Pencil Pushing

Exploring the Joys—and Mysteries—of Angles in Geometry

Teaching Guide

Video Resource

Access videos of *Pencil Pushing* lessons at [https://gdaymath.com/courses/gmp/.](https://gdaymath.com/courses/gmp/)

Student Handouts

All practice problems, and solutions, in an accompanying document.

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PENCIL PUSHING: Overview

Student Objectives

Students will develop a strong intuitive understanding of angles and the properties of interior and exterior angles of polygons. The context for choosing to believe that the three interior angles of a triangle are sure to add to 180 degrees will be laid bare!

The Experience in a Nutshell

Setting the Scene

View the welcome video to this GMP short experience at www.globalmathproject.org/courses/gmp, lesson 1.1.

This experience begins by developing solid intuitive understanding of what an angle is and explores the possibility that vertical angles across the intersection point of a pair of intersecting lines are congruent and that the interior angles of any triangle are sure to add to 180 degrees.

But matters are soon put into doubt with an exercise that challenges our intuitive understanding.

This leads us to make an axiomatic choice about the properties of our (Euclidean) geometry, from which do indeed agree to say that our intuitive understanding holds true for this particular geometry.

We quickly re-establish basic facts about interior and exterior angles of polygons, which then leads to an astounding property of bicycle tracks!

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Lesson 1: What is an Angle?

Welcome to a story about angles. The entire journey presented here is about the joy—and the mystery—of angles.

Loosely speaking, an angle is simply “an amount of turning.”

If I were to twirl in place and end up back facing you, we would say I turned through an angle of one full turn. If I did this a second time (in the same direction) we’d say I had turned through an angle of two full turns.

Twirling through an angle of half a turn would land me with my back to you, as would a twirl of one-and-half full turns, and an angle of 99.5 full turns.

Me after one full turn

Me after two full turns

Me after 99.5 full turns

Here’s pencil pointing to the right.

If we turn this pencil a quarter-turn about its end, say in a counter clockwise direction, then its new position relative to its starting position will be as shown.

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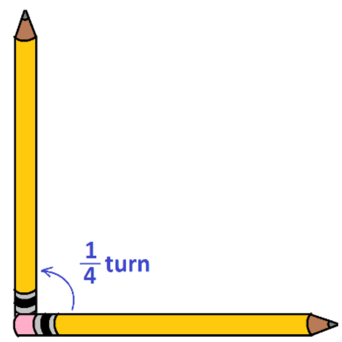
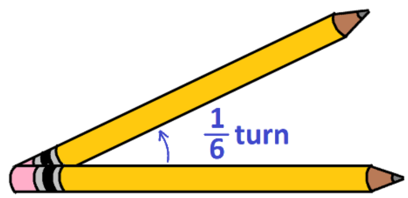
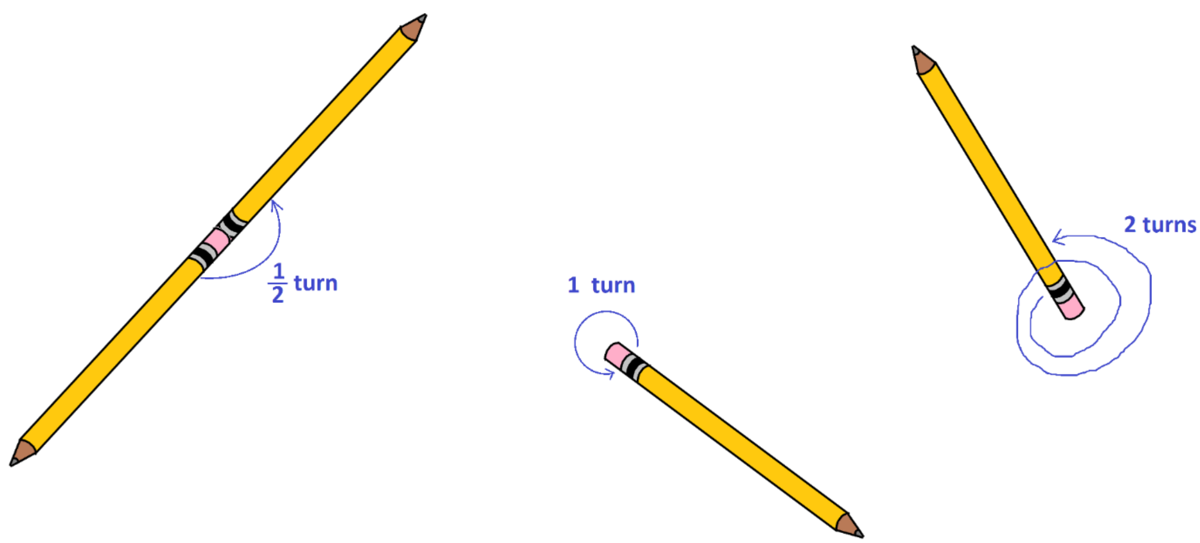


The following picture shows $\frac{1}{6}$ of a turn.



And the following shows a half turn, a full turn, and two full turns of pencils with different starting positions.



Notice that any full count of turns gives the same final effect as applying no turns at all. (Well, this statement can be questioned. See lesson 4 for some curious comments on this!)

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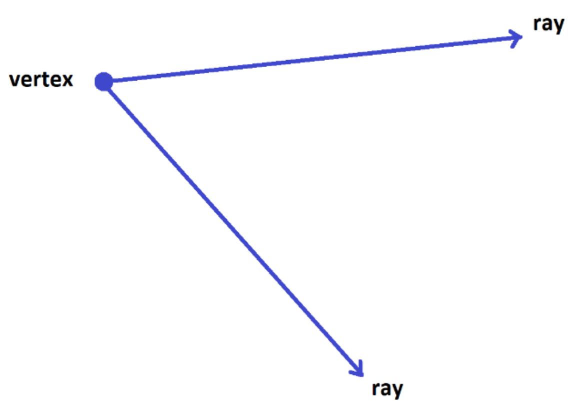
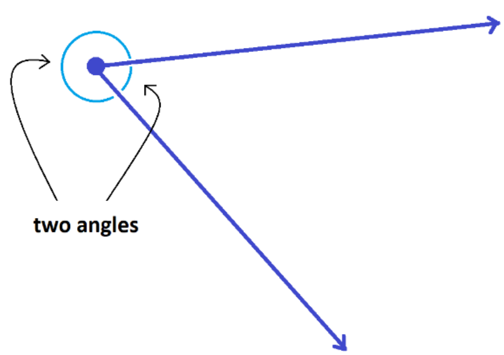
Formalities

In school books an angle is depicted with two rays (or just line segments) emanating from a common vertex and this description is often taken as *the definition* of an angle, stripped of the intuition.



Although "two rays with a common vertex" is a good formalized definition of an angle, it is still useful to keep the idea of an amount of turning explicitly in mind.



But there is a tiny problem: a diagram of two such rays actually depicts two angles (or more if you consider multilple full turns plus these angles). So one must always be careful to clarify exactly which angle is being discussed. If no clarification is offered, then one should assume that the smaller angle, the one representing the least amount of turning, is the one being considered.

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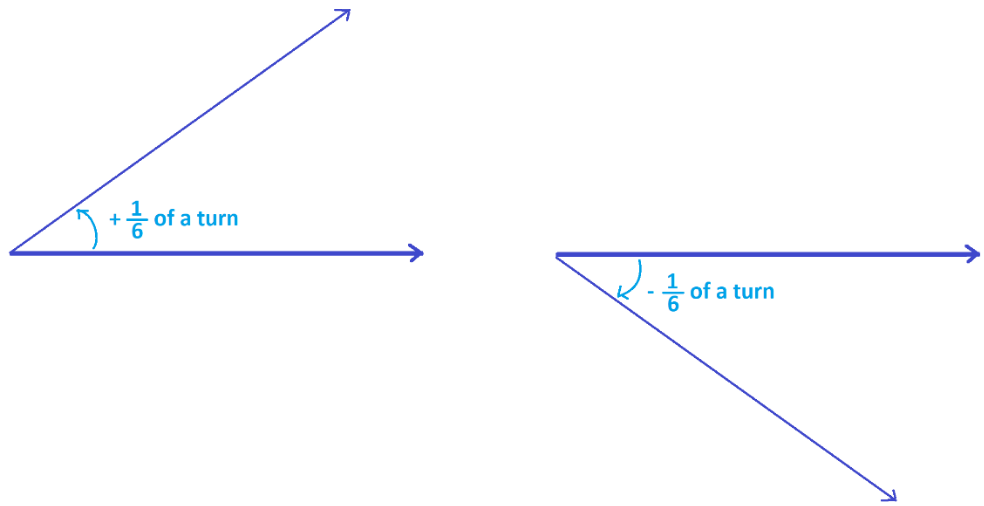
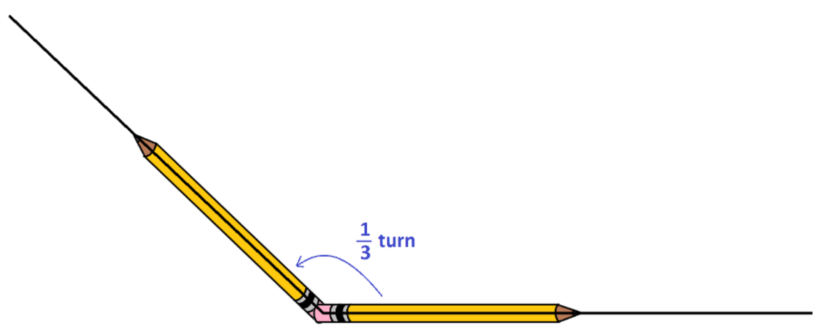
Aside: Mathematicians sometimes like to differentiate between positive and negative angles. It has become the convention to regard angles that turn in the counter-clockwise direction as positive and those turning in the clockwise direction as negative.



This distinction isn't made in a typical school course: all angles are considered positive.

Pencil Pushing

Pencils are a very good tool for encoding the physical effect of turning. To get a feel for the angle represented by two line segments with a common vertex drawn on a page simply choose one end of a pencil and place it at the vertex of the diagram, align the whole pencil with one line segment, and then turn the pencil about the chosen end until the pencil aligns with the second line segment. The amount of turning the pencil has undergone is the measure of the amount of turning that angle represents.



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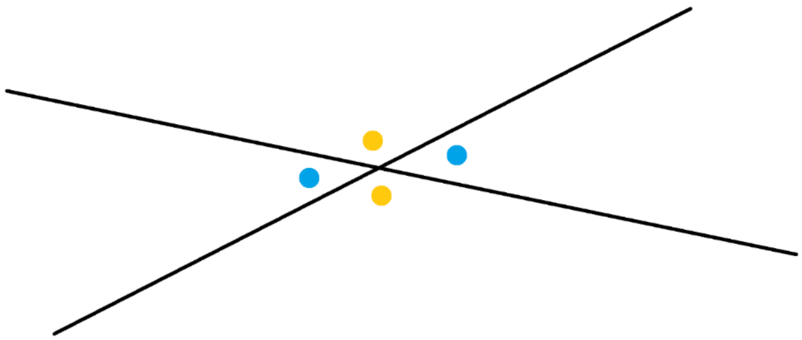
As a hint of the power of this idea, consider this.

Two straight lines that cross create four angles around the intersection point.

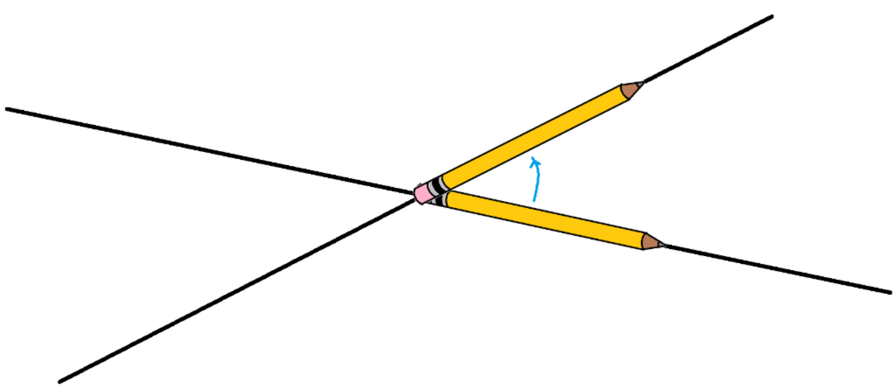


Each pair of angles opposite one another across the intersection point is called a pair of *vertical angles*–not because the angles are "not horizontal," but because they are in relation to a *vertex*, and people choose not to say "vertexical." (Some people use the term *vertically opposite angles*.)

So in our diagram the two blue angles are a pair of vertical angles. The pair of yellow angles are also vertical.

One learns in geometry class that vertical angles are *congruent*, that is, that they represent the same amount of turning. One can experience that with a pencil!

The rightmost blue angle affects a pencil as shown.



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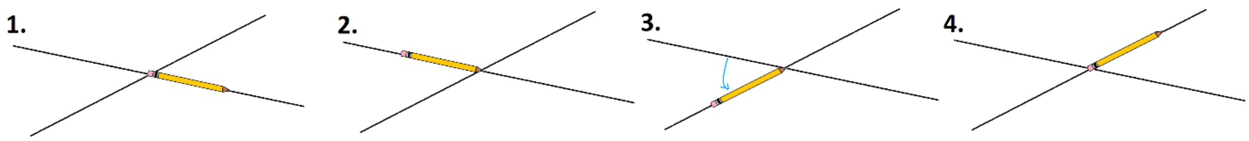
We can see the effect of the leftmost blue angle, with the pencil starting in the same initial position as follows.

1. Place the pencil in the same starting position.
2. Slide the pencil along the line on which it sits so that its tip now sits at the vertex. (This is not changing the direction in which the pencil points. No turning is being applied.)
3. Apply the turning of the leftmost blue angle. This will be tuning about its tip.
4. Slide the pencil along the second line. (This applies no turning to the pencil.)

We see we obtain the same final pencil position, and so both angles applied the same measure of turning to the pencil.



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Lesson 2: Putting Pencil-Pushing to Use

People like to say that one full turn corresponds to *360 degrees* of turning and use a superscript $\circ$ to denote a count of degrees.

> **Question:** *Why that number? Who chose the number 360 for the count of degrees in a circle? Would you think to choose that number? Let's talk about this in the next (optional) lesson.*

Thus half a turn corresponds to $\frac{1}{2}\cdot 360^{\circ} = 180^{\circ}$ of turning, quarter of a turn to $\frac{360^{\circ}}{4} = 90^{\circ}$ of turning, two full turns to $2\times 360^{\circ} = 720^{\circ}$ of turning, and so on.

> **Question:** *To how many degrees do the following fractions of turn correspond?*
>
> *a)* $\frac{1}{6}$ *of a turn* *b)* $\frac{3}{4}$ *of a turn* *c) one-and-a-half turns*
>
> *To what fraction of a turn do the following counts of degrees correspond?*
>
> *d)* 45° *e)* 20° *f)* 1080° *g)* 0.01°

People throughout the ages have noticed that there seems to be something special about the three angles inside a triangle—any triangle. It appears that

> *The three interior angles of a triangle have measures sure to add to half a turn (*180°*).*

And pencil pushing shows this. Try this next activity!

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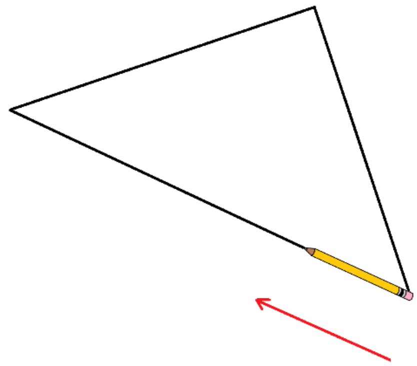
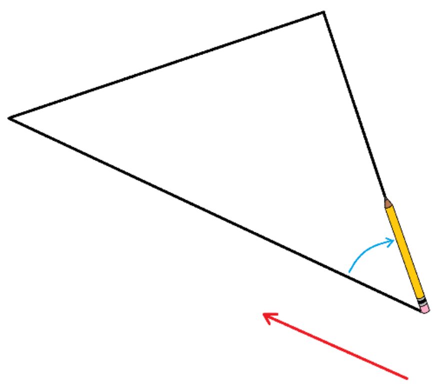
STEP 1: *Draw a large triangle on a piece of paper, one large enough to contain a small pencil. (Or draw a large triangle on a chalkboard, or in the sand.)*

Place a pencil with one end in the corner of the triangle, aligned with the side of a triangle. Take note of the direction the pencil is initially pointing.



STEP 2: *Apply to the pencil the amount of turning of the first angle.*



STEP 3: *Slide the pencil along the side of the triangle on which it sits. This action applies no turning to the pencil.*

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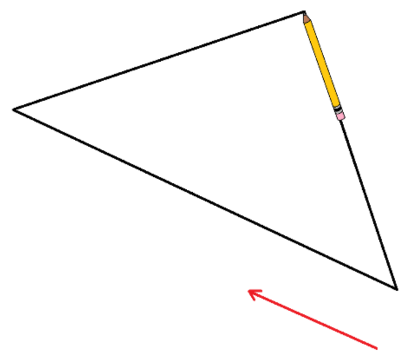
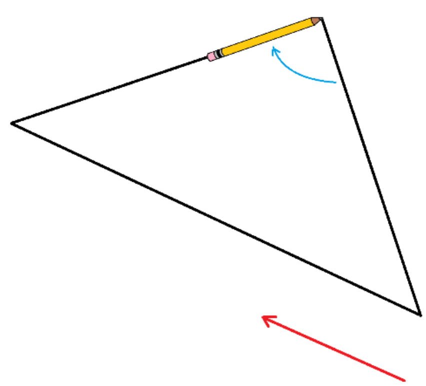
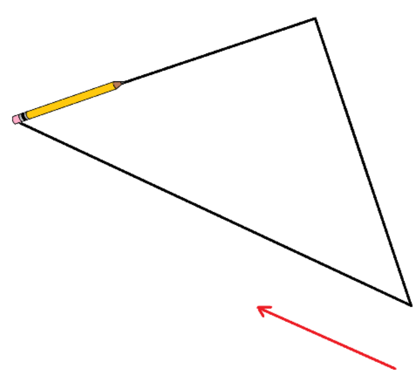


STEP 4: *Apply to the pencil the amount of turning of the second angle.*



STEP 5: *Slide the pencil along the side of the triangle on which it sits. This action again applies no turning to the pencil.*



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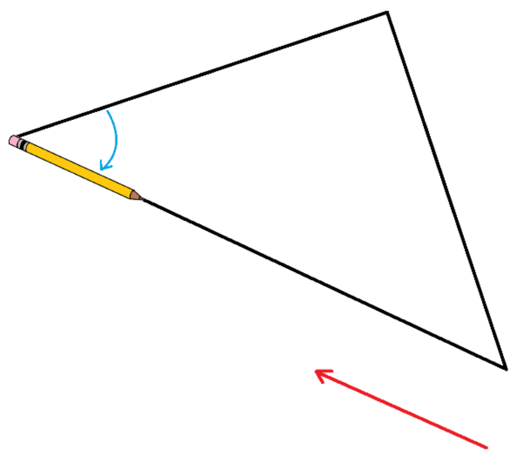
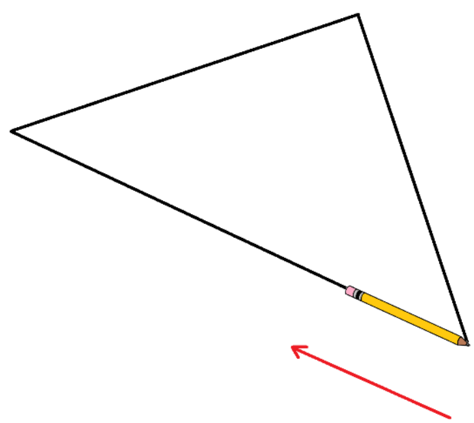
STEP 6: *Apply to the pencil the amount of turning of the third angle.*



STEP 7: *Slide the pencil back to start along the side of the triangle on which is sits.*



We see that the pencil is now pointing in the opposite direction to which it started!

The total effect of applying the amount of turning given by the three angles in a triangle is that of a half turn. So indeed,

The three interior angles of a triangle have measures that sum to 180°.

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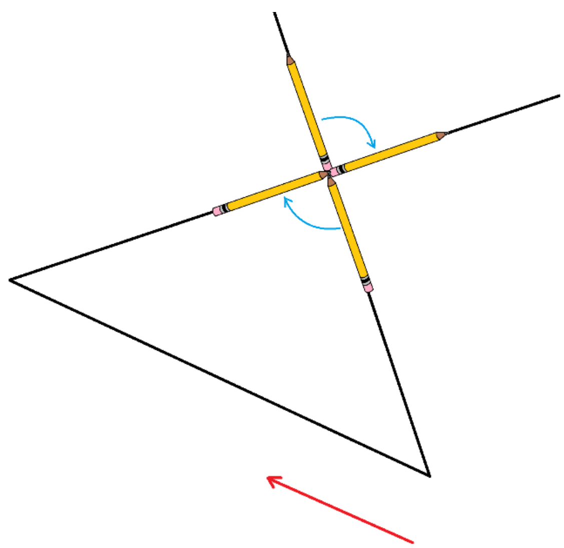
Can you see in your mind's eye that for any triangle you happen to draw on the page the same half turn of a pencil would occur? Can you see that the same phenomenon would occur for a pencil moving inside a triangle the size of the solar system? Can you see that it would occur too for a pencil inside a triangle the size of an atom (a sub-atomic sized pencil, of course)? It seems this pencil experiment demonstrates an idea that holds for all triangles.

Some might worry that the turning in step 4 is inconsistent with the other two actions of turning: here we are turning about the lead-tip of the pencil rather than the eraser end of the pencil.

To be consistent, in step 4 we could turn the pencil about the same eraser end too by sliding the pencil just past the vertex of the triangle and make use of the congruent vertical angles we noted in the last lesson.



Turning on the tip or turning on the eraser applies the same amount of turning.

Activity 1:
a) Draw a four-sided figure and apply the same pencil pushing idea to its interior four angles. Show that the four angles have measures that sum to two half turns, that is, add to 360°.

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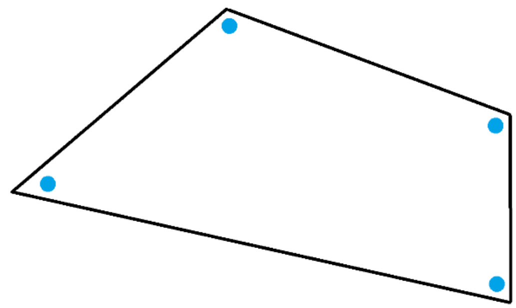
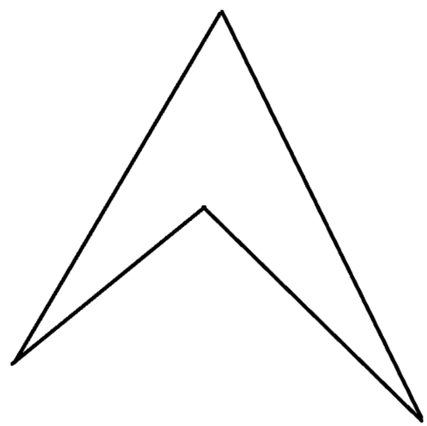


b) Consider the following quadrilateral.
Identify its four interior angles.
Show that its four interior angles also have measures that add to two half-turns.



Activity 2: Explore the measures of the interior angles of five-sided figures. What can you say about the sum of those measures? (WARNING: A pencil inside a pentagon undergoes more than just a single half turn.)

Activity 3: Draw a large 13-sided figure and use pencil-pushing to find the sum of the measures of its interior angles. How many half-turns?

Activity 4: Find a general formula that seems to hold for the sum of measures of the interior angles of an N-sided figure.

Activity 5: What can you say about the sum of measures of the (exterior) angles shown for this polygon?

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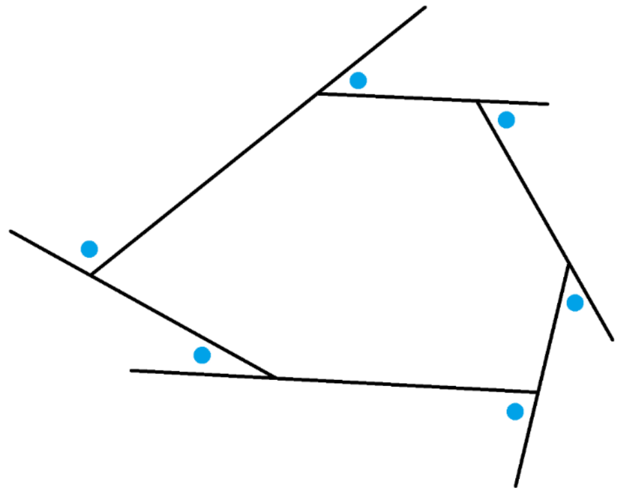
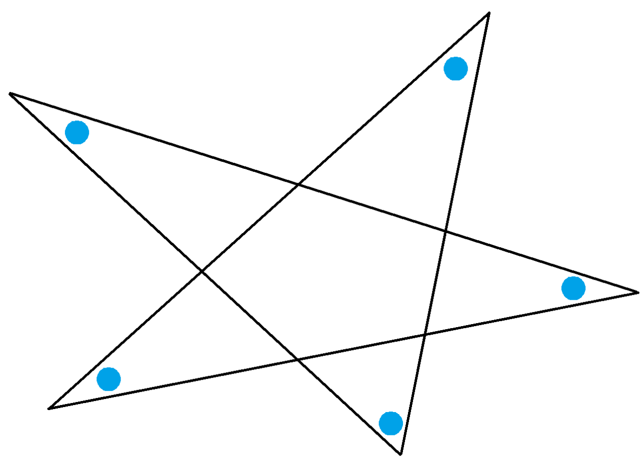


Activity 6: What does pencil pushing say about the sum measures of the five angles in a lop-sided five-pointed star?



Question 7: Does the following polygon exist if we insist that all the angles marked with a blue dot are congruent, that is, have the same measure?

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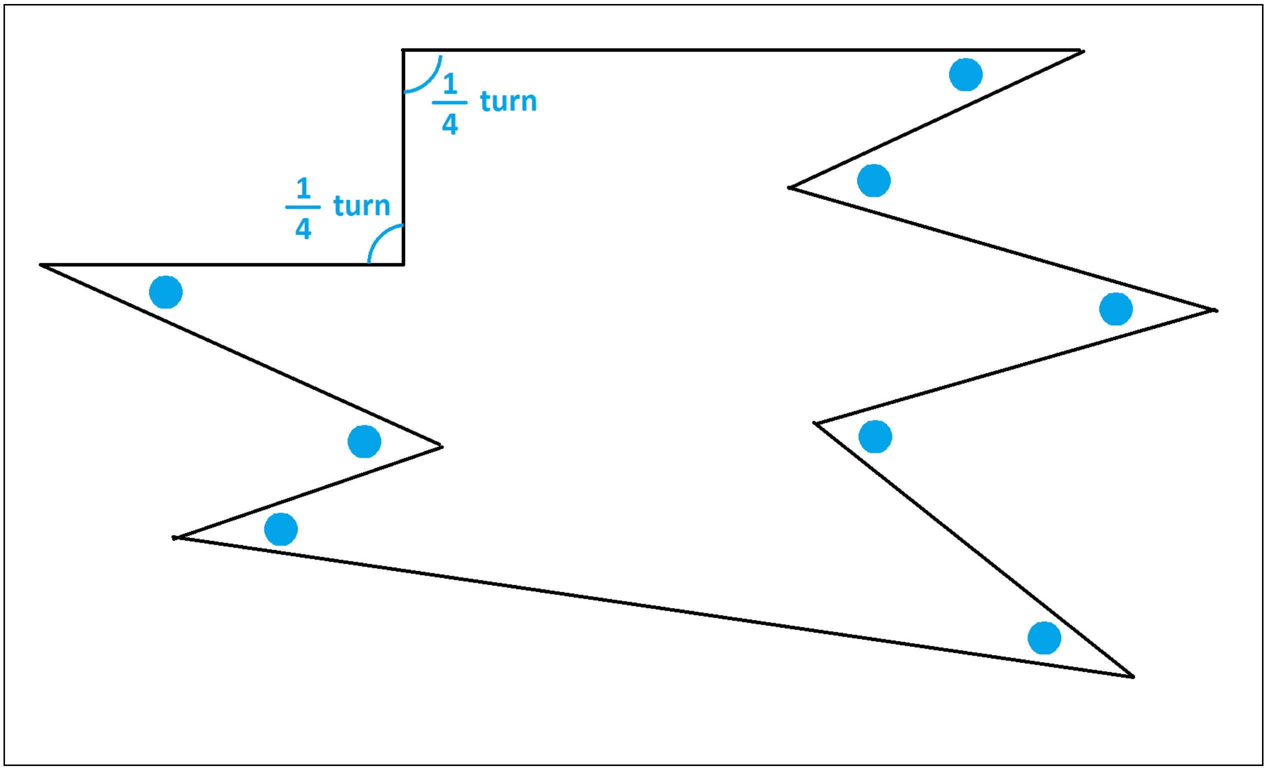

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Lesson ASIDE: How Many Degrees are in a Martian Circle?

This is a serious question. I want an actual answer!

How many degrees are in a Martian circle?

When presented with a problem in math (or in life!) there are two fundamental and important first steps to take.

STEP 1: Have an emotional reaction.

Acknowledge your human self by acknowledging your human reaction. If a problem looks scary, say "This looks scary." If it looks fun, say "Wow! Cool!" If you are flummoxed and don't have a clue what to do, say "I don't know what to do!"

Next, take a deep breath, and then

STEP 2: Do something! ANYTHING!

The key is to work past any emotional block that might be holding you back. Turn the page upside down, get up and go for a walk to mull on what to do, draw a diagram, draw a tree, circle some words, answer a different question that might or might not be related.

Our Martian question is very strange and you might be feeling flummoxed by it. What could it possibly be asking?

So let's change the question! Let's ask:

How many degrees are in an Earthling's circle?

Well, we Earthling's say that there are 360 degrees in a circle.

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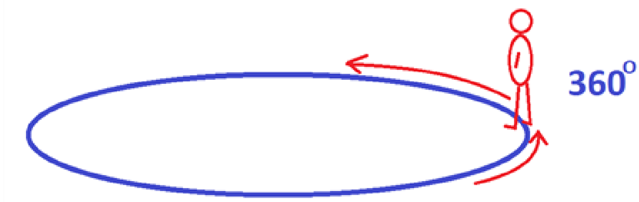


This now begs the question

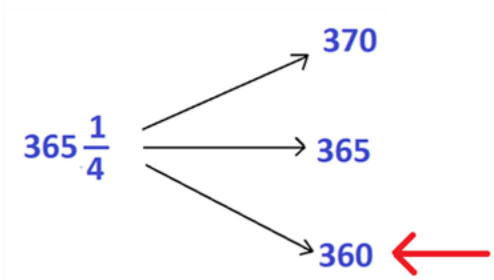
Why that number? Who chose the number 360 *for the count of degrees in a circle?*

And when you sit with this question for a while you might realize that this number is very close to a count in a large cyclic phenomenon we humans regularly experience: the count of days in a year.

Babylonian scholars of 4000 years ago were very much aware that the count of days in year is $365\frac{1}{4}$. *Shouldn't we be saying then that there are* $365\frac{1}{4}$ *degrees in a circle?*

The answer to this question might be clear: Who wants to do mathematics with the number $365\frac{1}{4}$? It's a very awkward number! So the natural thing to do is to round it to a friendlier value.

If we round the number $365\frac{1}{4}$ *to the nearest five or the nearest ten we get* 365 *and* 370*, not the number* 360*? Why did we humans decide to round* down *to* 360*?*



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Let's continue to be very human.

Thousands of years ago there were no calculators and all arithmetic had to be conducted by hand or in one's head. So let's work with a number that is amenable to easy calculation.

Often in mathematics we want to divide numbers by two, and we see already that choosing 365 as the count of degrees in a circle is unfriendly. Both 370 and 360 are even at least.

And then one realizes that 360 is a much friendlier number for arithmetic than 370: it is divisible by three, by four, by five, by six, by eight, by nine, by ten, be twelve, by fifteen, by eighteen, by twenty, and more!

> **Question:** *How many factors does the number 360 have? Compare that with the number of factors 370 possesses.*

So for two very human reasons—what we experience on this planet and our desire to avoid awkward work—we settled on the number 360 for the count of degrees in a circle.

> *Can we now answer how many degrees are in a Martian circle? What do we need to know?*

Martians might follow same reasoning we humans did, but in their context. So we need to know how many Martian days (we call them *sols*) are in a Martian year.

Each sol is 24 hours and 37 minutes long and Martians experience 667 sols in their year. So we might argue that Martians might initially say that there are 667 degrees in a Martian circle. But given that this an awkward number for basic arithmetic, they too will likely round that count to a much friendlier number.

So … What number do you think that might be?

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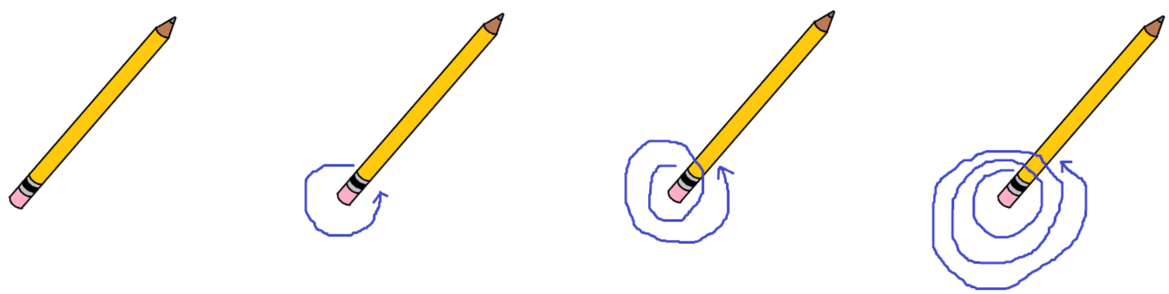
Lesson ASIDE: Turning Once, Turning Twice

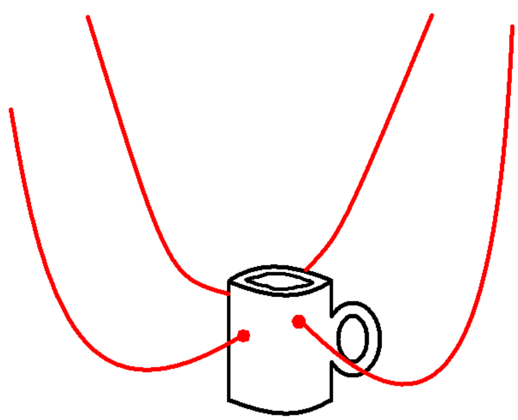
We said in lesson 1 that one full turn, two full turns, in fact, any count of full turns, has the same final effect as no turning at all.



No turns　　1 full turn　　2 full turns　　3 full turns

This is not true in some contexts! Try the following experiment.

Hold a teacup up in the air in the center of a room and have some friends tape four or five strings from the cup to various points about the room. Be sure to leave plenty of slack in the strings.



Rotate the teacup through one full turn, 360°*, tangling the strings in the process. (The handle serves as a pointer so that you can see when you have completed the turn.)*

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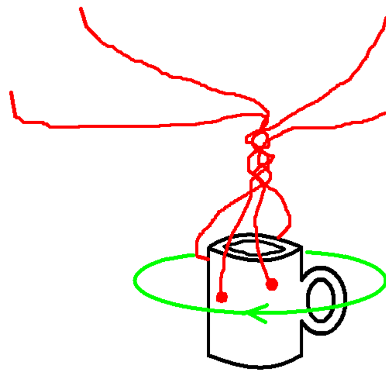

> *Now the cup is to be held fixed in space, never to move again!*
>
> *You and your friends' job now is to maneuver the strings up and under and around the cup and untangle them. (Whoever is holding the cup may have to move their hands out of the way, switch hands, and the like, but the cup itself is not to move.)*
>
> *Can you untangle the strings?*

Try as you might you will not succeed.
But let's make matters worse!

> *Give the cup another full turn IN THE SAME DIRECTION (* 720° *), tangling the strings even further.*
>
> *Holding the cup fixed in space, try again maneuvering the strings up and under and the around the cup. This time you will be able to untangle them!*

Absolutely do try this! (Perhaps try the experiment with three strings if you find yourself overwhelmed by strings.)

It turns out the that the tangle of strings from one full turn is fundamentally non-trivial—they cannot be untangled—but the tangle from two full turns is trivial and is equivalent to no tangle at all. (In fact, all tangles that result from an even number of full turns are trivial.)

So in this context, one-full turn is fundamentally different from two full turns.

Look up the WAITER'S TRICK for another physical demonstration of this phenomenon.

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Lesson 5: IT'S ALL A LIE! (Well, only sort of!)

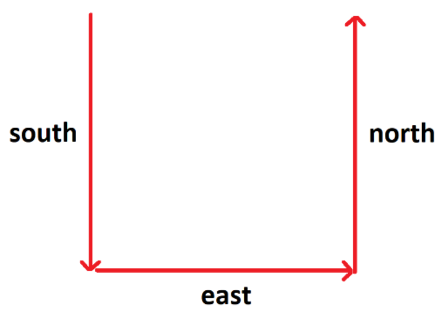
Here's a classic riddle.

One day a woman decides to take a three-mile walk.

She starts by heading directly south for one mile, trotting along at a happy pace, admiring the sunshine and the wildlife. Then she turns left 90° *and heads directly east for one mile, all the while enjoying the smell of the sweet air and the glorious sights of nature around her. Next, she turns left one more time, exactly* 90°, *and heads directly north just for one more mile. Surprisingly, after this third one-mile stretch she finds herself back to where she started!*

What color was the bear she saw on her walk?

What?! What an absurd question! Firstly, it is surely impossible to conduct the journey described and end up back to one's starting point.



And, secondly, none of the information provided mentions a bear nor provides any hint of the color of one. This puzzle is unanswerable nonsense!

Or is it?

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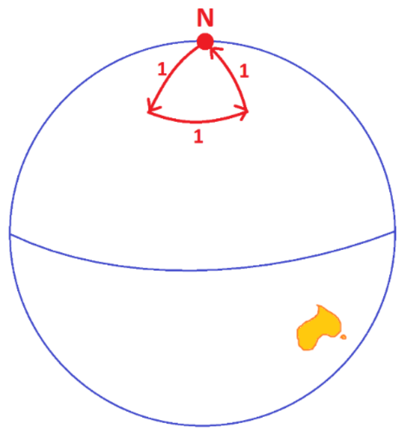
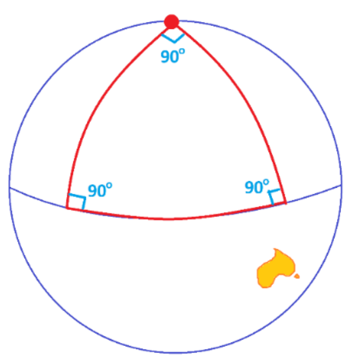
It turns out there is one place on Earth where one can walk a mile south, a mile east, a mile north and end up back to start, and where one might possibly see a bear. Start at the north pole! (In which case, any bear seen must be a polar bear and hence white.)



Comment: There are actually infinitely many different places in the southern hemisphere too at which one can conduct the journey described. (Choose a circle that circumnavigates the south pole with perimeter one mile, or with perimeter half a mile, or with perimeter one third of a mile, and so on. Now start the your three-mile journey one mile north of the circle you identify.) But as there are no bears in the Antarctic, the riddle cannot be set there. We must be in the northern hemisphere.

Let's extend the woman's walk. Suppose now she walks from the north pole south all the way down to the equator, then turns 90° east and walks an equal distance along the equator, to then turn 90° to head back to the north pole. This creates a large triangle with three 90° turns.

Notice that the angles in this triangle sum to 270°, not 180°!



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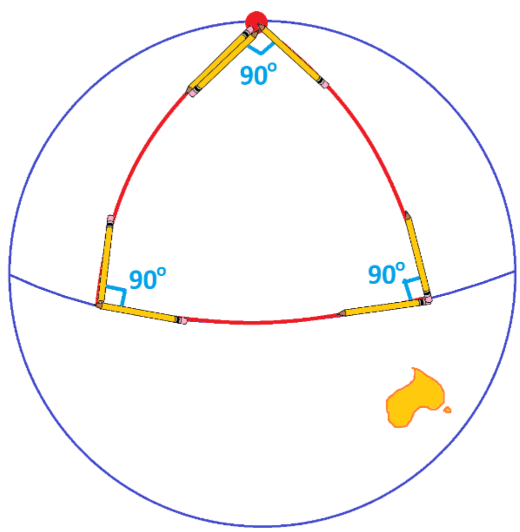
But here's the kicker.

> **Pencil pushing still works in this triangle and insists that the three angles of the triangle sum to 180°, half a turn!**

Imagine you walk the path of this big triangle, starting at the north pole and pushing a pencil straight in front of you towards the equator. When you reach the equator, rotate the pencil to point east. After pushing the pencil straight along the equator, you then turn it north. Next, you push the pencil straight north until you come back to the north pole, and then rotate the pencil through this third angle. Can you see in your mind's eye that if you could actually perform this experiment that the pencil will be back in its original alignment but facing the opposite direction, just as though it had undergone a 180-degree rotation?

What's going on?



Of course, you might well argue that this example does not apply because the Earth is not flat and the lines along which we are pushing the pencil are not straight. Fair enough. But then why are we willing to believe that the pencil trick is working properly for all the (small) triangles we are drawing on pieces of paper or on boards? How do we know the paper is flat? How do we know the board isn't slightly warped? The pencil trick will always say that the angles in a "triangle" sum to 180°!

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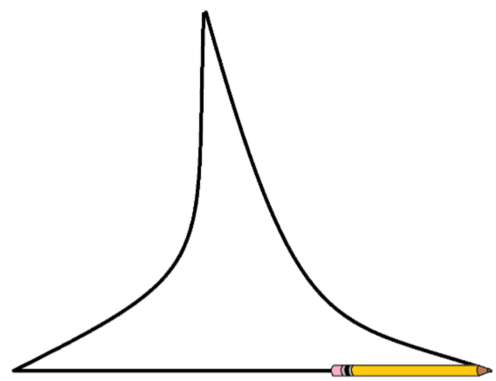
Question: *Here is a "triangle" with sides that are clearly not straight. Can you see that pushing the pencil along the sides of this triangle, and rotating it at each corner, still produces a net effect of a* 180° *turn on the pencil?*



So maybe the pencil is only speaking truth when we are sure the sides of the figure we are working with are, for certain, straight. After all, we did say that sliding the pencil along the side of a figure does not apply any turning to it.

But that now begs the question: How do we know whether or not a path is straight? If I were to go outside and walk due south my path, I am sure, will feel straight to me. Someone observing from space might say my path is curved, but here in Earth I am not sure how I would know.

For a triangle drawn on the classroom floor we might feel that its sides are straight. For a triangle drawn on a soccer field we might feel that its sides are straight. If we draw a triangle the size of the state of Iowa, would we still feel that its sides are straight?

As I question pencil-pushing more and more, more questions and worries come to the fore. My brain is starting to hurt!

So how do we extract ourselves from unending philosophical pickles?

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Let’s go back to what we like to believe.

In a theory of geometry we feel that there should be some notion of a line being “straight” and that we can make figures with segments of these straight lines. On a page of paper or on a chalkboard, which we like to believe is “flat,” we draw these lines with a ruler which we believe is straight. On the surface of the Earth, the line segments we feel are straight are different, they are sections of large circles, like a section of the equator. (These lines certainly feel “straight” when on the surface of the Earth.)

But these two geometries—geometry on paper and geometry on a sphere—will be different. We don’t think of the geometry on the surface of the Earth as “flat.”

And what is the difference we observed? It’s the nature of angles inside triangles.

So let’s turn all our confused and hazy thoughts into a definition and a statement of belief.

Definition: Call a geometry *flat* if we can be sure that the three interior angles of each and every triangle in the geometry have measures adding to 180°.

A Fundamental Assumption of Geometry:
We choose to believe that the geometry studied in school is flat.

There! Philosophical pickles resolved simply by being fully open and honest and declaring that we’re going to make a fundamental assumption about how our geometry shall work. Spherical geometry is not flat and shall work differently than the geometry we study in school, one that we presume is flat.

HISTORICAL COMMENT: The great Greek scholar Euclid (ca 300-260 BCE), hailed as the one of the greatest intellects of all time, is revered for his 13-volume series, *The Elements*. Here Euclid outlines the entire theory of geometry as based on a small set of fundamental beliefs and their logical consequences. (Euclid starts with 10 basic beliefs, but later scholars realized Euclid

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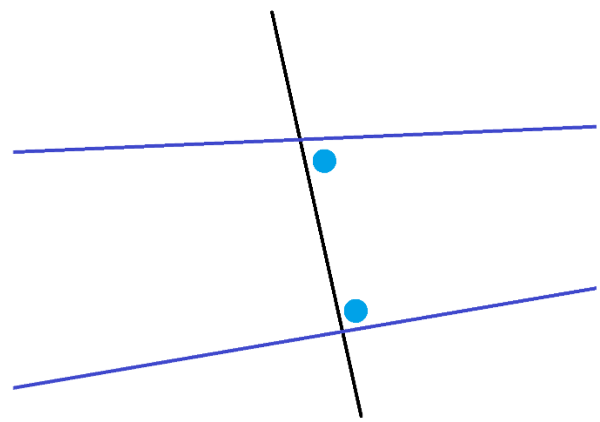
implicitly made additional assumptions in his work. In the late 1800s German mathematician David Hilbert adjusted the list of Euclid's beliefs to 28 axioms to produce a full set of initial beliefs needed for a logically complete theory of geometry.)

Euclid declared too, as one of his fundamental assumptions, a statement about "flatness." He chose a variant belief about triangles.

> **Euclid's Fifth Postulate:** *If the two angles shown have measures summing to less than a half turn, then the two blue lines shown, if extended, will meet to form a triangle.*



Euclid then used this assumption (and his other assumptions) to logically deduce that the measures of the three interior angles of any triangle sum to 180°. But it is also possible to start with our fundamental belief about angles in a triangle and deduce Euclid's Fifth Postulate from it (and the remaining basic assumptions). Thus, these two different fundamental beliefs about triangles are logically equivalent beginning assumptions to geometry.

The geometry studied in school textbooks, our "flat geometry," is usually called *Euclidean geometry.*

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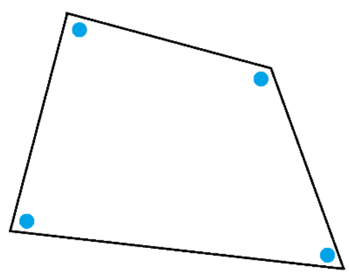
Lesson 6: Pencil Pushing Inspires Truth Nonetheless

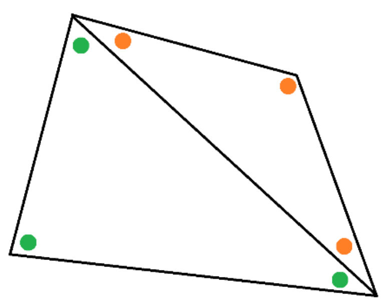
We are now assuming that geometry is flat, that is, that angles in triangles are sure to have measures that sum to half a turn. In other words, we are simply choosing to believe what pencil pushing says about angles in triangles.

Does this mean that what the pencil says about angles for other shapes might be true too in flat geometry?

Example: Back in lesson 2 pencil pushing suggested that the four angles in this quadrilateral have measures that sum to 360°. Is this true in flat geometry?



Answer: Yes, it is! Divide the quadrilateral into two triangles as shown. By our flatness belief, the measures of the three green angles sum to 180°, as do the measures of the three orange angles. And the sum of all six angles, which is 360°, matches the sum of the four original blue angles in the quadrilateral.



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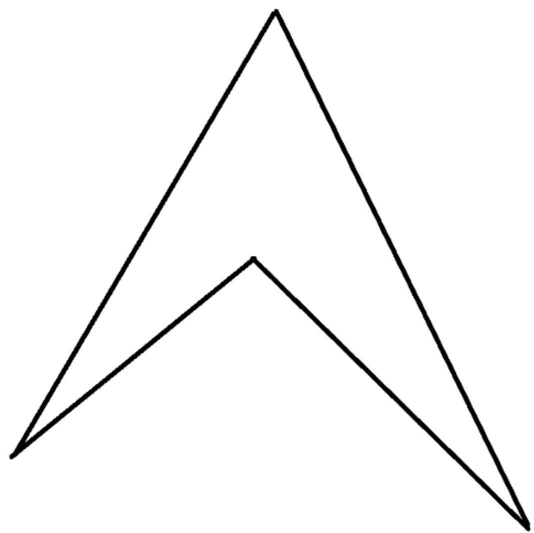
Question 1: Show, in flat geometry, that the sum of the measures of the interior angles of this next quadrilateral do indeed sum to 360°.



Question 2: Do you feel it is true that every 5-sided polygon subdivides into three triangles? If so, what does this say is the sum of the measures of the interior angles of a five-sided polygon?

Question 3: Do you feel it is true that every 13-sided polygon subdivides into eleven triangles? If so, what does this say is the sum of the measures of the interior angles of a 13-sided polygon?

What general formula might you suggest for the sum of measures of the interior angles of a polygon with N sides?

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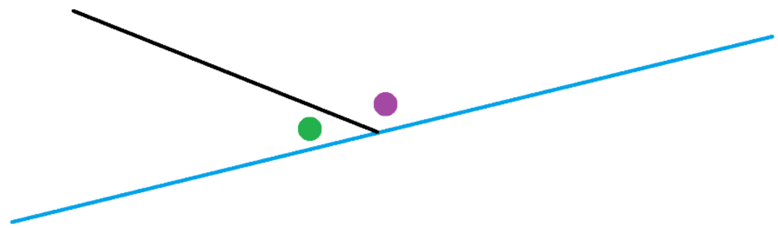
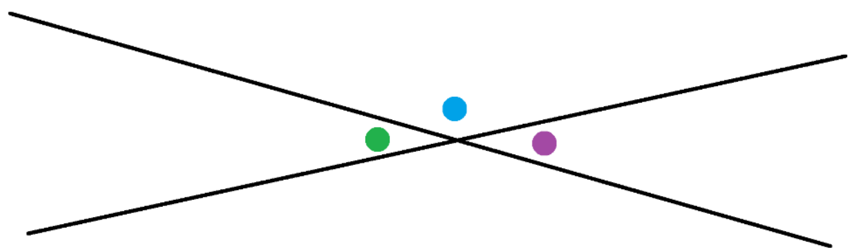
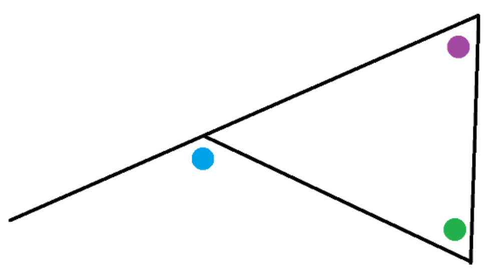
Question 4: The beginning assumptions in the theory of geometry show that for any straight line, such as the one shown in blue, with two angles constructed on it as shown have measures summing to 180°. (This feels intuitively correct!)



a) Use this next diagram to show that pencil-pushing is correct in suggesting that vertical angles must have the same measure. (Explain why the green and the purple angles have the same measure.)



b) Explain why the sum of the measures of the green and the purple angles in this diagram equals the measure of the blue angle.



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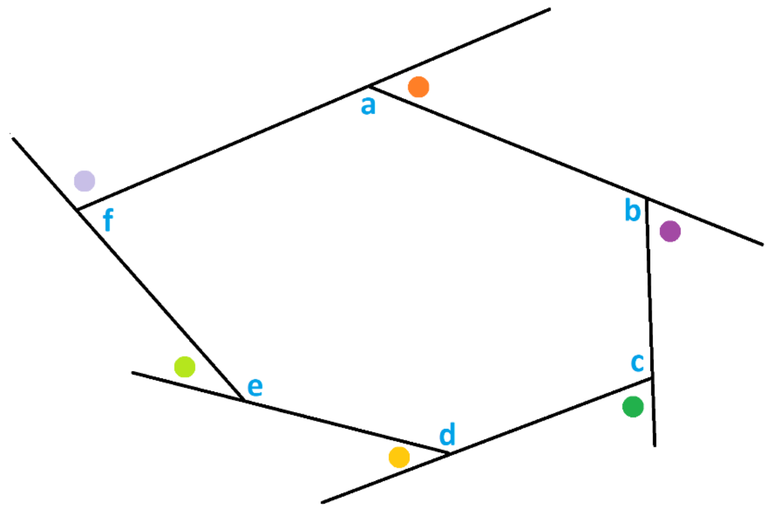
c) A hexagon subdivides into four triangles and so the measures of its six interior angles sum to $4\times180^\circ$. In this diagram where these measures are denoted a, b, c, d, e, and f and we have $a+b+c+d+e+f=4\times180^\circ$.



Use this to show that pencil pushing was correct to say that the six exterior angles (shown as dots) have measures that sum to 360°.

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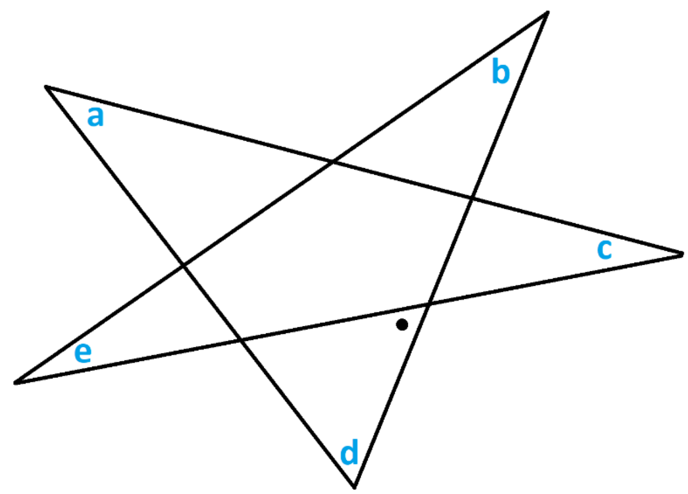
d) CHALLENGE: Show that pencil-pushing was correct to say that the measures of the angles shown in a lopsided star (represented as letters) add to half a turn, that is, $a+b+c+d+e=180^\circ$.

(Hint: Why is the measure of the angle with the dot equal to $b+e$?)



It seems we can use pencil-pushing nonetheless to guide us to possible true statements in flat geometry.

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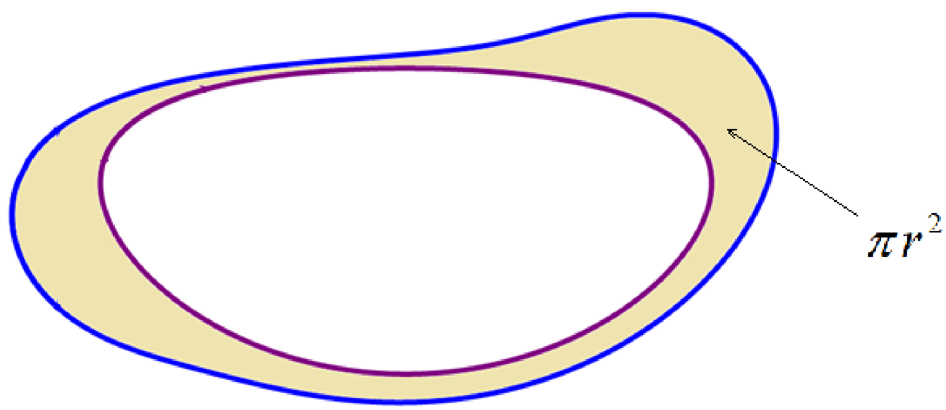
Lesson 7: An Inescapable Area

We'll now put some of our geometry results to good use. Our goal is to establish this next strange and curious fact.

Ride your bicycle in a large loop, starting in one position and ending in the same position. If you do this over sand or through snow or if your tires are dusty or wet, each tire will leave a track on the ground.

No matter which loop you choose to ride, the area between the two tracks is sure to be πr^2 *where* r *is the distance between the two axles of the wheels.*

You cannot escape and area of πr^2 *!*



Whoa!

Roll your bicycle out of your garage in the morning and roll it back to the same storing position in the evening. Then the amount of area you traversed between your front and back tires was πr^2, no matter what loop you rode that day!

Double whoa!

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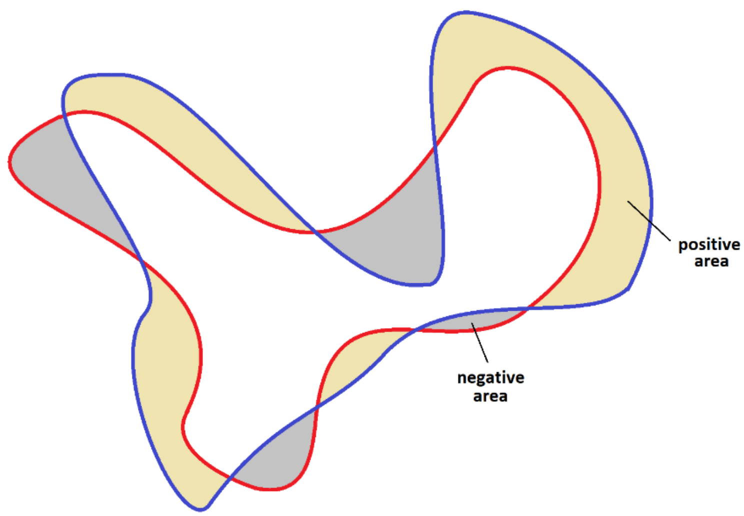
There are a few caveats to this claim.

If front and back wheel tracks cross, then there will be regions of area inside a given track and other regions outside of it. (In the sketch below, each grey region is inside the red track, for instance. If you choose to focus on the blue track, then each beige region lies inside it.) The total amounts of inside and outside areas are sure to differ by πr^2 !



We've also implicitly assumed a "riding a loop" means that the path you trace has you conduct one full revolution of turning in total. But if you ride you bike around your house twice and then park it back in the garage (that is, you perform a "double loop"), then the area between your two tracks shall be $2\pi r^2$. If you perform a triple loop, the area between the tracks shall be $3\pi r^2$, and so on. And if your tracks cross over as you ride, then you should regard some areas as positive area and others as negative area (like the grey and beige regions in the diagram above) and the total sum of areas in this case shall be πr^2 or $2\pi r^2$ or $3\pi r^2$, and so on.

Also, if you follow a figure 8 (a forward loop and a backward loop adjoined), for instance, then the total area between your tracks shall be $1\times\pi r^2 + (-1)\times\pi r^2 = 0$.

Let's now work to these bold claims about area. Let's study the mathematics of bicycle tracks next.

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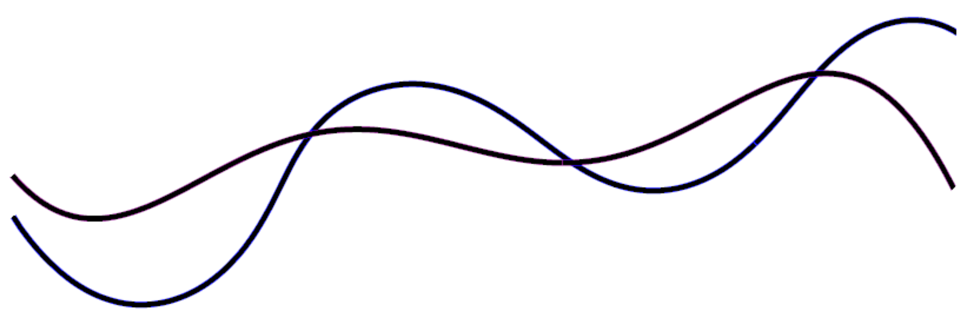
Lesson 8: Which Way Did the Bicycle Go? … and other bicycle ponderings.

Here's a famous puzzle.

> *You are brought to a crime scene. You are told that a thief just made off with a bag full of diamonds, escaping on a bicycle. You come across the following pair of bicycle tracks in the snow, no doubt made by the fleeing thief. But which way did the thief go?*



> *Just by looking at the shapes of the tracks (tread marks and splashes of snow are inconclusive) can you determine which way the thieving cyclist went: left to right or right to left?*

This charming curiosity appears in Joseph Konhauser, Dan Velleman, and Stan Wagon's text *Which Way did the Bicycle Go? … and Other Intriguing Mathematical Mysteries* (MAA, 1996). They learned of the problem from a geometry course being developed at Princeton in the 1980s. This problem is also described in Sherlock Holmes novel, but Doyle, according to the three authors above, gives an incorrect (non-mathematical) solution for determining the direction of travel.

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I usually present this puzzle as a group activity and guided discussion. To do so too, you will need the following supplies

A bicycle
Yard Sticks/Meter Sticks

and if want to conduct the activity inside

Sidewalk chalk
About 40 feet of poster paper
Thick red markers and thick blue markers
Painter's tape

Start by telling the story of the thief making off with a bag of diamonds while roughly sketching a pair of bicycle tracks on the board. Then ask

Does it make sense that there are two tracks?

Answer: Yes! A bicycle has two wheels and each wheel leaves a track.

Now mention that your sketch is highly inaccurate, that we should work with a genuine pair of bicycle tracks in order to properly observe their mathematical structure.

Ask participants to cover each wheel of a bicycle with chalk, one wheel with blue chalk say, and the other red chalk. Roll out the poster paper in a corridor and ask a volunteer to ride a wobbly path down the paper roll (making sure to not head off the paper, but also to give a path with lots of left and right turns).

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The chalk marks on the paper will likely be faint. Ask participants to draw over the colored traces with markers of the same colors.

Now bring the paper back into the room and hang it on the wall with painter's tape. Ask folk to forget what they just witnessed and ask

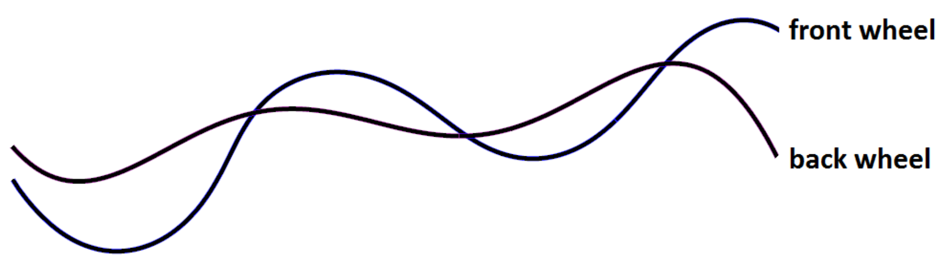
> *Okay. You just came across these very tracks. Is it possible to determine which way the bicycle went?*

(Of course, one can conduct this activity outside if you are able to ride a bicycle across a dirt ground, for instance, to make tracks.)

Two nudging questions

Ask these questions when/if frustration becomes too much.

> *Is it possible to say which track is most likely the back-wheel track and which is the front wheel track?*
>
> **Answer:** Think of how a bicycle is constructed. The front wheel of the bicycle can turn left and right, but the back wheel of a bicycle is fixed in its frame. This means that the more "stable" track is likely to be the back-wheel track and the wobblier one the front wheel track.



After some mulling time, next ask

> *You said that the back wheel is more "stable." In what sense? What makes it stable? What else do you notice about the construction of a bicycle?*

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Hold up a bicycle. Observe that the back wheel is indeed fixed in its frame, but more is true.

i) The back wheel always points towards the point of contact on the ground of the front wheel, no matter which way the front wheel is pointing.
ii) The distance to that point of contact is fixed: it is the distance between the axles of the wheels.

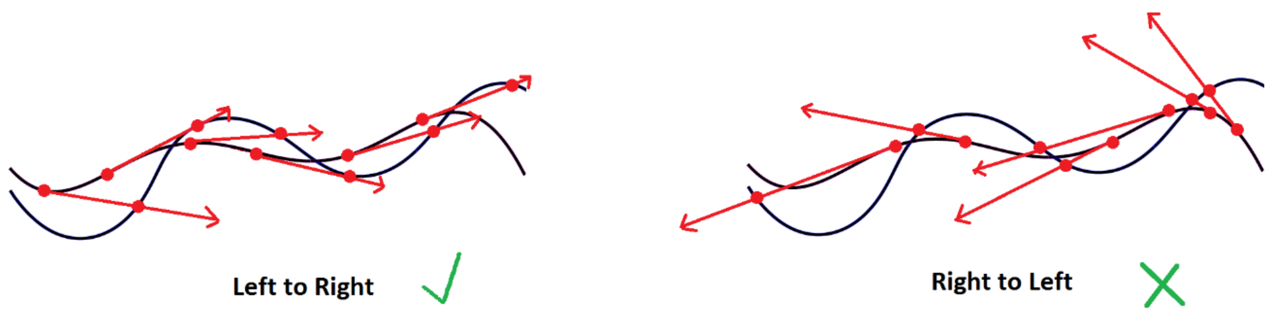
This is usually enough to prod the group conversation to the solution. Participants might rephrase these two observations as

> "So, each point on the back-wheel track points towards the front-wheel track at a fixed distance."

Help guide everyone to a more precise description, something along the following lines.

> "The tangent line at each point on the back-wheel track intercepts the front wheel track at a fixed distance."

Hold up a yard stick as a tangent line at a point on the back-wheel track. Demonstrate that, in moving the along the back track, the tangent line does indeed intercept the front wheel track at some consistent fixed distance in one direction of motion but not in the other.



We have

> *For a given pair of bicycle tracks, it is not only possible to determine in which direction the bicycle traveled, but also the length of the bicycle that made those tracks!*

Comment: Of course, we are presuming a mathematically ideal scenario here. As a bicycle turns left or right it tilts on its wheels and the points of contact to the ground shift to the side of the center-line of the wheels. Also, in riding a bicycle there are small degrees of slipping and sliding

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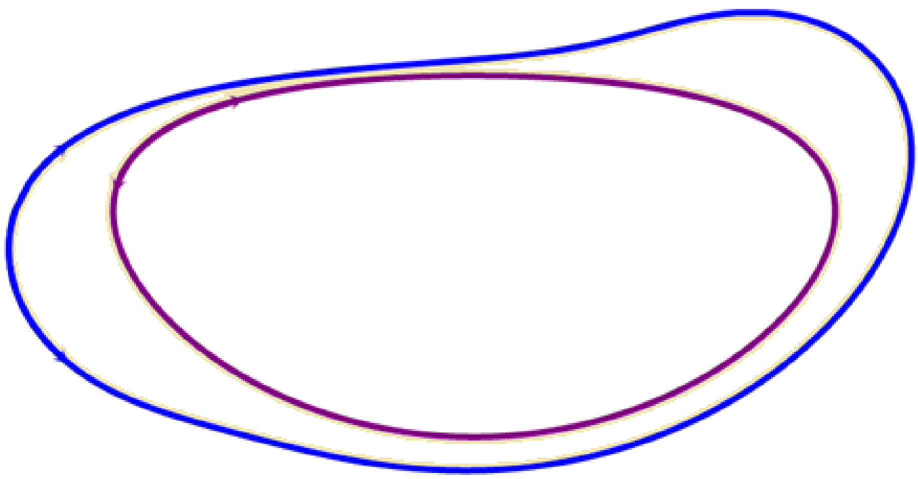
too. But despite these mathematical imprecisions, the technique described works extraordinarily well in practice! Next time you see mud tracks, water tracks, or sand tracks of a bicycle on the ground, examine them. You will be able to determine the direction of travel of the bike!

Here's another picture of mathematically precise tracks. Here the cyclist rode in a loop.

> *In this picture, which is the back-wheel track? Which is the front-wheel track? Which way did the cyclist travel?*



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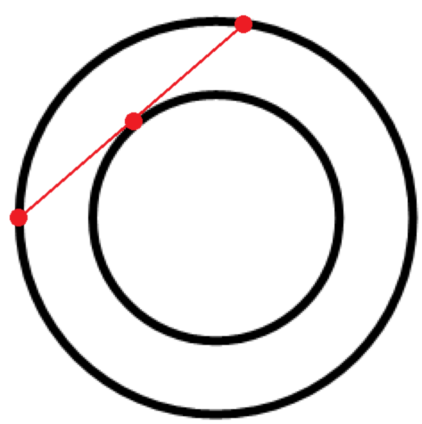
When Does the Method Fail?

There are instances in which we cannot determine the direction of motion of the cyclist.

> *If the rider rides a perfectly straight path, the two tracks of the wheels overlap along a single straight line. It is impossible in this case to ascertain the direction of motion from analyzing the tracks.*
>
> *What is another instance of rider motion for which one could not determine the direction of travel?*
>
> **Answer:** A cyclist could ride in perfect circle.

It is worth pausing on this answer to ask: In this case, what do the two tracks look like? Can you still determine which track is the back-wheel track and which comes from the front wheel?

Because of the symmetry of the situation we cannot determine the direction of motion in this case too.



Here's a surprisingly tough question.

1. *Are these the only two cases in which we could not determine the direction of travel?*

   More precisely,

   If a pair of smooth curves have the property that each tangent line to one curve always intercepts the other curve at two locations, a fixed distance r either side of the point of contact along the tangent line, must the curves each have constant curvature?

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Here are two more really tough questions.

2. *Could two bicycles of different lengths produce the same pair of (non-straight) bicycle tracks?*

3. *Could a bicycle produce a single non-straight track? (That is, is it possible to ride a non-straight path on a bicycle so that back wheel track covers the front wheel track? Alternatively, could a bicycle and a unicycle produce exactly the same non-straight track?)*

All three questions can be united as one.

Could two bicycles of lengths r and s produce the same non-straight tracks?

Allow the cases $s = -r$ and $s = 0$.

These questions are fun to explore, but they will require some technology help in order to draw bicycle tracks with the correct mathematical structure. (Write a set of parametric equations for a back-wheel curve, compute the unit tangent vector at a location on this curve and plot the endpoint of this vector. The trace of this point is the front wheel track.)

It might be good to mull on these questions only a short while. The next part of the lesson continues the activity and brings matters back down to a manageable level.

Comment: A solution to Problem 3 appears in Dr. David Finn's 2002 paper "Can a Bicycle Create a Unicycle Track?". This question, along with question 1, is also discussed in Stan Wagon's book *Mathematica In Action: Problem Solving Through Visualization and Computation* (Springer). And the 2013 paper "Tractrices, Bicycle Tire Tracks, Hatchet Planimeters, and a 100-year-old Conjecture" by Robert Foote, Mark Levi, and Serge Tabachnikov explores the mathematics of tracks in great depth.

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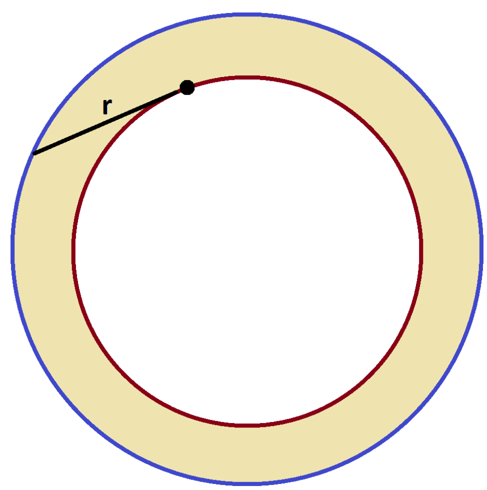
THE AREA BETWEEN TRACKS

Let's now attend to the goal of explaining why, after riding a complete loop, the area of the regions between the two tracks is sure to be πr^2 where r is the distance between the two axles of the bicycle.

Optional Geometry Exercise:

Suppose a cyclist rides a perfect circle.

If you are familiar with basic circle theorems from geometry class and the Pythagorean theorem, you can prove that the area between the two circular tracks formed is sure to be πr^2 no matter the size of the circles. How?



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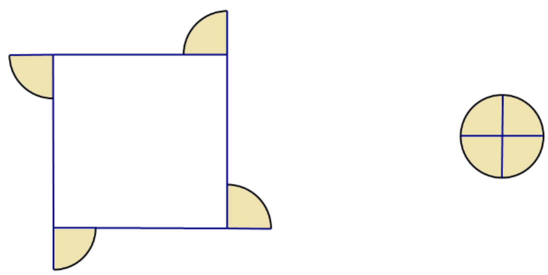
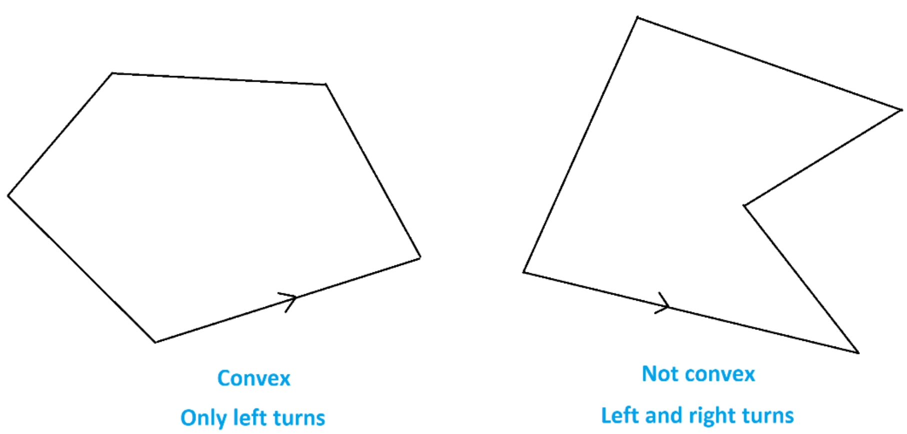
Let's start by riding bicycles along loops that are a polygons!

> *Suppose we force the back wheel of the bicycle to follow a square path, pivoting on the back wheel at the corners and swinging the front wheel round. Can you see what with the front wheel path will look like? Can you see that the area between these tracks is, for sure,* πr^2 *?*



Think about rectangular back-wheel paths, triangular back-wheel paths, and other polygonal back-wheel paths.

For Consistency …
Let's assume we're always pushing the back wheel of a bicycle along a looped path in a counterclockwise direction. And to make sure the front wheel always stays on the outside of the loop, let's assume we have a path that only makes left turns when we pivot on the back wheel. This means we'll work with, just for now, *convex* shaped polygonal paths.



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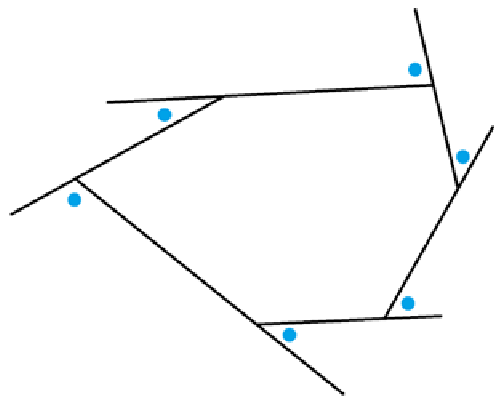
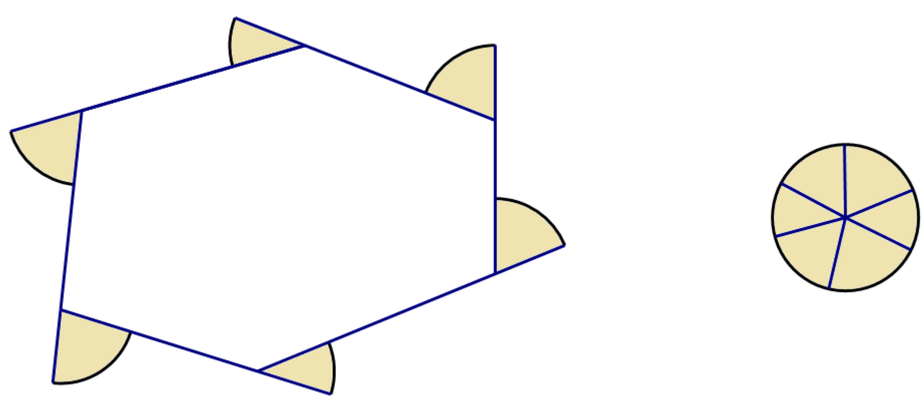
Our pencil-pushing work from lessons 2 and 6 now helps us out. We saw there that the measures of the exterior angles of a convex polygon always sum to one full turn, 360°.



This allows us to conclude the following.

> If we force the back wheel to follow a convex polygonal path, then the region between the front wheel and back wheel tracks is composed of a set of sectors of a circle of radius r. Because the exterior angles of a regular polygon sum to 360°, these sectors fit together perfectly to make one full circle.
>
> Thus the area between the front-wheel and back-wheel tracks for a cyclist riding in a convex polygonal path is πr^2.



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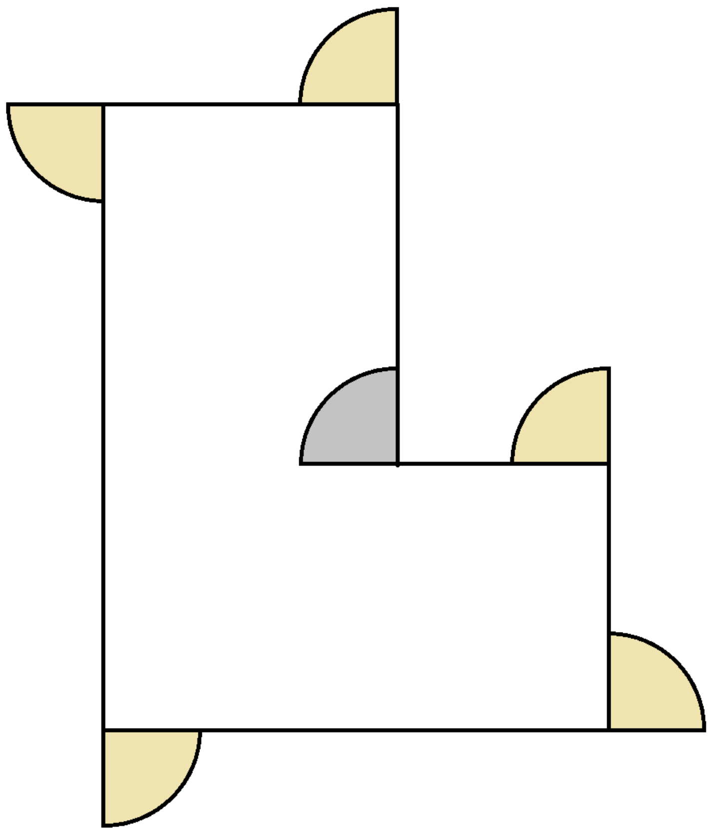
EXPLORATION: Examine what happens for a cyclist riding a non-convex polygonal path.



If you regard areas swept out in a counterclockwise direction as positive and areas swept out in a clockwise direction as negative, does it seem that the sum of positive and negative sector areas still make one full circle of positive area?

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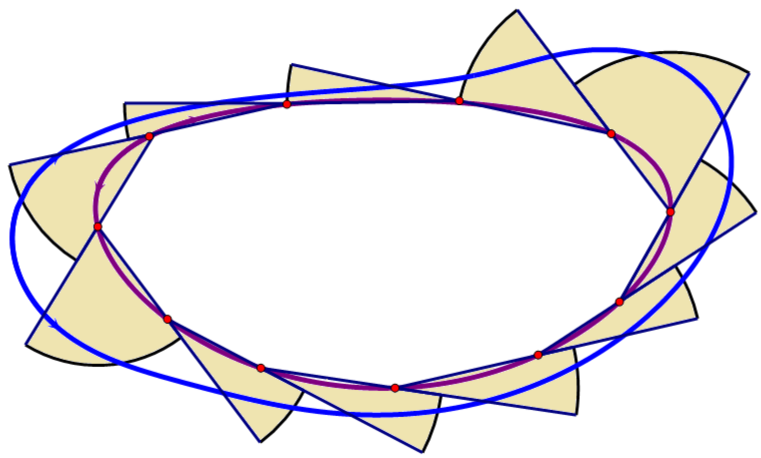
Smooth Curves

Any smooth convex curve can be approximated as a convex polygon by drawing short line segments between points of the curve. And if we ride the bicycle along the convex polygonal approximation, we know the area between the bicycle tracks approximating the original tracks is sure to be πr^2. This approximates the area between the original pair of tracks.



We can get a better approximation of the true area between the tracks by using a polygon to approximate the back-wheel track with many more sides composed of much shorter line segments. But we know that area between the tracks in this approximation will be πr^2 too. We can do finer and finer approximations, but each time these approximations have value πr^2. Since the approximate values converge to the true value of the area between the original tracks, we can only conclude that the area between the original tracks must be πr^2 too!

This does it for convex bicycle paths!

And if we develop a sufficiently robust theory of positive and negative area, one can indeed prove the general result. For any loop that has you turn a total of k full turns as you traverse it (for example, a "double loop" has $k = 2$, and figure 8 has $k = 0$), the total area between the tracks, summing positive and negative areas, is sure to be $k \times \pi r^2$.

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