different from what we do!

213 =


8.4 The number 2|11|3 is the number 313.

8.5 "Twelvety" is twelve tens. That's 120.
"Eleventy" is eleven tens. That's 110.
"Thirteenty," if you were to say it, is thirteen tens. That's 130.

8.6



8.7 I would love to know how this number is said in other languages. Do people say "one ten thousand, two thousand, and three" or do they say "twelve thousand, and three," or something else?

8.8 a) Twelve-thousand twelve-hundred, twelvety, twelve.

b) Indeed. The only part of this that sounds weird in English is "twelvety."

c) It's 13,332.

This read as "thirteen-thousand, three-hundred, thirty, two." We don't say "one ten-thousand, three-thousand, three-hundred, threety, two" as we should!

9.1 You get the same answer if you go right to left. Perhaps this problem is too nice. Doing a trickier one might show that going left-to-right is a problem?

9.2 If you keep reading, you'll see I answer my own question on that page.

9.3 $148+323=4\,|\,6\,|\,11=471$

$567+271=7\,|\,13\,|\,8=838$

$377+188=4\,|\,15\,|\,15=5\,|\,5\,|\,15=565$

$582+714=12\,|\,9\,|\,6=1\,|\,2\,|\,9\,|\,6=1296$

$310462872+389107123=6\,|\,9\,|\,9\,|\,5\,|\,6\,|\,9\,|\,9\,|\,9\,|\,5=699569995$

$87263716381+18778274824=9\,|\,15\,|\,9\,|\,13\,|\,11\,|\,9\,|\,8\,|\,10\,|\,11\,|\,10\,|\,5=\ldots\;=106041991205$

9.4 Can you copy the explanation I gave on the previous pages? Do you have a better explanation?

9.5 a) The largest sum one computes in a column of two numbers is 9+9, plus maybe a 1 from a previous explosion, giving 9+9+1=19. This requires carrying only a 1.

b) 199 + 9, for example, works.

c) 10009+2, for example works.

d) 9999999999+1 works.

10.1 You really do get 8352.

10.2

a) $2784 \times 2 = 4 \,|\, 14 \,|\, 16 \,|\, 8 = 5568$
b) $2784 \times 4 = 8 \,|\, 28 \,|\, 32 \,|\, 16 = 11136$
c) $2784 \times 5 = 10 \,|\, 35 \,|\, 40 \,|\, 20 = 13920$
d) $2784 \times 10 = 20 \,|\, 70 \,|\, 80 \,|\, 40 = 27840$

11.1 Do you feel you can explain it?

11.2 a) 4760   b) 47600. Multiplying by one hundred is the same as multiplying by ten (which gives 4760) and then by ten again (which gives 47600).

11.3 a) $313 \times 10 = 3130$ b) $97800 \times 10 = 978000$

12.1 69|161 really is 851.

12.2 My pictures are not drawn to scale. Does that matter?

	30	1
40	1200	40
1	30	1

31 x 41 = 1200 + 40 + 30 + 1 = 1271

	100	30	4
10	1000	300	40
2	200	60	8

134 x 12 = 1000 + 300 + 40 + 200 + 60 + 8
= 1608

12.3

$$37 \times 11 = 33 \mid 77 = 40 \mid 7 = 4 \mid 0 \mid 7 = 407$$

$$375 \times 11 = 33 \mid 77 \mid 55 = 4125$$

12.4 Is it making some sense?

12.5

a) We have $23 \times 24 = 400 + 60 + 80 + 12 = 552$.

	20	3
20	400	60
4	80	12

b) The "8" is really $4 \times 20 = 80$. It's eight tens.

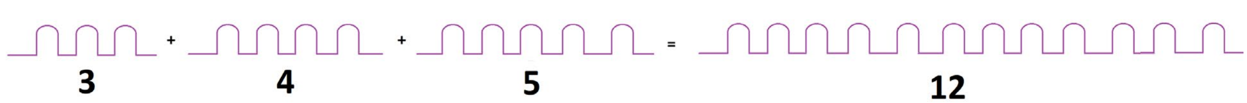
c) The "4" is really 20×20. It's four hundreds.

d) Are you starting to see that we're just using rectangles without drawing rectangles?

12.6 I got it!

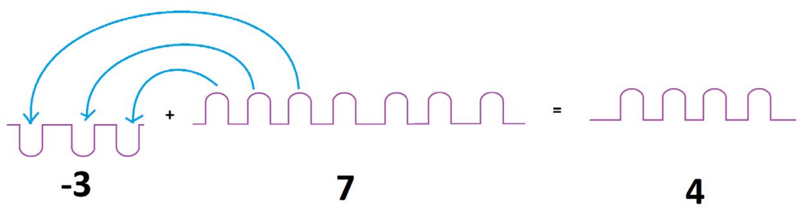
```
              3   1   2   5
      x           8   3   2
=           6 | 2 | 4 | 10
        9 | 3 | 6 | 15 | 0
   24 | 8 | 16 | 40 | 0 | 0
   24 | 17 | 25 | 48 | 19 | 10
= 2 | 6 | 0 | 0 | 0 | 0 | 0
```

13.1



13.2 Two piles and six holes combine to leave four holes: $2 \;+\; -6 \;=\; -4$. (Can you draw a picture of this?)

13.3



13.4

a) and c) $6 \;+\; -7 \;+\; -1$. This is six piles and seven holes and one hole. It gives two holes, that is, -2.

b) and c) $5-1+3-2-1=4$. This is five piles and one hole and three piles and two holes and one hole. They leave four piles.

13.5 I personally wish we didn't use a little "negative sign" for the opposite numbers. I think it is just confusing! (Or is that just me?)

14.1 Do you see that three dots annihilate with three tods to leave just two tods behind?

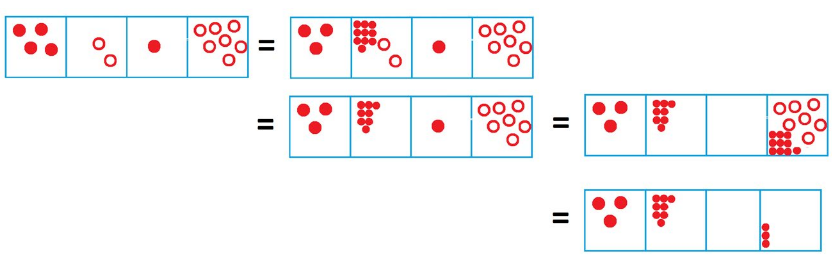
14.2 In a picture of one dot and five tods there would be just one annihilation—POOF!—to leave behind four tods: $1 \;+\; -5 = -4$. (Can you draw the picture?)

14.2 One thousand tods would "cancel" a thousand of the 2000 dots. We'll be left with $1000+65$ dots.

15.1 Read on!

15.2 Is it possible to describe a sound effect in words?

15.3



15.4 They really are. Here are the unexplosions to do:

$$2\,|-3\,|-5 = 1\,|\,7\,|-5 = 1\,|\,6\,|\,5$$

(A dots-and-boxes picture would help. Can you draw one?)

15.5 I am guessing that Raj realizes that the 2 in 2|-3|-5 is really 200, the -3 is really -30, and the -5 is, well, -5. So, the answer is 200-30-5, which is 165.

15.6

$$\begin{aligned} 6328-4469 &= 2\,|-1\,|-4\,|-1 \\ &= 1\,|\,9\,|-4\,|-1 \\ &= 1\,|\,8\,|\,6\,|-1 \\ &= 1\,|\,8\,|\,5\,|\,9 \quad = 1859 \end{aligned}$$

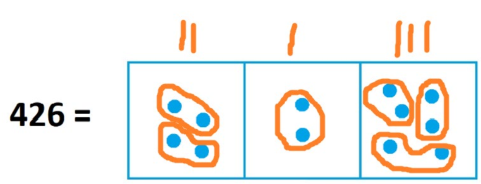
$$\begin{aligned} &78390231-32495846 \\ &\quad = 4\,|\,6\,|-1\,|\,0\,|-5\,|-6\,|-1\,|-5 \\ &\quad = 4\,|\,5\,|\,9\,|\,0\,|-5\,|-6\,|-1\,|-5 \\ &\quad = 4\,|\,5\,|\,8\,|\,10\,|-5\,|-6\,|-1\,|-5 \\ &\quad = 4\,|\,5\,|\,8\,|\,9\,|\,5\,|-6\,|-1\,|-5 \\ &\quad = 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,4\,|-1\,|-5 \\ &\quad = 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,3\,|\,9\,|-5 \\ &\quad = 4\,|\,5\,|\,8\,|\,9\,|\,4\,|\,3\,|\,8\,|\,5 \\ &\quad = 45894385 \end{aligned}$$

I personally find it much easier to do the unexplosions from left to right.

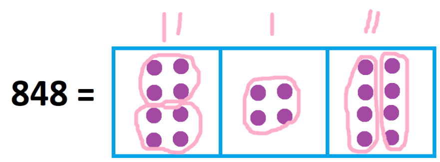
15.7 The answer is $-5\,|-3\,|\,1$, which is -500 and -30 and 1, which makes -529. (Is this what a calculator gives?)

Or, maybe you might say that $-5\,|-3\,|\,1$ is the opposite of $5\,|\,3\,|-1$ (which is 529). So, the answer is -529.

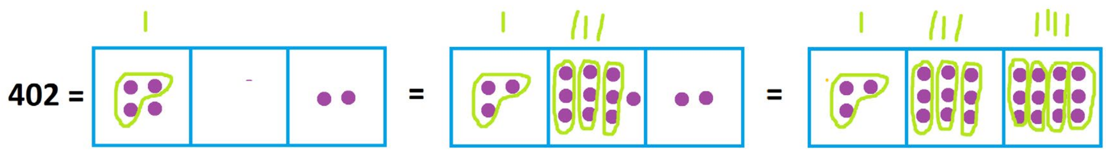
16.1



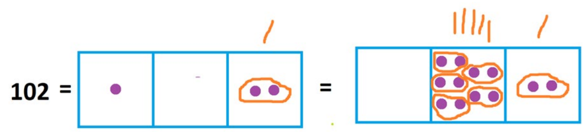
16.2



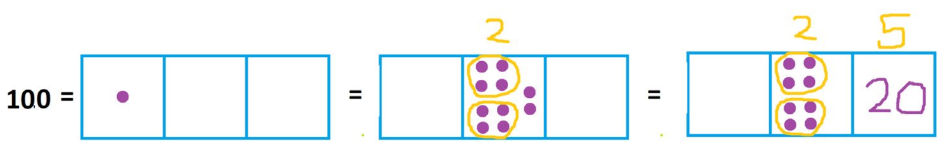
16.3



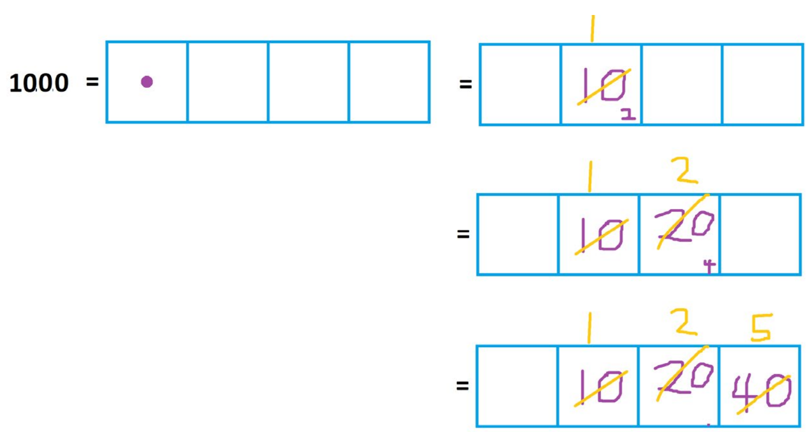
16.4



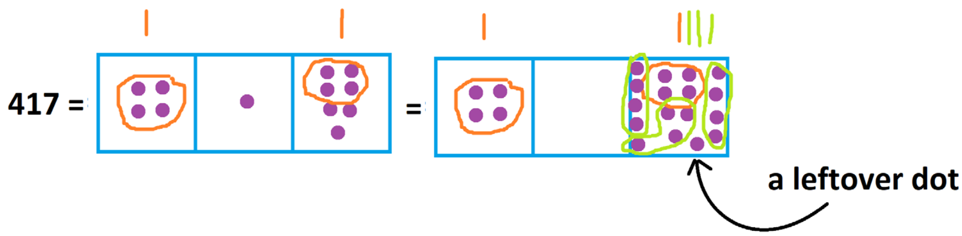
16.5 I am starting to get tired of drawing dots.



16.6 Can you see what I am doing here as I hunt for groups of eight?



16.7 We see that $417 \div 4 = 104$ with a remainder of 1.



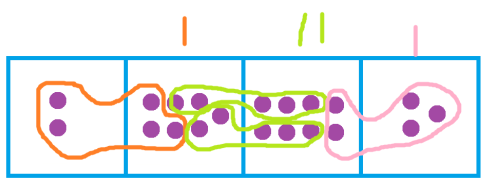
17.1

a) Jad first shows that each loop is really a set of 12 dots. It's an explosion from that group of 12 that spills a dot over one place (leaving two dots behind).

b) Jad is now trying to show that each group of 12 is really coming from a single dot that got multiplied by 12. In fact, it is each dot in a picture of 23 that got multiplied by 12. That is, a picture of 276 is really a picture of 23 that got multiplied by 12.

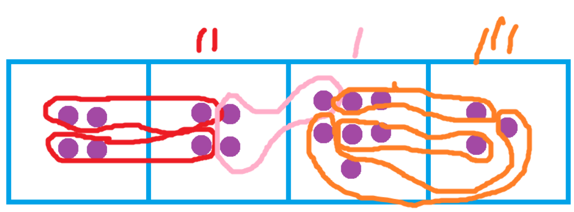
17.2 Really do try to find the specific parts that confuse you. And really do try to think of a question to ask about each of these parts. (Sometimes just doing this is enough to make things "click." And if not, now you know what to ask a friend or a teacher.)

17.3

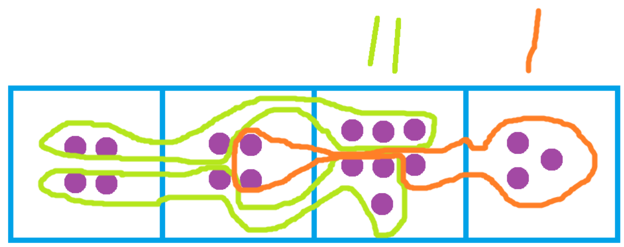


17.4

a)



b)



17.5 Everything Ricky has done is absolutely beautiful and correct.

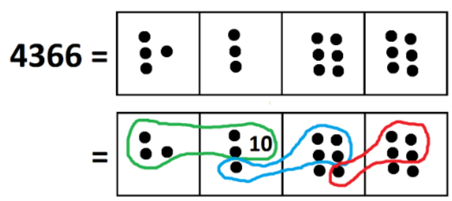
17.6 Read the next page after you try.

17.7 Can you make sense of it? Perhaps drawing loops over my picture will help.

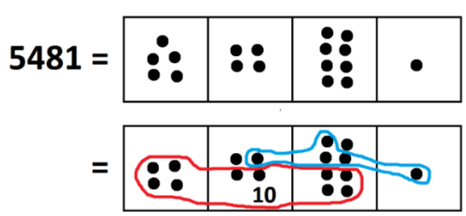
18.1 Hopefully, by doing the two different methods of calculation side by side, you are starting to see how they are doing the same thing.

19.1 We see 2789 ÷ 11 = 253 with a remainder of 6. That is, $2789 \div 11 = 253 + \frac{6}{11}$.

19.2 $4366 \div 14 = 311 + \frac{12}{14}$.



19.3 $5481 \div 131 = 41 + \frac{110}{131}$.



19.4

a) 4840 ÷ 4. This equals 1210.

b) 721 ÷ 7. This equals 103. (One needs to unexplode two dots.)

c) 126 ÷ 6. This equals 21. (Unexplode one dot.)

d) 126 ÷ 3. This equals 42. (Unexplode one dot.)

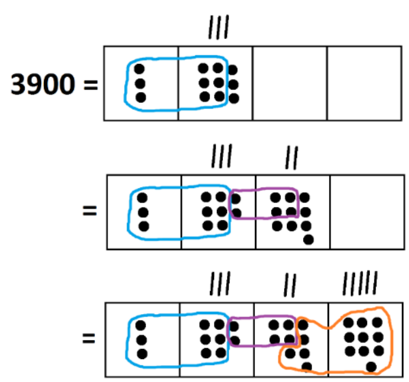
e) 126 ÷ 2. This equals 63. (Unexplode one dot.)

f) 126 ÷ 1. This equals 126. (There is one group of one at the hundreds level, two at the tens level, and six at the ones level!)

g) 3641 ÷ 11. This equals 331.

h) 3632 ÷ 11. This equals 331 with a remainder of 1, or $331\frac{1}{11}$.

i) 3639 ÷ 11. This equals 331 with a remainder of 8, or $331\frac{8}{11}$ .j) 3900 ÷ 12. This equals 325. (I got efficient with my loops.)



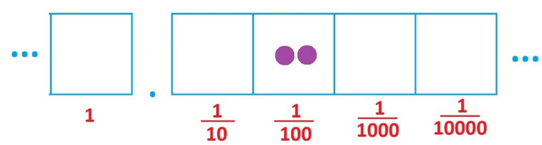
k) 100 ÷ 9. This equals 11 with a remainder of 1, or $11\frac{1}{11}$.

l) 1000000000 ÷ 9. This equals 111111111 with a remainder of 1, or $111111111\frac{1}{9}$.

21.1 I don’t think there is a name for such a point in other bases. But for a base-two machine perhaps “bimal point” would be a good thing to call it?

21.2 Many countries in the world use a comma instead of a point! (And where we use commas to separate groups of three digits, they use points! For example, “twelve-thousand, three-hundred thirty three point two” is written 12.333,2.)

21.3

0.02 = 

21.4 It’s $\frac{6}{10}$, six tenths. (Which happens to be the same as $\frac{3}{5}$.)

21.5 Saying “six tenths” is saying what it really is. That seems better to me.

21.6 I hope you do.

21.7 a) $0.009=\frac{9}{1000}$ b) $0.26=\frac{26}{100}$ (two unexplosions)

c) $0.3007=\frac{3007}{10\ 000}$ (I imagined a lot of unexoplosions here!)

21.8 Both are indeed correct. You need to do some explosions or unexplosions to see that one is the same as the other.

22.1 Now 0.03 is $\frac{3}{100}$. Multiplying this by ten gives $\frac{3}{10\times \cancel{10}}\times \cancel{10}=\frac{3}{10}$, which is 0.3.

22.2

10 x …			2	2	.	3	7			…
	1000	100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$	

= …			20	20	.	30	70			…
	1000	100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$	

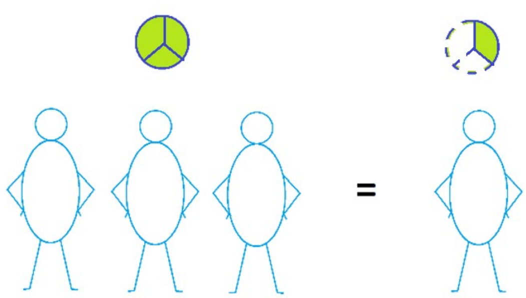
After explosions, this is

…		2	2	3	.	7				…
	1000	100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$	

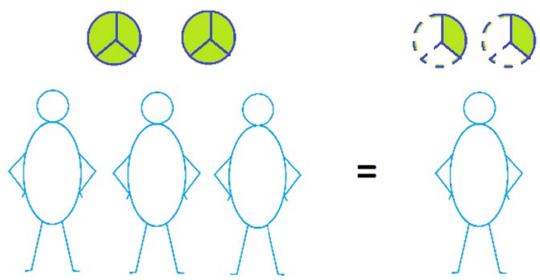
22.3 Multiplying 22.37 by one-hundred is the same as first multiplying it by 10 to get 223.7, and then multiplying that by 10, which would give 2237. (One can draw a dots-and-boxes to show that 223.7×10 is 2237.)

23.1 20 pies are being shared equally among 5 students to give 4 pies per student.

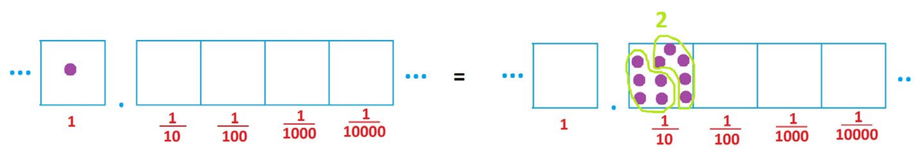
23.2 Each student really does get a third of the pie, just as we were taught what a third is in early grades.



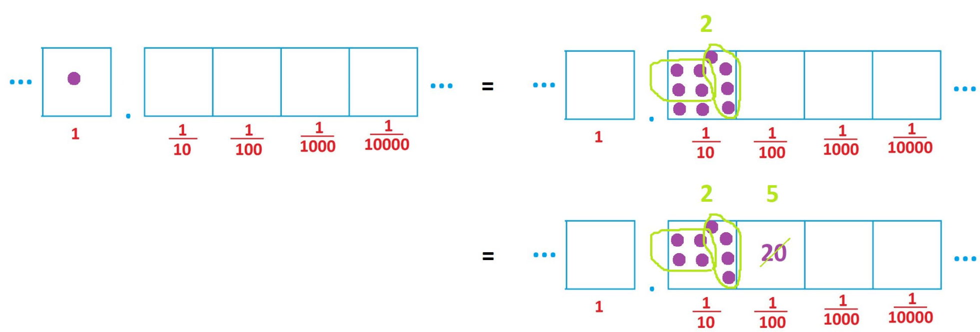
23.3 Two pies for three students gives each student a third of a pie, twice, That's two-thirds!



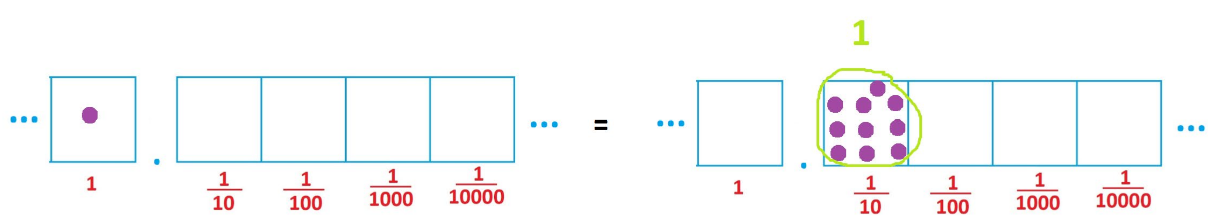
24.1 $\frac{1}{5} = 0.2$



24.2 $\frac{1}{4} = 0.25$

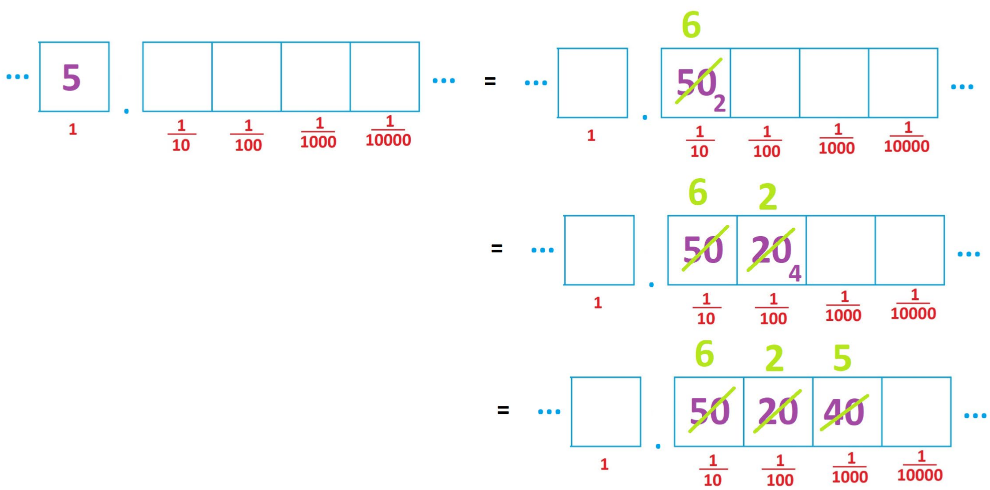


24.3 The decimal 0.1 is $\frac{1}{10}$. So, of course, $\frac{1}{10} = 1 \div 10$, has to give 0.1. (And it does!)



24.4 $\frac{125}{1000} = \frac{25 \times 5}{200 \times 5} = \frac{25}{200} = \frac{5 \times 5}{40 \times 5} = \frac{5}{40} = \frac{1 \times 5}{8 \times 5} = \frac{1}{8}$.

24.5

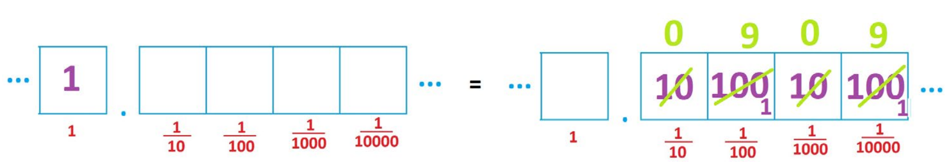


24.6 I am curious about your thoughts here.

24. 7 $\frac{1}{9} = 0.111111\cdots$. (Hopefully you see this in a dots-and-boxes picture.)

24.8 $\frac{5}{6} = 0..866666\cdots$. (Hopefully you see this in a dots-and-boxes picture.)

24.9



24.10 The fractions $\frac{1}{3}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}$, and $\frac{1}{12}$ have infinitely long decimal representations that repeat.

25.1 a) $41.3737373737\cdots$ b) $534.6\overline{012}$

25.2 a) $0.4\overline{9305}$ b) $2.35\overline{517}$ c) $719.320\overline{65783}$

25.3 It's $0.2\overline{635089}$.

$$
\begin{array}{cccccccccccccccccc}
 & 0. & 1 & 2 & 5 & 1 & 2 & 5 & 1 & 2 & 5 & 1 & 2 & 5 & 1 & 2 & 5 & \ldots \\
+ & 0. & 1 & 3 & 8 & 3 & 8 & 3 & 8 & 3 & 8 & 3 & 8 & 3 & 8 & 3 & 8 & \ldots \\
\hline
= & 0. & 2 & 5 & 13 & 4 & 10 & 8 & 9 & 5 & 13 & 4 & 10 & 8 & 9 & 5 & 13 & \ldots \\
= & 0. & 2 & 6 & 3 & 5 & 0 & 8 & 9 & 6 & 3 & 5 & 0 & 8 & 9 & 6 & 3 & \ldots
\end{array}
$$

25.4 Despite being silly, it is correct.

25.5 You first repeat a 5.

25.6 You won't have a remainder of 7 because you are looking for group of seven and that is one!

You won't have a remainder of 8 because you are looking for group of seven and there is a group of seven within that eight. (You'd get a remainder of 1, instead.)

25.7

Imagine a dots-and-boxes picture of $21 \div 345$.

We'd be looking groups of 345 dots.

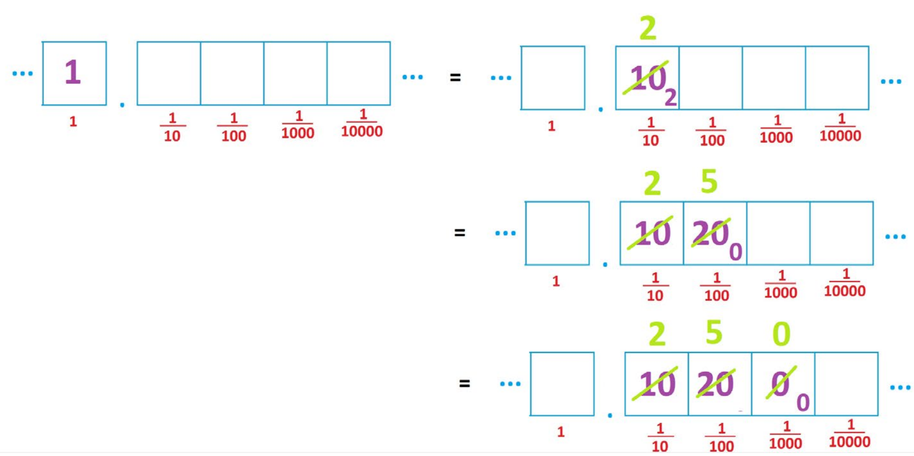
There are likely to be remainders.

There are only 345 possible remainders: 0, 1, 2, 3, …, 344, 345.

We can't keep getting different remainders: we must repeat at least one of them.

As soon as we do, we're in a cycle.

25.8



25.9 $\dfrac{23}{45} = 0.5\bar{1}$

26. 1 It is. It's the fraction $\dfrac{2}{1}$. (Sharing two pies among one student gives the one lucky student two whole pies!)

Its decimal representation is "2" and this is the same as $2.00000\cdots = 2.\overline{0}$. It has repeating zeros.

26.2 No, No, No, and No.

26.3 I find this freaky!

26.4 She is (assuming the pattern she indicates keeps happening).

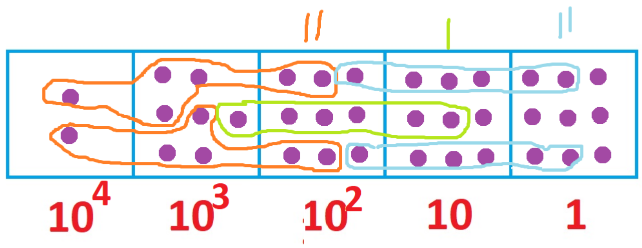
It is a number without a repeating decimal pattern and it is bigger than $0.3333333333\cdots$, which is one third.

26.5 I bet you can do it! Just write down an infinitely long decimal that is clear to anyone who looks at it that it does not have a repeating pattern.

27.1 a) $5\times5\times5=125$ b) $3\times3=9$ c) $1\times1\times1\times\cdots\times1=1$

27.2 $4,8,16,32,64,128,256,512,1024$.

28.1



29.1 Indeed. What did you find?

29.2 We have that $2x^2+7x+6$ is $2\times25+7\times5+6$, which is 91.

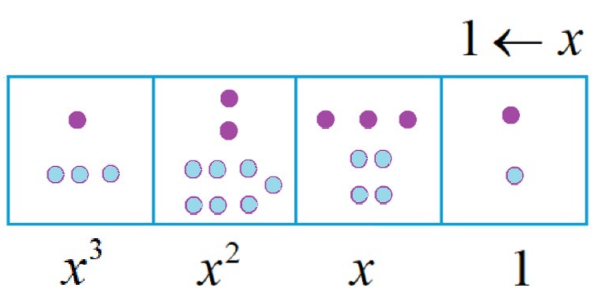
And $x+2$ is $5+2$, which is 7.

And $2x+3$ is $2\times5+3$, which is 13.

So $\left(2x^2+7x+6\right)\div\left(x+2\right)=2x+3$ is saying this time that $91\div7=13$, which is true!

29.3 Did you get it?

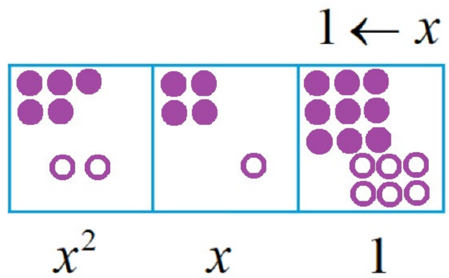
29.4



Adding these together gives $4x^3+9x^2+7x+2$.

For $x=10$ we are saying: $1231+3741=4971$.

29.5 Don't forget tods!



We get $3x^3 + 3x + 3$.

For $x = 10$ we are saying that $549 - 216 = 333$.

29.6. a) It's $2541 \div 21 = 121$ b) It's $\left(2x^3 + 5x^2 + 4x + 1\right) \div (2x + 1) = x^2 + 2x + 1$

29.7. a) $x^3 + x^2 + 2x + 1$ b) $x^2 + 2x + 3$
If $x = 10$, we computed $23541 \div 21 = 1121$ and $13653 \div 111 = 123$.

29.8. You can do it!

For $x = 10$ it says 14641 ÷ 11 = 1331

For $x = 2$ it says 81 ÷ 3 = 27

For $x = 3$ it says 256 ÷ 4 = 64

For $x = 4$ it says 625 ÷ 5 = 125

For $x = 5$ it says 1296 ÷ 6 = 216

For $x = 6$ it says 2401 ÷ 7 = 343

For $x = 7$ it says 4096 ÷ 8 = 512

For $x = 8$ it says 6561 ÷ 9 = 729

For $x = 9$ it says 10000 ÷ 10 = 1000

For $x = 11$ it says 20736 ÷ 12 =1728

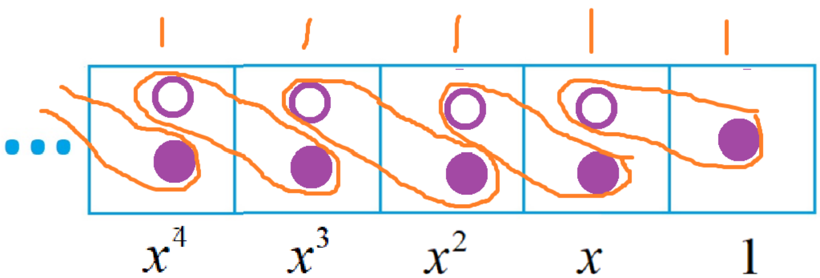
29.9 In an $1 \leftarrow x$ machine we don't actually know how many dots to draw when we do an unexplosion. But in the picture, we can deduce what x has to be.

30.1 Read on!

31.1 31.2 31.3 31.4 All the answers are given in the questions. Keep at it. Try to get those answers.

31.5 This one is indeed wild. It is an infinitely long answer. You get that $\frac{1}{1-x}$ is

$$1+x+x^2+x^3+x^4+\cdots.$$



Can you now work out $\frac{1}{1-x-x^2}$ too?

Of course, the video shows all the work for these two examples.