Garden Paths
STUDENT HANDOUTS

Use these handouts for students who would like to practice the problems presented throughout the experience. These are entirely optional, are NOT homework, and are to be savored and mulled on just for mathematical joy.

Video Resource
Access videos of *Garden Paths* lessons at https://gdaymath.com/courses/gmp/.

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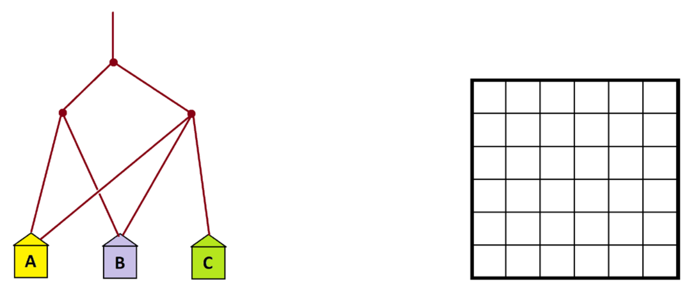
Lesson 1 Handout: Garden Strolls

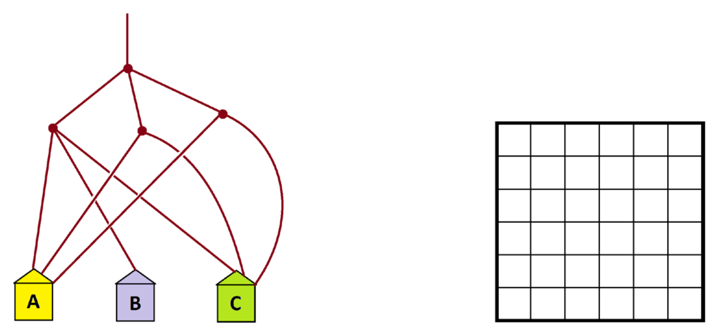
Do as many of these questions as feel interesting and fun to you.

Practice 0: In each of the following examples determine the fraction of people that end up in each house.

a)



b)



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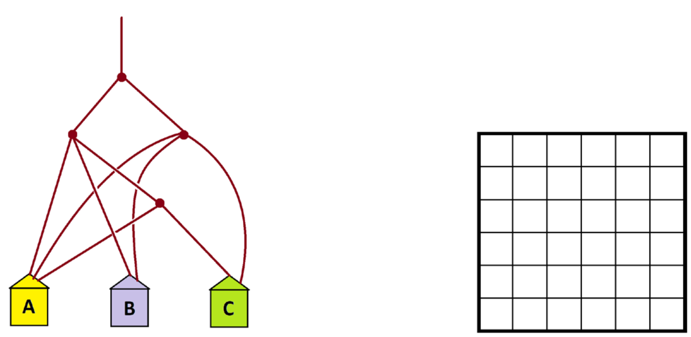

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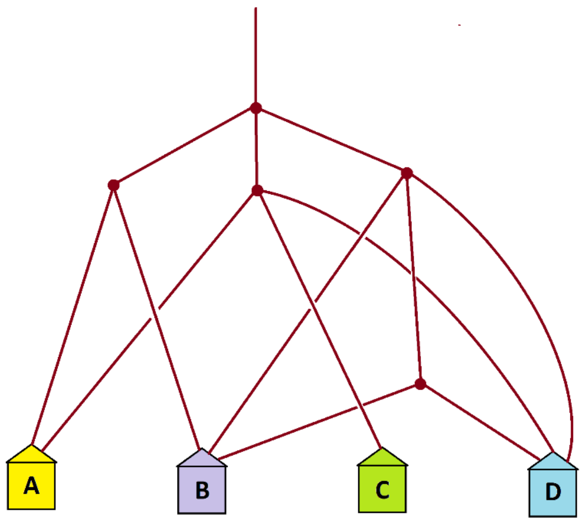
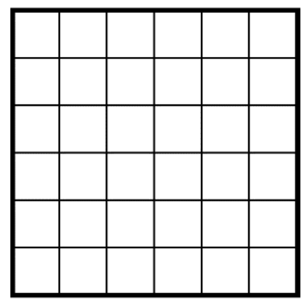
Practice 1:

a) A set of people walk down the following system of paths. Any prediction on which house will end up with the most people in it? The house that will end up with the least?

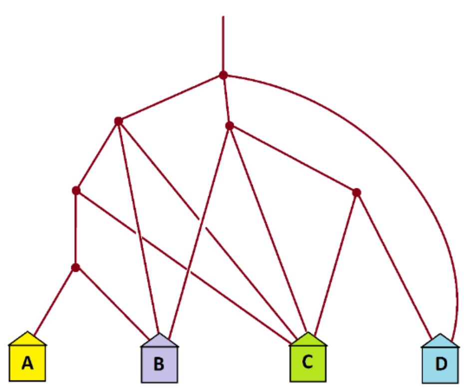


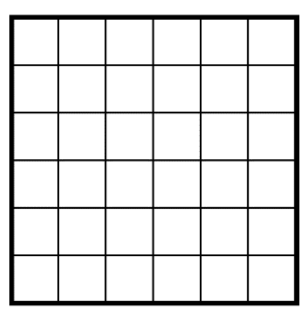


b) Show that the proportions of people who end up in each house are $\frac{5}{18}$, $\frac{6}{18}$, $\frac{2}{18}$, and $\frac{5}{18}$, assuming that people arriving at a fork split themselves equally in count in each direction. (Surprisingly, equal proportions of people end up in houses A and D.)

We've provided here a six-by-six square to help with this question.

Practice 2: Answer the same questions for this garden path system.





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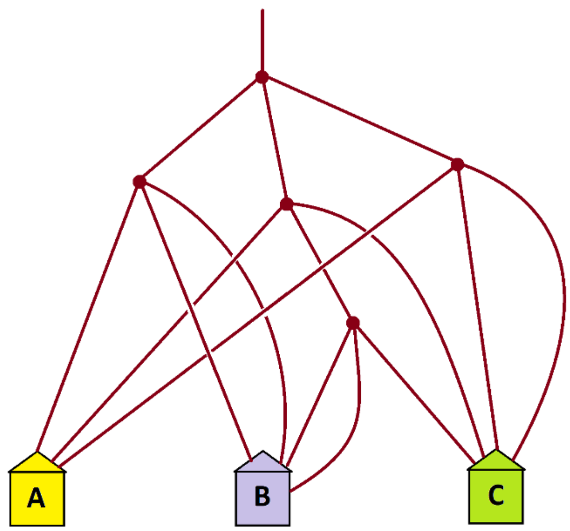
Practice 3: And answer the same questions for this garden path system. Here some forks have two paths that go to the same house.



Practice 4: Design a garden-path system such that the house with least number of paths leading to it actually ends up with the most number of people in it.

Keep Going!

Have some fun making up your own garden path systems for others to solve. (Make sure you can solve them yourself!)

In your examples it might be best to use a blank square rather than one already divided into subsquares.

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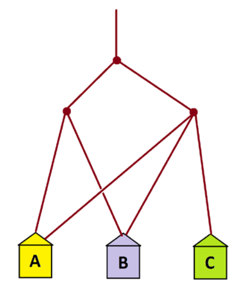
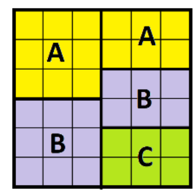
Solutions to Lesson 1 Handout

0.

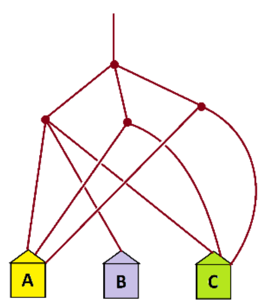
a)

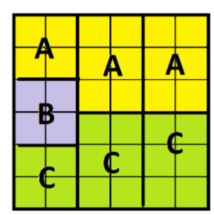




House A $= \frac{1}{4}+\frac{1}{6} == \frac{5}{12}$ House B $= \frac{1}{4}+\frac{1}{6} == \frac{5}{12}$ House c $= \frac{3}{12}$

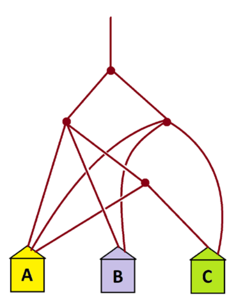
b)

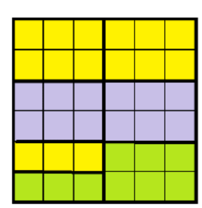




House A $= \frac{1}{9}+\frac{1}{6}+\frac{1}{6} = \frac{8}{18}$ House B $= \frac{1}{9} = \frac{2}{18}$ House c $= \frac{1}{9}+\frac{1}{6}+\frac{1}{6} = \frac{8}{18}$

c)





House A $= \frac{1}{6}+\frac{1}{12}+\frac{1}{6} = \frac{5}{12}$ House B $= \frac{1}{6}+\frac{1}{6} = \frac{4}{12}$ House c $= \frac{1}{12}+\frac{1}{6} = \frac{3}{12}$

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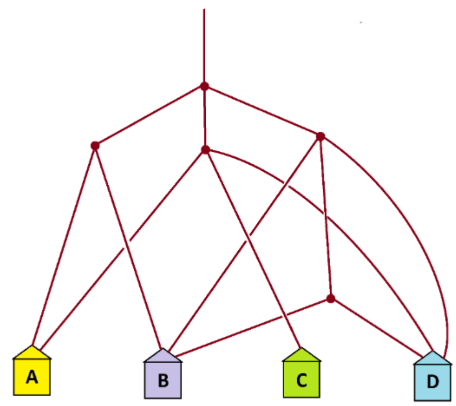
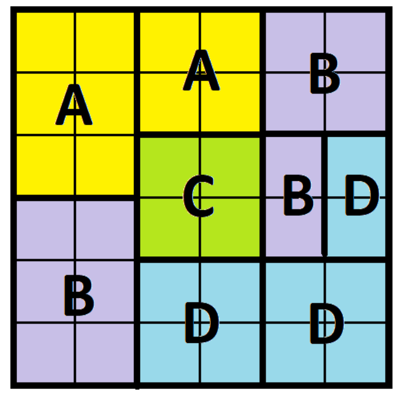
1. We have





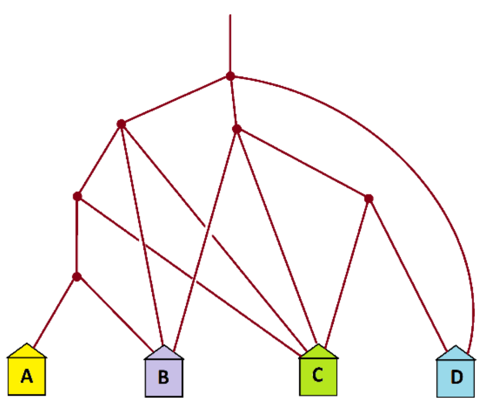
House A $= \dfrac{5}{18}$ of the people.

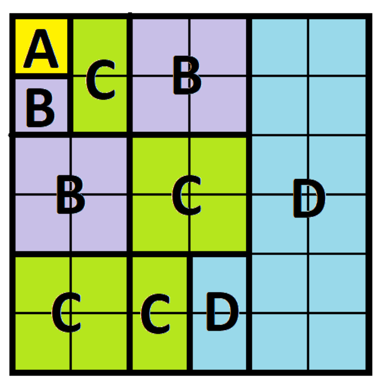
House B $= \dfrac{6}{18}$ of the people.

House C $= \dfrac{2}{18}$ of the people.

House D $= \dfrac{5}{18}$ of the people.

2. We have





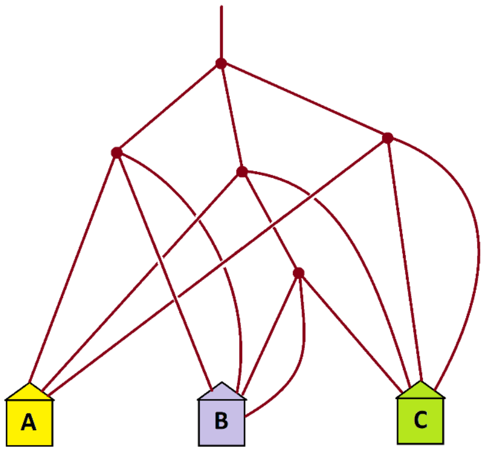
House A $= \dfrac{1}{36}$ of the people.

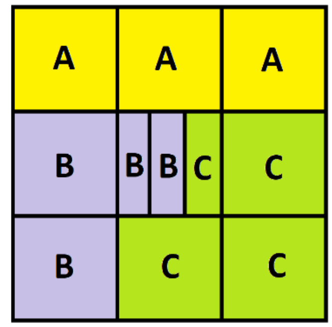
House B $= \dfrac{9}{36}$ of the people.

House C $= \dfrac{12}{36}$ of the people.

House D $= \dfrac{14}{36}$ of the people.

3. We have





(Did you notice that it is better to use a blank square in this example? This example splits into twenty-sevenths)

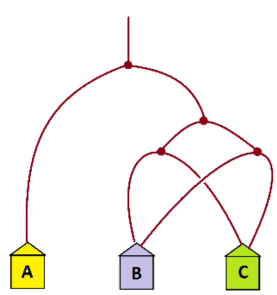
House A $= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{9}{27}$ of the people.

House B $= \frac{1}{9} + \frac{1}{9} + \frac{2}{27} = \frac{8}{27}$ of the people.

House C $= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} = \frac{10}{27}$ of the people.

(As a quick check, these fractions do add to 1.)

4. Something like the following does the trick. Here house A ends with half the people in it and houses B and C each a quarter of the people in them.



Lesson 2 Handout: Flipping Coins, Rolling Dice, and Such

Do as many of these questions as feel interesting and fun to you.

Practice 1: I roll a ruby die and then I roll an emerald die.

a) What are the chances that I will see an even number followed by a six?
b) What are the chances I will see only composite numbers on my rolls?

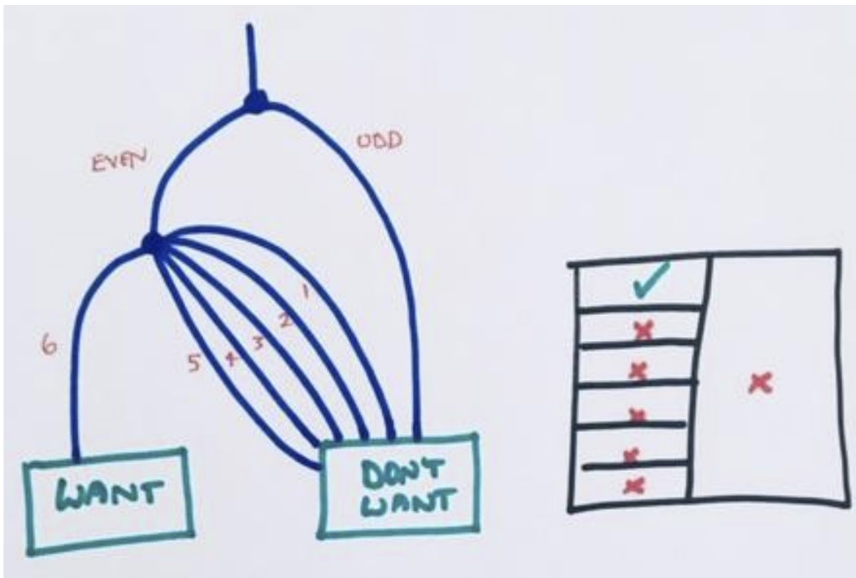
Practice 2: A bag contains two red balls and three white balls. I pull out a ball at random, note its color, and put it back in the bag. I then shake the bag with all five balls in it and pull out a ball again at random and note its color. What are the chances I see a red ball each time?

Practice 3: There is a 30% chance that I will sneeze at least once on any given day and a 60% chance I will yawn at least once.

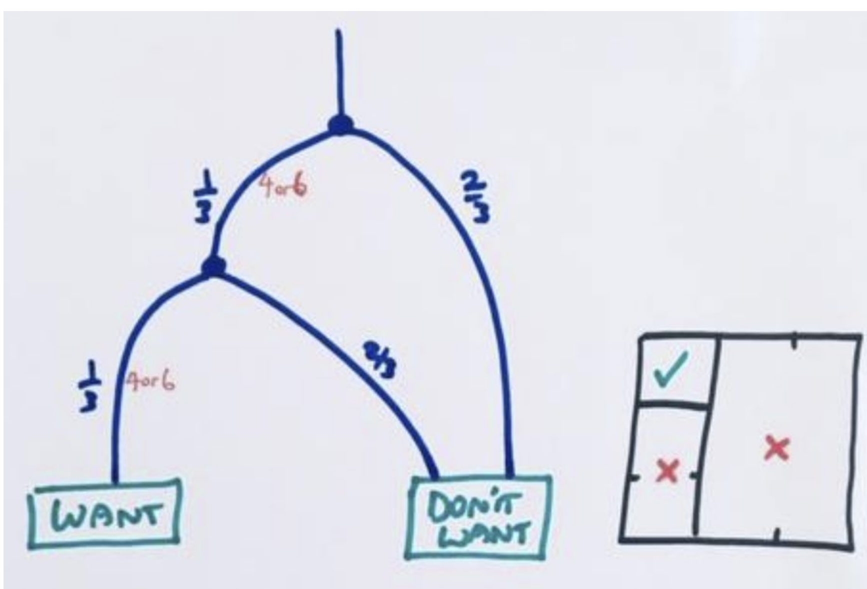
a) What are the chances that I will both sneeze and yawn tomorrow?
b) What are the chances I will sneeze but not yawn?
c) What are the chances I will either sneeze or yawn but not both tomorrow?

Solutions to Lesson 2 Handout

1. a) The answer is $\frac{1}{12}$.



b) The answer is $\frac{1}{9}$.



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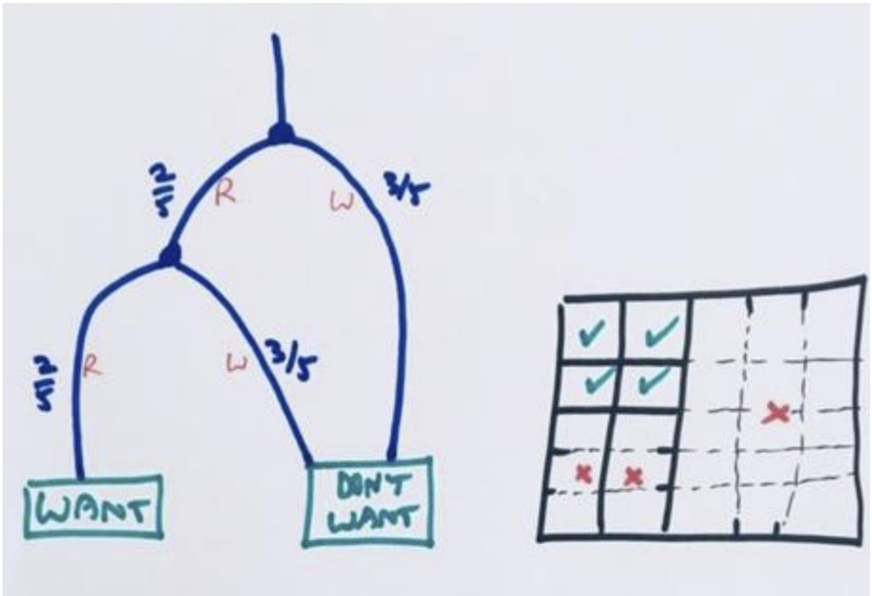
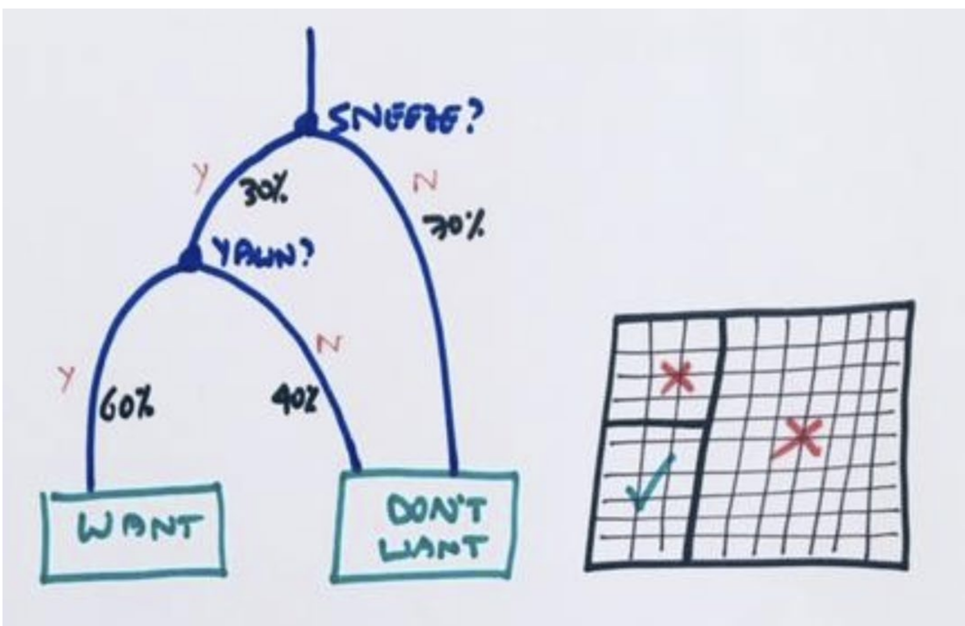
2. The answer is $\frac{4}{25}$.



3. a) The answer is $\frac{18}{100} = 18\%$.



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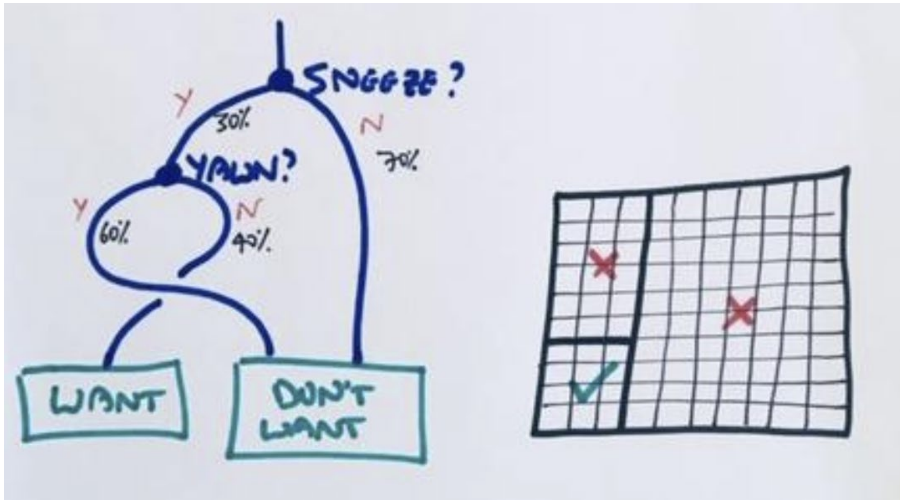
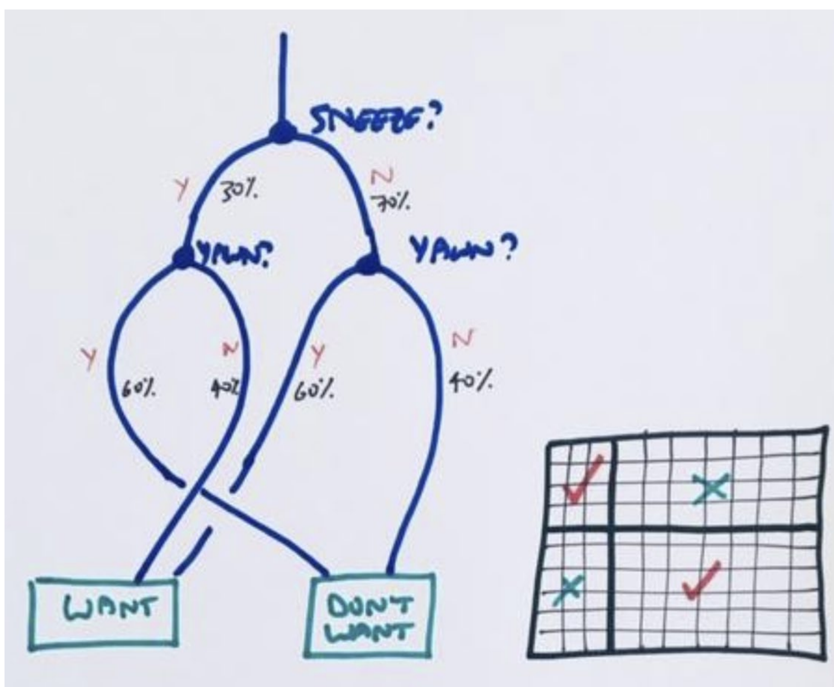
b) The answer is $\frac{12}{100} = 12\%$.



c) The answer is $\frac{54}{100} = 54\%$.

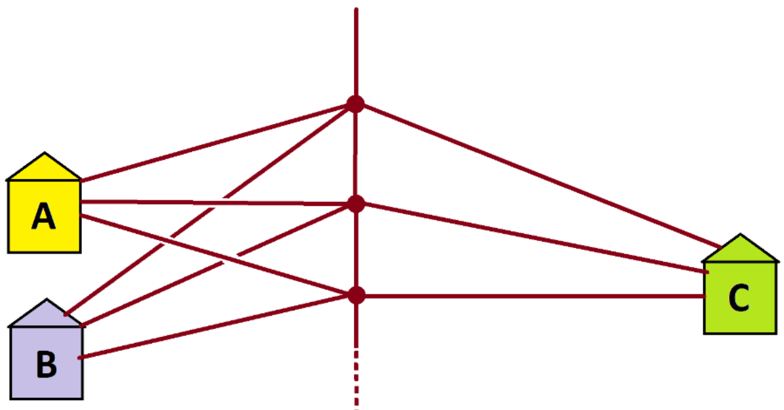


Lesson 3 Handout: Infinite Garden Paths

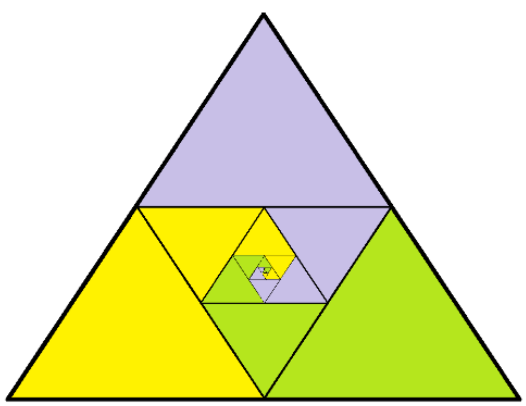
Do as many of these questions as feel interesting and fun to you.

Practice 1:

a) Argue that this next infinite garden path suggests that $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots$ equals $\frac{1}{3}$.



b) Anu was having trouble creating an area model for this system using a square. Then she thought to use an equilateral triangle instead. This is what she drew. What do you think of it?



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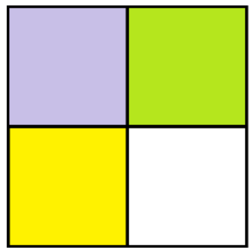
c) Chee Wei said it is actually possible to use the area of a square to represent this garden-path system nicely. He gave a hint as to what he was thinking by sharing this start to picture on the board. How do you think we will continue the picture?



Practice 2: a) Draw a garden path system that leads to the sum $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}+\cdots=\frac{1}{4}$.

b) What do you think might be value of $\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\frac{1}{N^4}+\cdots$ for a positive integer N ?

Solutions to Lesson 3 Handout

1. a) The picture looks non-symmetrical, but is actually is in concept! At each fork each of the three houses ends up with one-quarter of the people from that fork (and the remaining folk keep moving down the path). So the count of people that end up in each house is identical. As everyone ends up in a house eventually (what at the chances that someone will keep walking down straight down the garden path forever?), each house must end up with one-third of the people.

In particular, $\frac{1}{3}$ of the people end up in house A.

But we can compute this number by noting that, first, a quarter of the people go to house A, then a quarter of a quarter of the people, then a quarter of a quarter of a quarter of the people, and so on. That is, the count of people in house A must be $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots$.

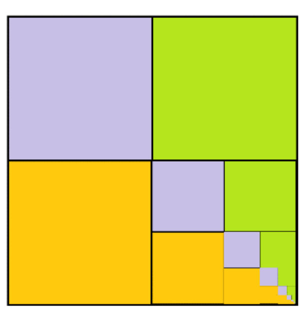
These two counts must be the same!

b) Look at the yellow spiral, say, for house A. The first triangle of the spiral is one-quarter of the total area. The next triangle in the spiral is one-quarter of a quarter of the total area. The next triangle in the spiral is one quarter of a quarter of a quarter of the total area. And so on. This matches our computation in part a).

But clearly this yellow spiral is, in totality, one third of the total area. This infinite sum must equal one-third.

Anu's picture is lovely!

c) Perhaps this way.



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2. a) How about this?

b) We might guess $\dfrac{1}{N-1}$. (Though we have to assume N is not 1!)

Lesson 5 Handout: What we Choose to Believe

Do this question if it is interesting and fun to you.

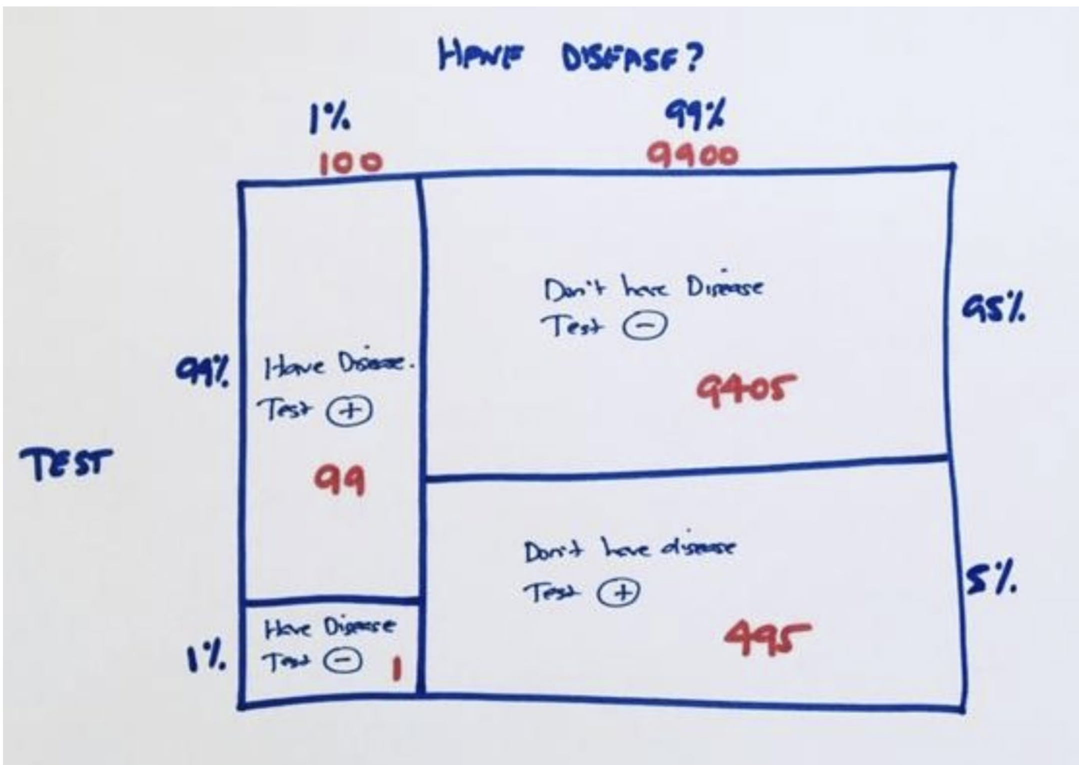
Practice 1: Suppose that 1% of the population has a certain disease. A test for the disease has been developed. This test will return a correct positive result for 99% of the people who actually have the disease (meaning that there is a 1% chance of a “false negative”) and a correct negative result for 95% of the people who do not have the disease (meaning that there is a 5% chance it will produce a “false positive”).

You have just been tested positive for the disease! Show that there is one a one-in-six chance that you actually have the disease.

Solutions to Lesson 5 Handout

1. Let's work with a population of 10,000 people. We expect 100 of people to have the disease (one-percent of them) and 9900 of them to be disease free.

This table shows counts of people we expect with positive and negative test results.



Of the $99+495=594$ people who test positive, we see that only 99 of them actually have that disease, and $\frac{99}{594}=\frac{1}{6}$ as claimed.

Lesson 6 Handout: The Infamous Two-Girls Paradox

Do as many of these questions as feel interesting and fun to you.

Recall the three scenarios.

> Albert, who you just met, tells you that he is the father of two children and that his oldest child is a girl. What are the chances that his other child is also a girl?
>
> Bilbert, who you just met, tells you that he is the father of two children and that one of his children is a girl. What are the chances that his other child is also a girl?
>
> Cuthbert, also a new acquaintance, tells you that he is the father of two children and that one of his children is a girl who was born on a Tuesday. What are the chances that his other child is also a girl?

Practice 1: Suppose Bilbert lives in a society that requires you to mention that you have a male child if you legitimately can. Given that Bilbert said "One of my children is a girl," what now are the chances that Bilbert has two girls?

Practice 2: Suppose Cuthbert lives in a society that requires you to mention that you have a male child if you legitimately can. Given what Cuthbert said what are the chances that Cuthbert has two girls?

Practice 3: Suppose Cuthbert lives in a society that requires you to mention that you have a female child if you legitimately can. Further, assume people like to add the day of the week their daughters were born and when given a choice between two daughters to mention, they use the flip of a coin to decide. Given what Cuthbert said what are the chances that Cuthbert has two girls?

Solutions to Lesson 6 Handout

1. Under this condition is must be the case that Bilbert has two girls. The answer is 100%.

2. Under this condition is must be the case that Cuthbert has two girls. The answer is 100%.

3. Every father with a girl born on a Tuesday and a boy, and every father with two girls each born on a Tuesday will utter the words "One of my children is a girl born on a Tuesday."

Of the fathers with two girls, one born on a Tuesday, the other not, half will say those words.

None of the other fathers will utter those words.

Here's a table of all the counts of fathers who make that statement. The probability that such a father actually has two girls is $\dfrac{10+30+30}{10+30+30+70+70}=\dfrac{1}{3}$.

YOUNGEST CHILD

OLDEST CHILD

	GIRL born on TUESDAY	Girl not born on TUESDAY	BOY
GIRL born on TUESDAY	10	30	70
Girl not born on TUESDAY	30	X	X
BOY	70	X	X

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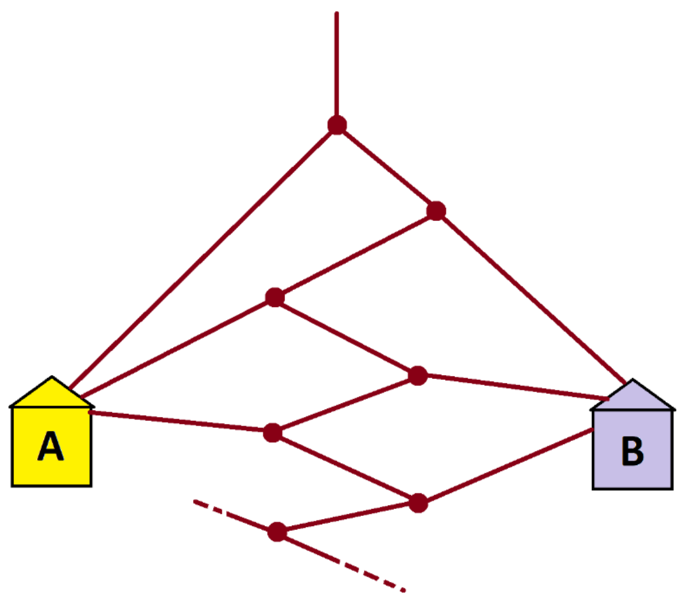
Lesson 8 Handout: Infinite Garden Paths, Again

Do as many of these questions as feel interesting and fun to you.

Practice Problem 1: Consider the following infinite garden-path system with two types of two-way forks: each fork sends half the people walking through it to a house and the other half to a next fork.



a) Using only a start node S, a second node "1", and the two houses A and B, redraw this garden path system much more succinctly.
b) In moving though this system, what proportion of people end up in house A and which proportion of people end up in house B?
c) OPTIONAL: On average, how many steps does a person take in moving to a house?

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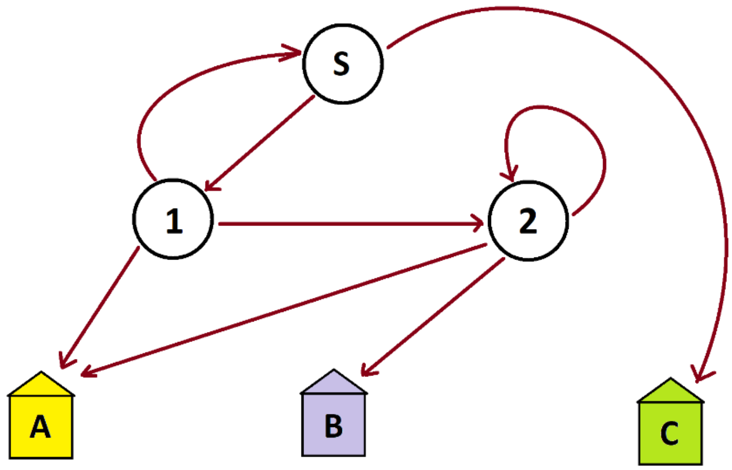
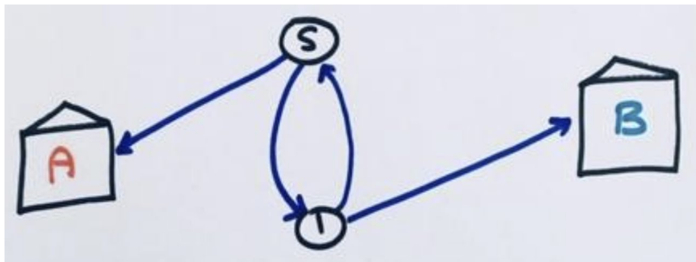
Practice Problem 2: Here is a very complication system!



Show that 30% of the people moving through this system will end up in house A, 10% in house B, and 60% in house C.

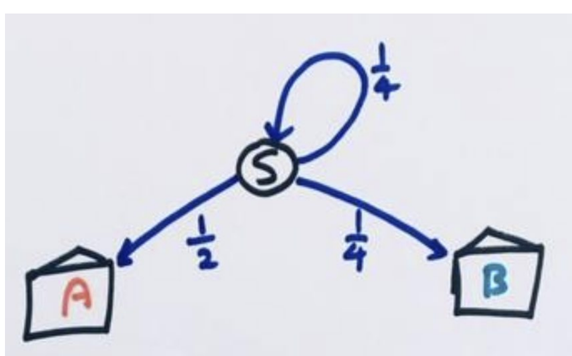
Solutions to Lesson 8 Handout

1. a)

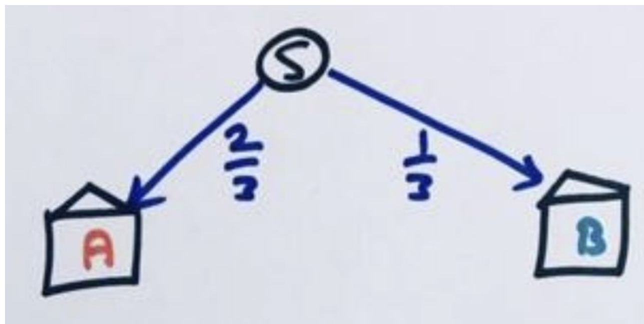


b) Here’s a square model of the first few iterations of this system.

We see from the second iteration that our system is equivalent to this set of garden paths.



But we can argue that the start node is sure to send two-thirds of the people to house A and one third to house B. We have



c) From the first three iterations we see that the average number of steps taken, m, satisfies

$m=\frac{1}{2}(1)+\frac{1}{4}(2)+\frac{1}{4}(2+m)$ giving $m=2$.

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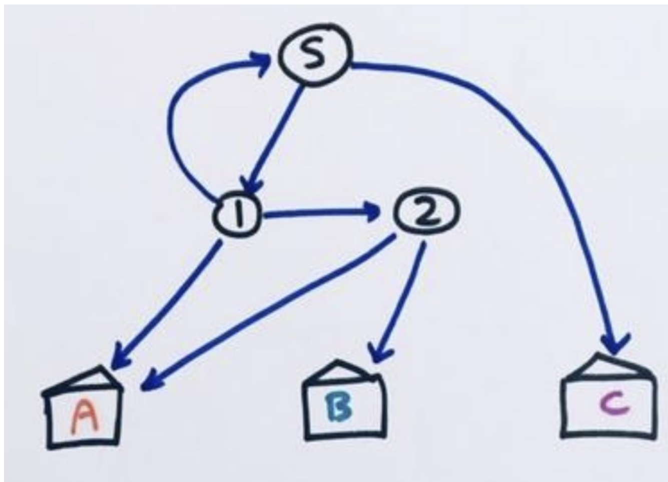
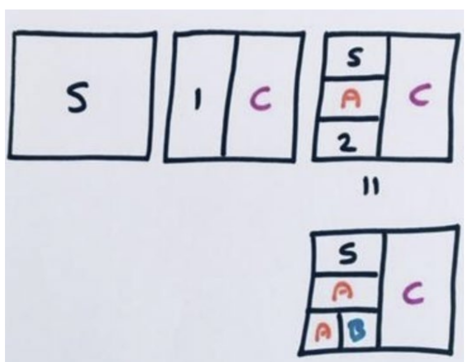
2. First look at the loop from node 2 back to node 2. This means node 2 is going to keep forcing people to either go to house A or house B (equally likely). This our system is equivalent to this one



The first few iterations of this appear as follows.



But node 2 is equivalent to simply pushing folk into houses A and B, equally likely. Our system is thus philosophically equivalent to one based on the final square I just drew.

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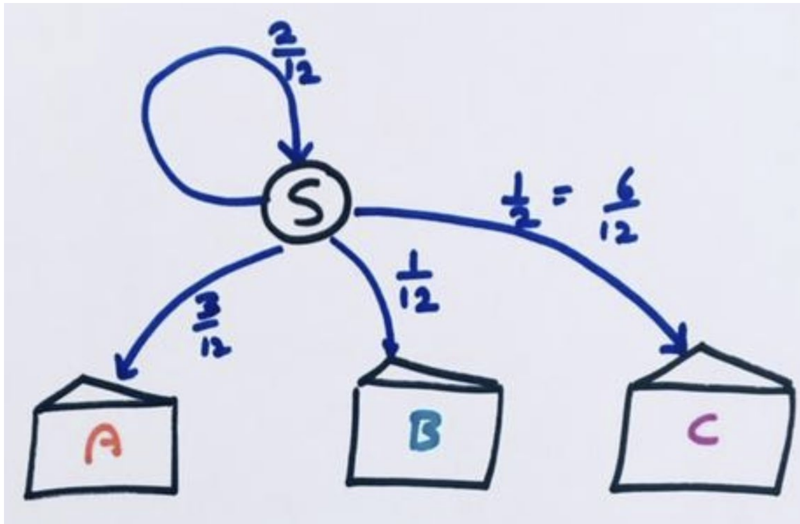
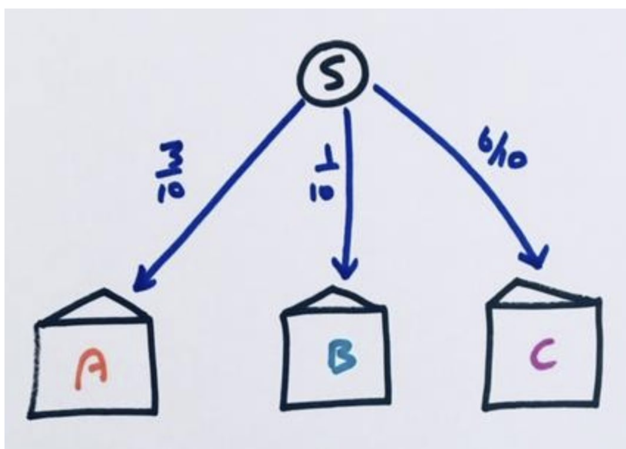


But that loop from S back to S only forces folk into houses A, B, C in the proportion 3:1:6. So the system is philosophically equivalent to



This explains the proportions given in the question.

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Lesson 9 Handout: Infinite Probability Problems

Do as many of these questions as feel interesting and fun to you.

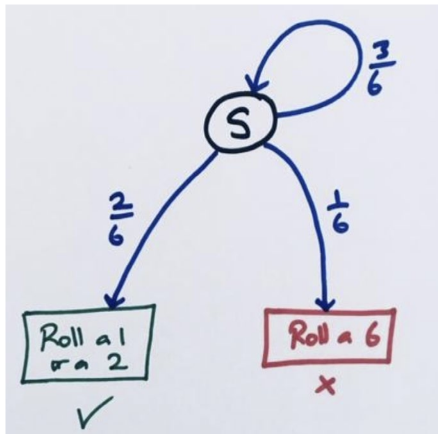
Practice Problem 1: I will repeatedly roll a die. What are the chances that I will see either the roll of a 1 or the roll of a 2 before seeing a roll of a 6?

Practice Problem 2: I will repeatedly roll a die. What are the chances that I will see the roll of a 1 and a roll of a 2 and a roll of a 3 before seeing a roll of a 6?

Practice Problem 3: I will repeatedly toss a coin. What are the chances I will see a head immediately followed by a tail (HT) before seeing two consecutive tails (TT)?

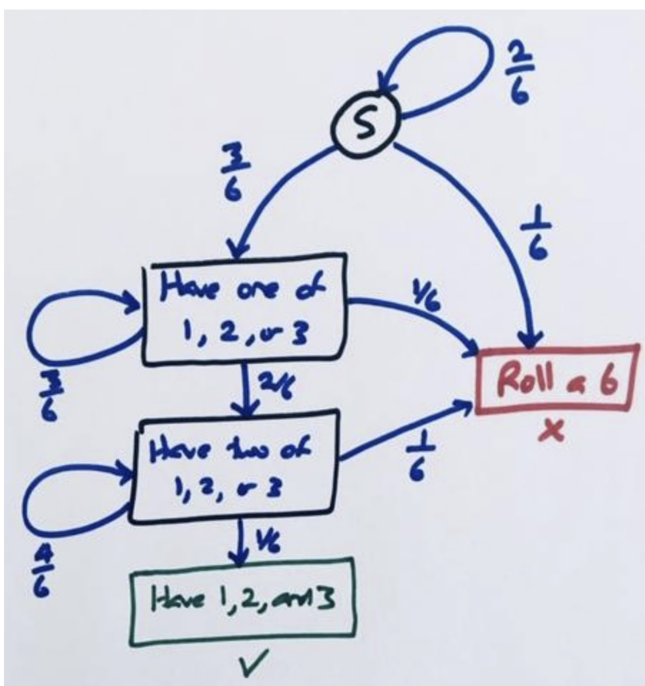
Solutions to Lesson 9 Handout

1. Here's a path diagram representing this action.

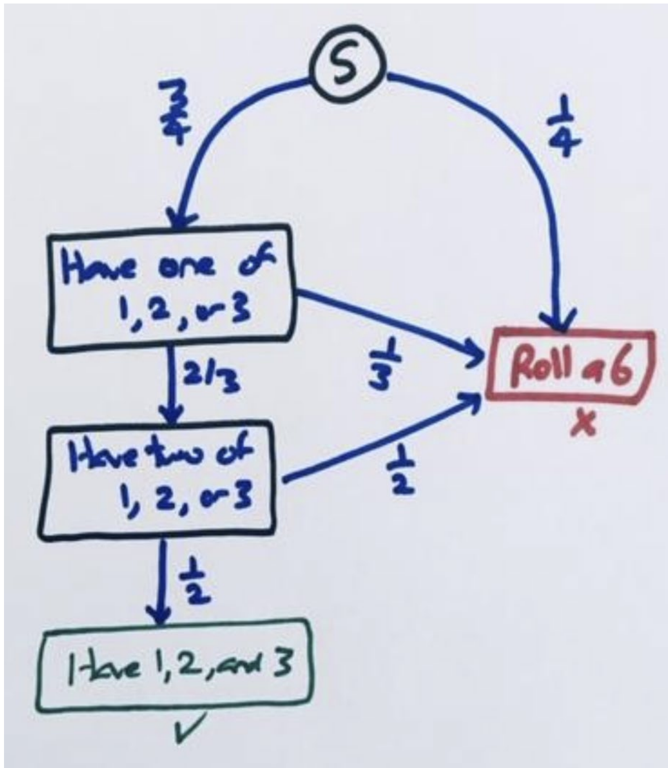


We see that double the count of folk end up with the desired result than the undesired result. The probability of thus seeing a 1 or a 2 before ever seeing a 6 is $\frac{2}{3}$.

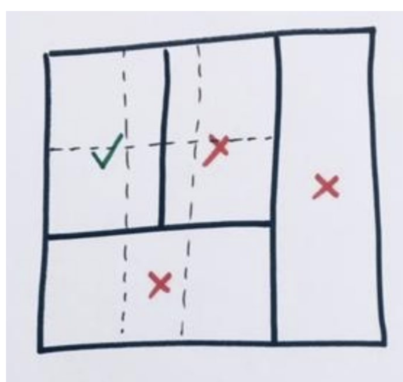
2. Here's a path system for this problem.



Is it philosophically equivalent to this system:



The square model shows that we win $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}=\frac{1}{4}$ of the time.



GARDEN PATHS

3. The only way to lose is to first toss a TAIL and then immediately toss another TAIL. (If you toss a HEAD on the first or second toss you are guaranteed to win.) There is a $\frac{1}{4}$ chance of this happening. Thus there is a $\frac{3}{4}$ chance of winning.

Can you also get this answer from the following diagram?