GLOBAL MATH PROJECT

Pencil Pushing

Exploring the Joys—and Mysteries—of Angles in Geometry

STUDENT HANDOUTS

Video Resource

Access videos of *Pencil Pushing* lessons at https://gdaymath.com/courses/gmp/.

PENCIL PUSHING

Handout A

Here are some practice questions from lessons 1 and 2. Play with the pencil!

These questions are just for fun practice. Give them a try – but if you get stuck on a problem, don't worry about it. Just skip it and try a different one.

Practice Question: *To how many degrees do the following fractions of turn correspond?*

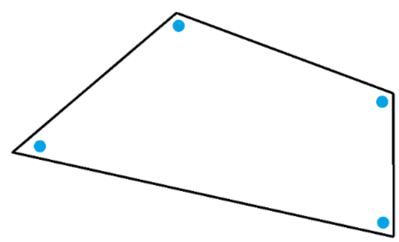
a) $\frac{1}{6}$ *of a turn* *b)* $\frac{3}{4}$ *of a turn* *c) one-and-a-half turns*

To what fraction of a turn do the following counts of degrees correspond?

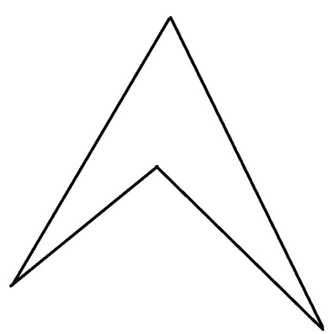
d) 45° *e)* 20° *f)* 1080° *g)* 0.01°

Activity 1:

a) Draw a four-sided figure and apply the pencil pushing idea to its interior four angles. Show that the four angles have measures that sum to two half turns, that is, add to 360°.



b) Consider the following quadrilateral. Identify its four interior angles. Show that its four interior angles also have measures that add to two half-turns.



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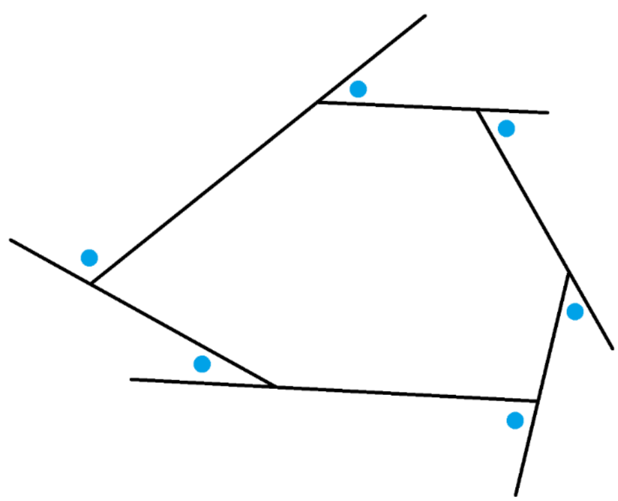
Activity 2: Explore the measures of the interior angles of five-sided figures. What can you say about the sum of those measures? (WARNING: A pencil inside a pentagon undergoes more than just a single half turn.)

Activity 3: Draw a large 13-sided figure and use pencil-pushing to find the sum of the measures of its interior angles. How many half-turns?

Activity 4: Find a general formula that seems to hold for the sum of measures of the interior angles of an N -sided figure.

Activity 5: What can you say about the sum of measures of the (exterior) angles shown for this polygon?



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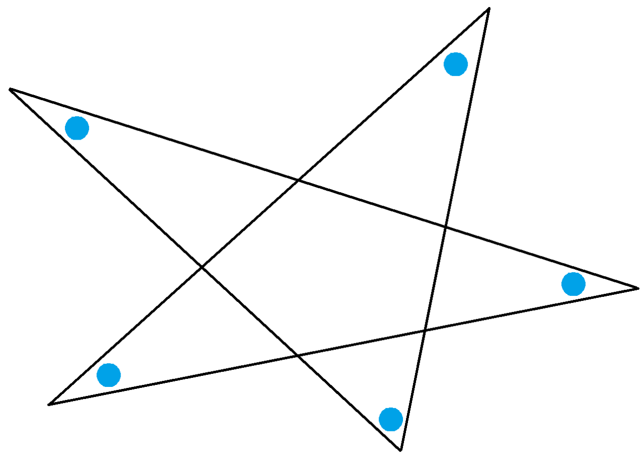
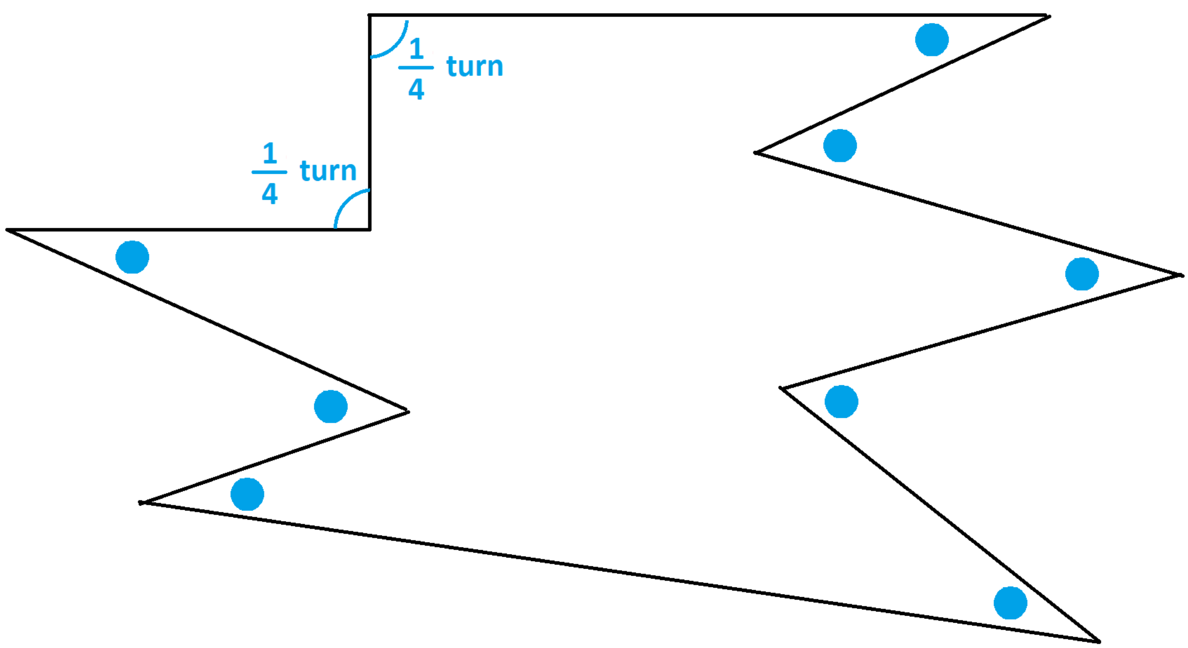
Activity 6: What does pencil pushing say about the sum measures of the five angles in a lop-sided five-pointed star?



Question 7: Does the following polygon exist if we insist that all the angles marked with a blue dot are congruent, that is, have the same measure?



Solutions to Handout A

Practice Question:

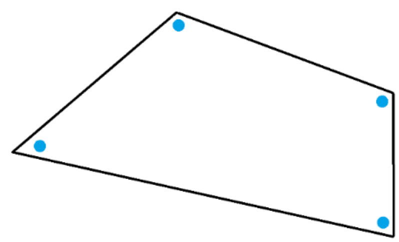
a) 60° b) 270° c) 540°

d) One eighth of a turn e) One eighteenth of a turn

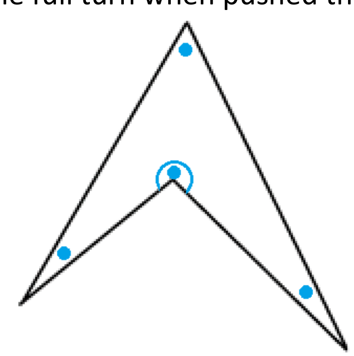
f) Three full turns g) $\frac{1}{36,000}$ of a turn.

Activity 1:

a) Do indeed show that a pencil undergoes one full turn.



b) The interior angles are as shown. Making sure the pencil stays inside the figure, we see that it again undergoes one full turn when pushed through all four angles.



Activity 2: It seems that the five interior angles of a pentagon have measures that sum to one-and-a-half turns, that is, to 540°.

Activity 3: It seems that the 13 interior angles of a 130sided polygon have measures that sum to eleven half turns, that is, to five-and-a-half full turns.

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Activity 4: We might conjecture the interior angles of an N-sided polygon have measures that add to $N-2$ half turns, that is, to $(N-2)\times 180^\circ$.

Activity 5: It appears these exterior angles have measures that add to one full turn, 360°.

Activity 6: It appears these five angles have measure that sum to one half turn, that is, to 180°.

Question 7: According to activity 4 we believe that the measures of the ten interior angles of this ten-sided shape sum to $8\times 180^\circ = 1440^\circ$. Thus we have

$$90^\circ + dot + (360^\circ - dot) + dot + (360^\circ - dot) + dot + dot + (360^\circ - dot) + dot + 270^\circ = 1440^\circ.$$

That is,

$$2dot + 1440^\circ = 1440^\circ$$

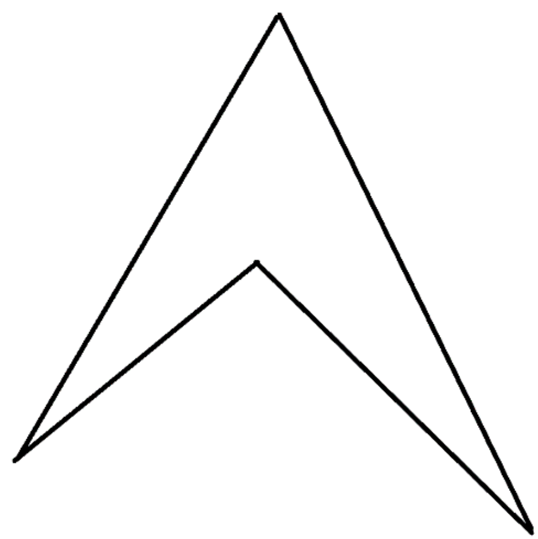
showing that measures of all the angles marked by a dot are zero! The figure cannot exist.

Handout B

Here are some practice questions from lessons 5 and 6. This time we'll simply take as a given that that the measures of the three interior angles of any triangle sum to 180°.

These questions are just for fun practice. Give them a try – but if you get stuck on a problem, don't worry about it. Just skip it and try a different one.

Question 1: Show, in flat geometry, that the sum of the measures of the interior angles of this next quadrilateral do indeed sum to 360°.



Question 2: Do you feel it is true that every 5-sided polygon subdivides into three triangles? If so, what does this say is the sum of the measures of the interior angles of a five-sided polygon?

Question 3: Do you feel it is true that every 13-sided polygon subdivides into eleven triangles? If so, what does this say is the sum of the measures of the interior angles of a 13-sided polygon?

What general formula might you suggest for the sum of measures of the interior angles of a polygon with N sides?

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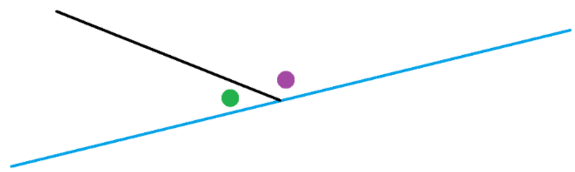
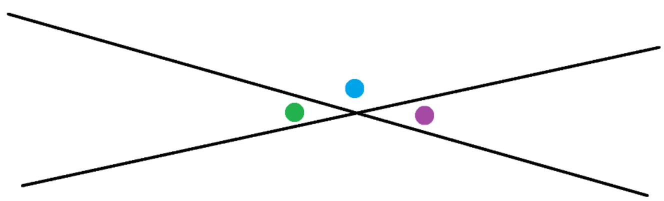
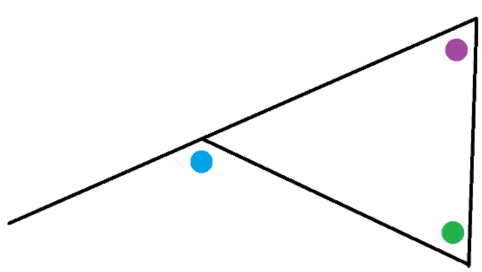
Question 4: The beginning assumptions in the theory of geometry show that for any straight line, such as the one shown in blue, with two angles constructed on it as shown have measures summing to 180°. (This feels intuitively correct!)



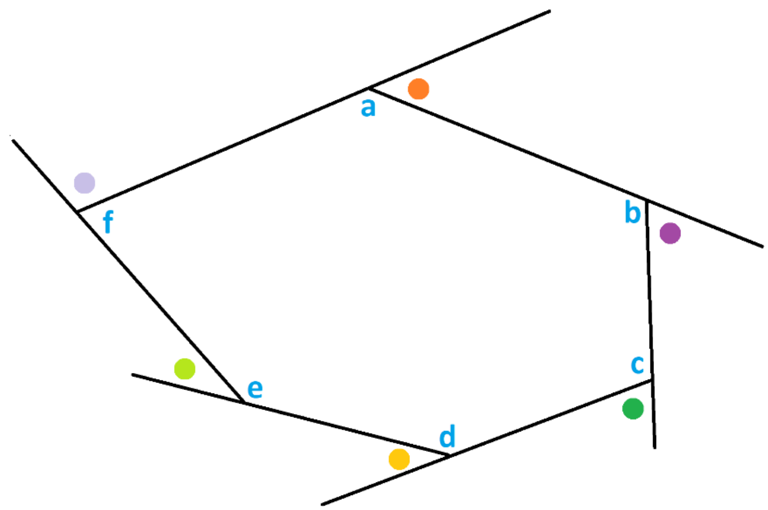
a) Use this next diagram to show that pencil-pushing is correct in suggesting that vertical angles must have the same measure. (Explain why the green and the purple angles have the same measure.)



b) Explain why the sum of the measures of the green and the purple angles in this diagram equals the measure of the blue angle.



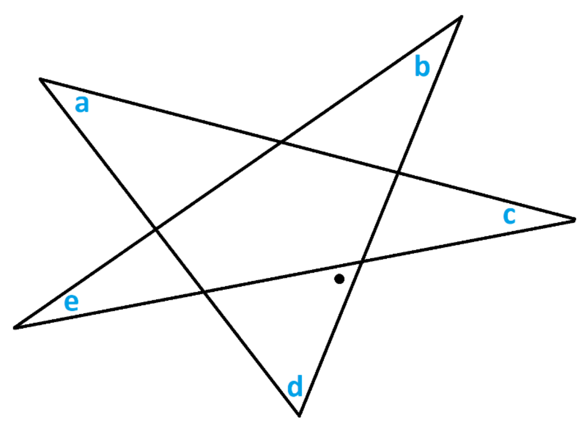
c) A hexagon subdivides into four triangles and so the measures of its six interior angles sum to $4 \times 180^\circ$. In this diagram where these measures are denoted a, b, c, d, e, and f and we have $a+b+c+d+e+f = 4 \times 180^\circ$.



Use this to show that pencil pushing was correct to say that the six exterior angles (shown as dots) have measures that sum to 360°.

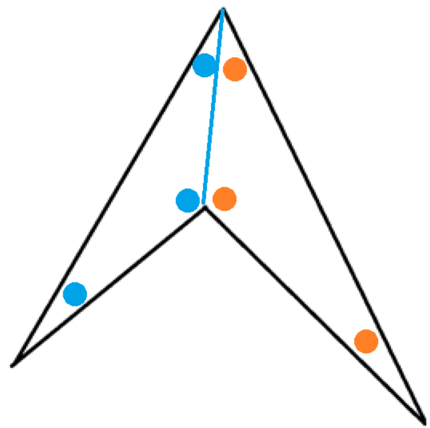
d) CHALLENGE: Show that pencil-pushing was correct to say that the measures of the angles shown in a lopsided star (represented as letters) add to half a turn, that is, $a+b+c+d+e = 180^\circ$.

(Hint: Why is the measure of the angle with the dot equal to $b+e$?)



Solutions to Handout B

Question 1: The tree blue-dot angles sum to 180°, as do the three orange-dot rectangles.



It follows that the four interior angles of the quadrilateral do indeed sum to 360°.

Question 2: This seems to be true. In which case the five interior angles of any pentagon add to $3\times180^\circ = 540^\circ$.

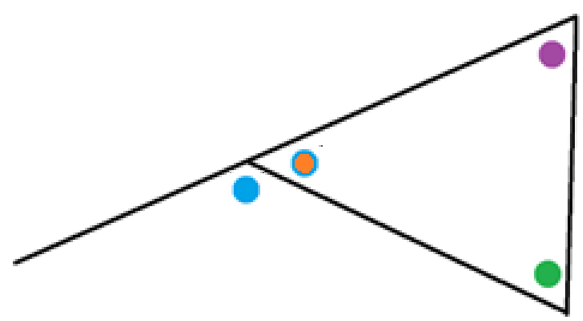
Question 3: It seems believable to say that any 13-sided polygon divides into 11 triangles and so has interior angles that sum to $11\times180^\circ$.

In general, an N-sided polygon divides, it seems, into $N-2$ triangles and so has interior angles that sum to $(N-2)\times180^\circ$.

Question 4: a) We have $green+blue=180^\circ$ and $purple+blue=180^\circ$. It follows that

$$green = 180^\circ - blue = purple.$$

b) Identify the orange-dot angle shown.



We have that $orange + blue = 180^\circ$ and $orange + green + purple = 180^\circ$. It follows that $blue$ and $green + purple$ both equal $180^\circ - blue$, and so are equal in value.

c) The six exterior angles sum to

$$\begin{aligned}&(180^\circ - a) + (180^\circ - b) + \cdots + (180^\circ - f)\\ &= 6 \times 180^\circ - (a + b + \cdots + f)\\ &= 6 \times 180^\circ - 4 \times 180^\circ\\ &= 360^\circ.\end{aligned}$$

d) From part b) it follows that $dot = b + e$. (Look at the triangle that contains angles b and e. In the same way, the third angle in the triangle that contains dot and d equals $a + c$.

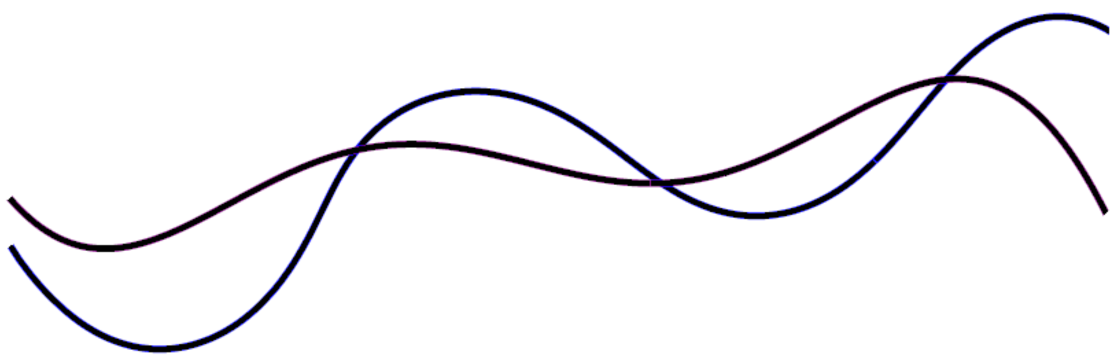
Since the three angles in a triangle sum to 180°, we have $d + (b + e) + (a + c) = 180^\circ$, which is what we were asked to prove.

Handout C

Here are some practice questions from lessons 7 and 8.

These questions are just for fun practice. Give them a try – but if you get stuck on a problem, don't worry about it. Just skip it and try a different one.

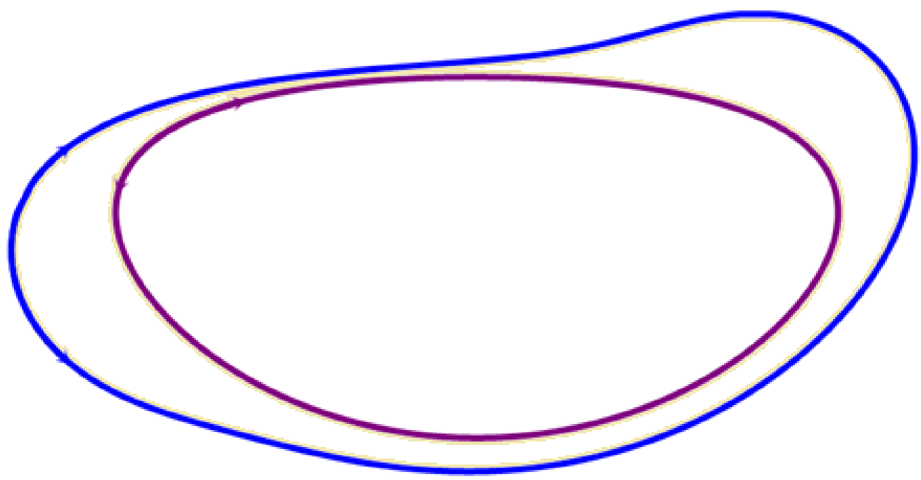
Question 1: You are brought to a crime scene. You are told that a thief just made off with a bag full of diamonds, escaping on a bicycle. You come across the following pair of bicycle tracks in the snow, no doubt made by the fleeing thief. But which way did the thief go?



Just by looking at the shapes of the tracks (tread marks and splashes of snow are inconclusive) can you determine which way the thieving cyclist went: left to right or right to left?

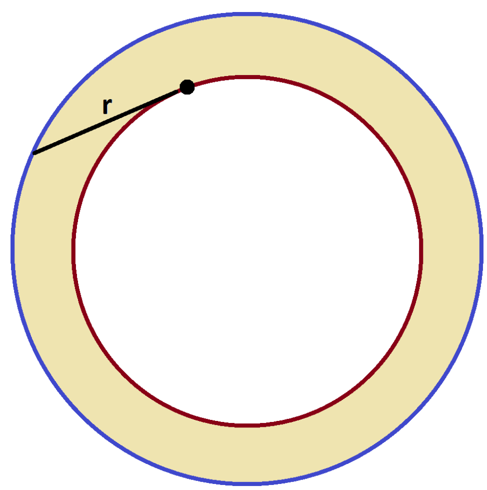
Question 2: Here's another picture of mathematically precise tracks. Here the cyclist rode in a loop.

In this picture, which is the back-wheel track? Which is the front-wheel track? Which way did the cyclist travel?



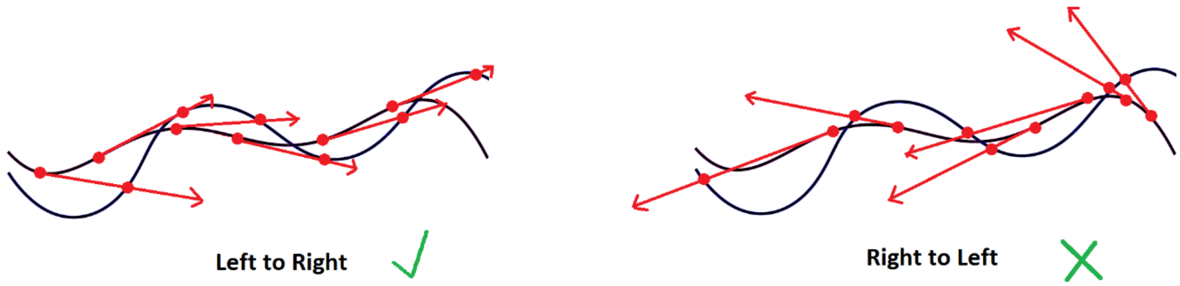
Question 3: Suppose a cyclist rides a perfect circle.

If you are familiar with basic circle theorems from geometry class and the Pythagorean theorem, you can prove that the area between the two circular tracks formed is sure to be πr^2 no matter the size of the circles. How?

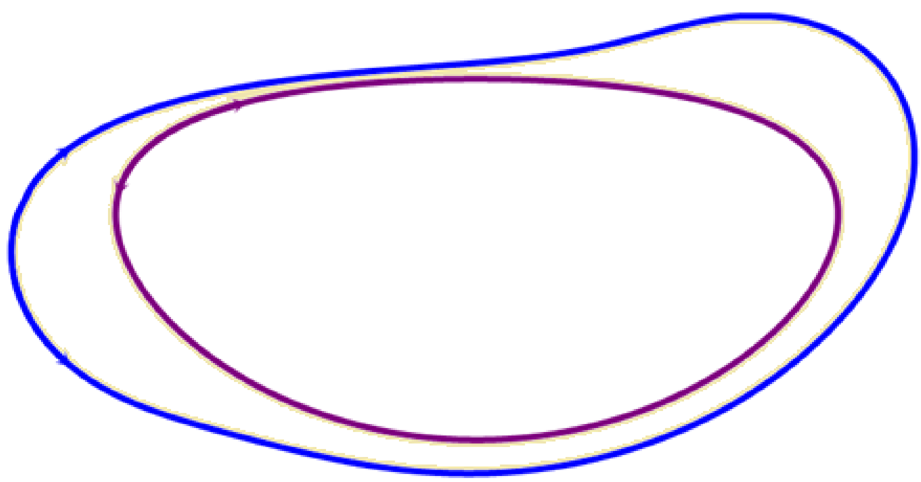


Solutions to Handout C

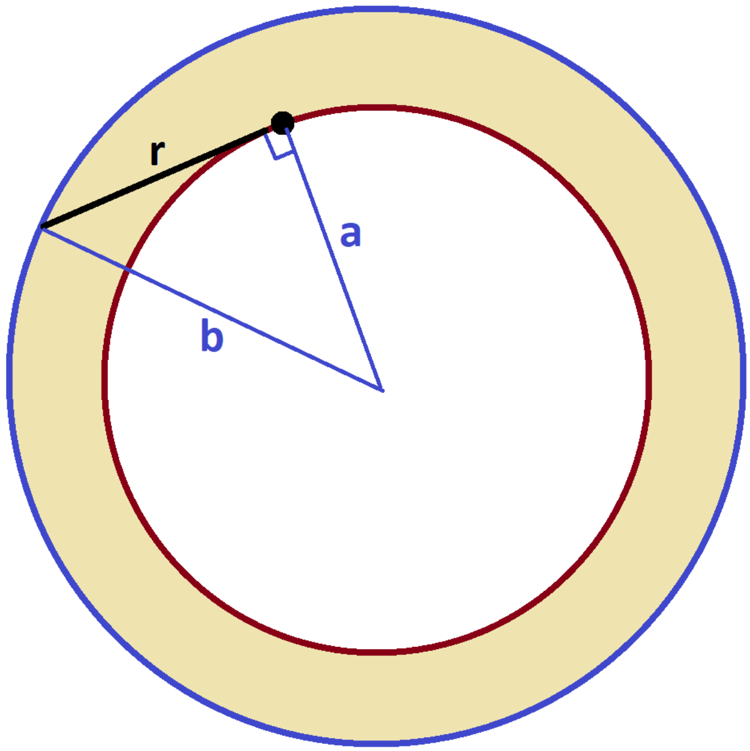
Question 1: The cyclist went left to right.



Question 2: The inner track (red) is the back wheel track. The outer (blue) track is the front wheel track. The cyclist went counter clockwise.



Question 3: Suppose the inner circle has radius a and the outer circle radius b. Draw the radii shown.



By the radius/tangent theorem in circle geometry, we have a right triangle and so $b^2 = a^2 + r^2$.

The area between the two circular tracks is thus

$$\pi b^2 - \pi a^2 = \pi r^2,$$

as claimed.