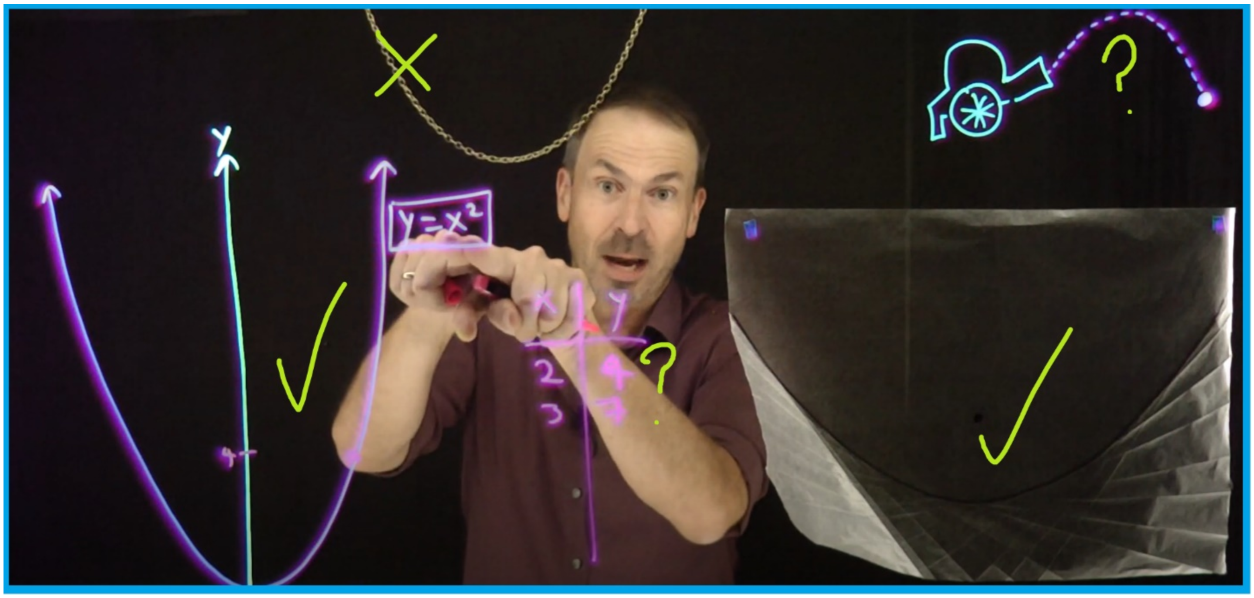
GLOBAL MATH PROJECT

The Power of Symmetry



QUADRATICS

ABOUT THE COURSE

This course is about the beauty and the thinking behind the quadratics: how to think about them and do them -- both their algebra and their graphing -- with natural ease and abundant joy. How? By being absolutely true to their story!

An exploration of quadratics is really an exploration of the power of symmetry, of seeing how this elegant idea so profoundly unites and simplifies. Symmetry is our friend! There really is no need to memorize anything once you realize this core feature of the topic.

This swift course will cover much of the traditional high-school algebra work on quadratics, but do so in a way that teaches how to see through mathematical clutter and to think like a mathematician.

THE LESSONS

Lessons are subject to the beautiful give-and-take of conversation and so the topics covered in any individual period described here may sway to some degree.

There are 65 practice problems for the course (too many!) and solutions to them appear in green as the second half of this document.

Oct 20, 11 am CDT:

Lesson 1: **Setting the Scene: What to do if you trust patterns**

Okay: What's the next number 2 4 6 8 __?
If you trust pattens you'd likely say 10.

Let's start off the course by playing the game of trusting patterns and seeing what mathematics we can develop. We'll also see how Galileo was hoping to use a pattern to verify a theory of gravitation.

Oct 22, 11 am CDT:

Lesson 2: **Still trusting patterns?**

Let's keep playing the game of trusting patterns. But some dubious examples might start creeping in.

REFERENCES FOR LESSONS 1 and 2:
If you want to look at some videos of me explaining what we covered in lessons 1 and 2 and/or read some material on it, have a look at my website here. This a part of another little course I put together, but this middle section stands alone and covers what we did together.

PRACTICE MATERIALS: Try **PRACTICE PROBLEMS 1 THROUGH 8.**
Pick and choose the questions that you think would be most helpful to try. Solutions are provided!

Oct 27 ,11 am CDT

Lesson 3: **The play of Area**

Sometimes schoolyards are in the shapes of rectangles and they are called quadrangles. Notice the prefix "quad."

While we're in the mood for play, let's start playing with quadrangles to see how our beliefs about area are intimately connected to arithmetic and algebra. (Have you ever wondered why negative times negative is positive?)

REFERENCES FOR LESSON 3:
The first bit of this video covers what we did in lesson 3. In fact, the text you see under this video matches the lesson too!

PRACTICE MATERIALS: Try **PRACTICE PROBLEMS 9 THROUGH 14.**

Oct 29, 11 am CDT

Lesson 4: **The name "quadratic" and the Quadrus Method**
Symmetry is our friend! Let's start playing with symmetrical quadrangles and finally start this course on quadratics! Let's solve quadratic equations. (Why not?)

Nov 3 11 am CDT

Lesson 5: **Mastering the Quadrus Method**
Let's become masters at solving every quadratic equation possible!
(Oh. If you've heard of something called the "quadratic formula" we can discuss that too if you want – but we don't actually need it.)

REFERENCES FOR LESSONS 4 and 5:
Check out the videos here and here and here.

To see written material on this content (maybe you prefer to read?), it's all in this document on pages 12 to 35. (That seems like a lot of pages! But there are many pictures and pictures take up a lot of space.)

PRACTICE MATERIALS: Try **PRACTICE PROBLEMS 15 THROUGH 28**. But that is too many questions! So just pick and choose one or two to do from each set of problems presented. Do enough until you feel you "get it."

Nov 5, 11 am CDT

Lesson 6: **What is graphing?**
People say that "math is a language." What does that actually mean?
Let's chat about that. And let's also figure out what it means to "graph an equation." (It's tied to our answer about language.)

REFERENCES FOR LESSON 6:
The video here covers the content of this lesson.

To see written material on this content (maybe you prefer to read?), it's all in this document on pages 42 to 48..

PRACTICE MATERIALS: Try **PRACTICE PROBLEMS 29, 30, 31.**

Nov 10, 11 am CST

Lesson 7: **Balancing a Quadratic on your head**
Let's start graphing quadratics. Can we get a graph to balance on my head?

Nov 12, 11 am CST

Lesson 8: **A factor of steepness**
Ooh! Our equations have been "too nice." Let's start graphing more complication quadratic equations now.

REFERENCES FOR LESSONS 7 and 8:
The videos here and here cover these two lessons.
See pages 49-60 of this document .

PRACTICE MATERIALS: Try **PRACTICE PROBLEMS 32 THROUGH 46**. Again, too may. So please just pick and choose.

Nov 17: 11 am CST

Lesson 9: **The previous too lessons were too hard. Ignore them!**

We've lost sight of symmetry. We need to bring back symmetry thinking!

*Doing so sets us up for a ridiculously easy way to graph *all* quadratic equations just by using common sense.*

REFERENCES FOR LESSON 9:
Look at the videos here, here, here, and here! (That's a lot of videos!)
See pages 61-73 of this document .

PRACTICE MATERIALS: Try **PRACTICE PROBLEMS 47-65**. As before, pick and choose!

Nov 19: 11 am CST

Lesson 10: **How to spell your name in math**

Let's wrap things up. We've basically mastered it all now. So let's just have some fun writing absurd equations that spell our names!

REFERENCES FOR LESSON 10:

Just have at the website https://globalmathproject.org/personal-polynomial/ and have fun!

There are videos there to watch if you want.

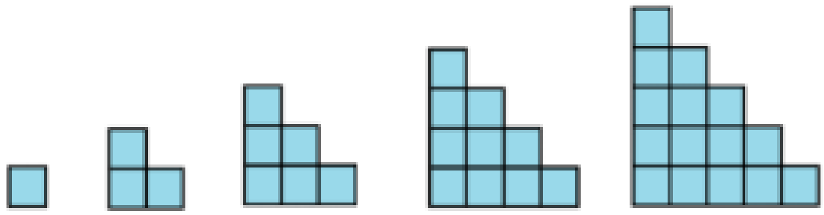
You can also read pages 81-88 of this document if you want and try the practice problems there.

PRACTICE PROBLEMS

Practice 1: Make an intelligent guess as to the next number in the following sequence.

2 3 6 11 18 27 38 __

Practice 2: Consider the following sequence of diagrams each made of squares 1 unit wide.



If the implied geometric pattern of these first five figures continues …

a) What would be the perimeter of the tenth figure?
b) What would be the area of the tenth figure?

Practice 3: a) Show that for the following sequence it seems that the third differences are constant. Make a prediction for the next number in the sequence.

0 2 20 72 176 350 612 …

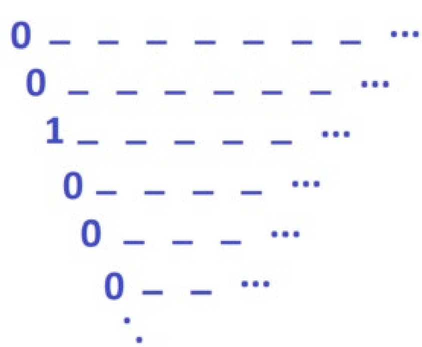
b) Use differences to make an intelligent guess as to the next number in this sequence.

-1 4 7 8 7 4 -1 __

c) How many difference rows must one complete in the sequence below (the powers of two) to see a row of constant differences?

1 2 4 8 16 32 64 128 …

Practice 4: What sequence has **0 0 1 0 0 0 …** as its leading diagonal?



Practice 5: a) Use difference methods to find a formula for the sequence of numbers

2, 2, 4, 8, 14, 22, 32,…

(Just so you have it, the answer is $n^2 - 3n + 4$. Can you see how to get this answer by looking at the leading diagonal?)

b) Use difference methods to show that **0, 2, 10, 30, 68, 130, 222, …** follows the formula $n^3 - 3n^2 + 4n - 1$.

Practice 6: Find formulas for as many of these sequences as you feel like doing.

5, 8, 11, 14, 17, 20, 23, …

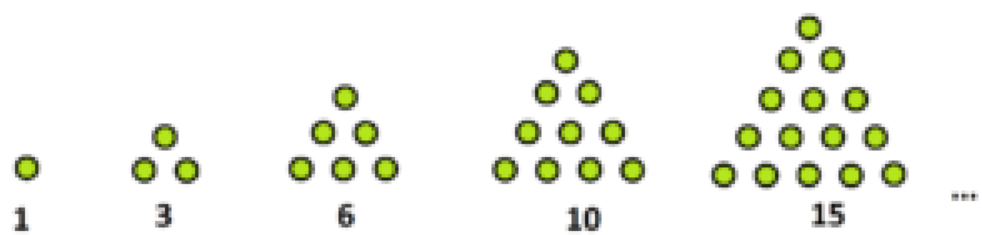
3, 3, 3, 3, 3, 3, 3, 3, 3, …

1, 3, 15, 43, 93, 171, 283, …

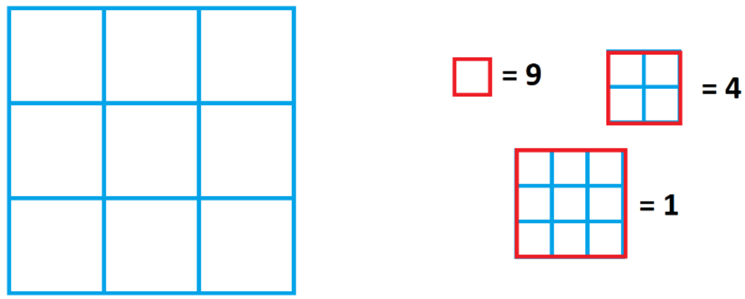
1, 0, 1, 10, 33, 76, 145, 246, 385, …

3, 3, 7, 21, 51, 103, 183, 297, ….

Practice 7: Find a formula for the n th triangular number: 1, 3, 6, 10, 15, 21, 28, 36, ….
(Don't be afraid of fractions!)



Optional Practice 8: Let $S(n)$ be the total number of squares, of any size, one can find in an $n \times n$ grid of squares. For example, $S(3) = 14$ because in a three-by-three grid of squares one can find nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of $9 + 4 + 1 = 14$ squares in the grid.



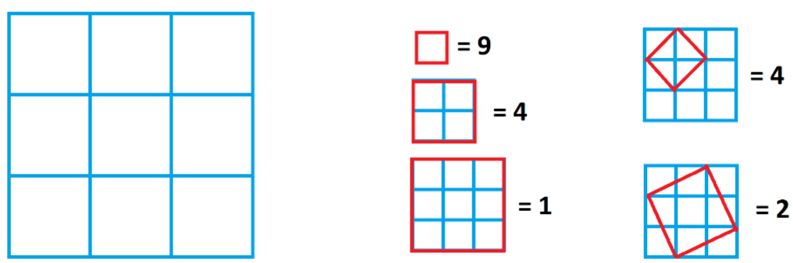
a) Find $S(1)$, $S(2)$, $S(4)$, and $S(5)$.

b) What do difference methods suggest is the general formula for $S(n)$?

c) What is the value of $1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2$?

OPTIONAL:

d) Care to count both tilted and non-tilted squares? For example, there are a total of 20 tilted and non-tilted squares to be drawn on a three-by-three grid.



Practice 9: Use the area model to compute 3721×223.

(Into how many pieces might you divide your rectangle?)

Practice 10: Use the traditional long multiplication algorithm to compute 845×387. And use it again to compute 387×845, but as you do so this second time ask yourself: Is it obvious the algorithm will give the same final answer?

Practice 11: Use the area model to compute $4\frac{1}{3} \times 10\frac{2}{5}$.

Practice 12: Compute 16×15 four different ways to conclude that $(-4) \times (-5)$ is positive 20.

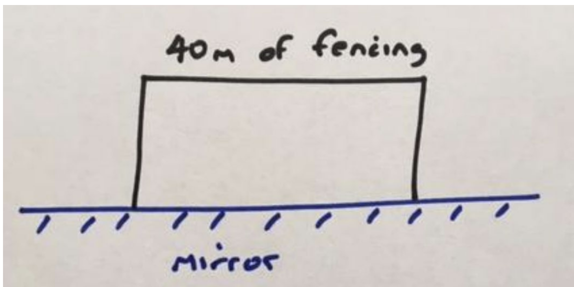
Practice 13:

a) Use the area model to compute $(2x^2 + x + 1)(3x + 2)$.

b) Use your answer to quickly see the value of 211×32.

c) Put $x = -10$ into your answer from a). What multiplication problem is this the answer to?

Practice 14: (CHALLENGE) A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. But she has huge mirrored wall in her field and wants to use the mirror as one side of her rectangular pen.



What should the dimensions of the rectangle be in order to obtain a pen of maximal area?

Practice 15: Solve

a) $2x^2 = 50$
b) $y^2 + 5 = 14$
c) $100x^2 = 1$
d) $4p^2 = 0.25$
e) $a^2 + 7 = 7$
f) $x^2 = 20$
g) $x^2 = -20$

Practice 16: Solve

a) $(4x-6)^2 = -7$
b) $(4x-6)^2 = 0$
c) $(4x-6)^2 = 4$
d) $(4x-6)^2 = 5$

Practice 17: Solve

a) $(y+1)^2 - 2 = 23$
b) $4(p-2)^2 - 16 = 0$
c) $9 + \left(34x - 77\frac{1}{2}\right)^2 = 0$
d) $\left(x - \sqrt{2}\right)^2 = 5$

Practice 18: Solve

a) $p^2 - 6p + 9 = 9$
b) $x^2 - 4x + 4 = 1$
c) $x^2 - 20x + 100 = 7$
d) $r^2 - 16r + 64 = -2$
e) $x^2 + 2\sqrt{5}x + 5 = 36$
f) $x^2 - 2\sqrt{2}x + 2 = 19$

Practice 19: Solve for x giving your answer in terms of A and B.

$$x^2 + 2Ax + A^2 = B^2$$

Practice 20: Solve

a) $f^2 + 8f + 15 = 80$
b) $w^2 + 90 = 22w - 31$
c) $x^2 - 6x = 3$.

Practice 21: Solve as many of these as you feel like doing.

a) $w^2 - 5w + 6 = 2$
b) $x^2 + 9x + 1 = 11$
c) $p^2 + p + 1 = 0.75$
d) $x^2 = 10 - 3x$
e) $x^2 - x - 1 = 2\frac{3}{4}$
f) $x^2 + 3 = 9$

Practice 22: Solve as many of these you feel like doing.

a) $2x^2 = 9$
b) $4 - 3x^2 = 2 - x$
c) $\alpha^2 - \alpha + 1 = \frac{7}{4}$
d) $3x^2 + 3x + 1 = 19$
e) $-3x^2 + 3x + 1 = 19$
f) $10k^2 = 1 + 10k$

Practice 23: Consider $4x^2 + 6x + 3 = 1$. Does it look like this quadratic equation will have problems when solving it? Does it have problems as you try to solve it? What can you do to obviate the difficulties you encounter?

Practice 24:

a) Design a quadratic equation that has two negative solutions.

b) Design a quadratic equation with just one solution, namely, $x = 4$.

c) Design a quadratic equation with $x = 2$ and $x = 10$ as solutions.

Practice 25:

a) A rectangle is twice as long as it is wide. Its area is 30 square meters. What are the dimensions of the rectangle?

b) A rectangle has one side 4 meters longer than the other. Its area is 30 square meters. What are the dimensions of the rectangle?

Practice 26: Consider $y = 2(x-4)^2 + 6$. What value for x produces the smallest possible value for y? Why?

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Practice 27: Find one solution to $(x+1)^3 = 27$.

Practice 28 (OPTIONAL): This problem will require you to multiplying through by 4 many times!

a) Solve $x^2 + x = 2$.
b) Solve $2x^2 + x = 3$.
c) Solve $4x^2 + x = 5$.
d) Solve $8x^2 + x = 9$.
e) Solve $16x^2 + x = 17$.

If you are game …

f) Find the solutions to $2^N x^2 + x = 2^N + 1$.

Practice 29:

a) Sketch a graph of the one-variable equation $x^2 = 4$. (So it's graph will require only one number line, one for x values.)

b) Sketch a graph of the one-variable inequality $x^2 \geq 4$.

Practice 30: Sketch a graph of the two-variable equation $x^2 = y^2$.

Practice 31:

a) Sketch a graph of the one-variable equation $x = 3$.

b) Sketch a graph of $x = 3$ thinking of it as a two-variable equation. (Imagine it as $x + 0 \cdot y = 3$ if you like.)

c) Sketch a graph of $x = 3$ thinking of it as a three-variable equation. (Imagine it as $x + 0 \cdot y + 0 \cdot z = 3$ if you like.) How will you draw your three number lines?

Practice 32:

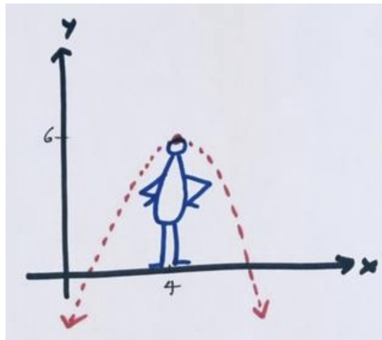
a) Sketch the graph of $y = (x-10)^2$.
b) Sketch the graph of $y = (x+5)^2$.

Practice 33:

a) Sketch the graph of $y = (x-5)^2 + 2$.
b) Sketch the graph of $y = (x+5)^2 - 2$.

Practice 34: When we say that the graph of $y = x^2$ is a U-shaped graph, is that a correct analogy? The two sides of the letter "U" are vertical. Does the graph of $y = x^2$ possess vertical lines?

Practice 35: Find three different equations that give U-shaped graphs that balance on my head this way.



Practice 36: Draw, on the same sets of axes, rough sketches of each the following equations.

$y = x^2$	$y = 1.1x^2$	$y = 0.9x^2$
$y = -x^2$	$y = -1.1x^2$	$y = -0.9x^2$

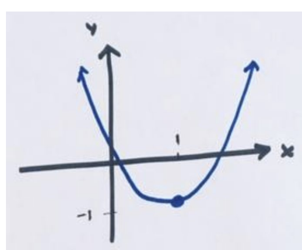
Practice 37: Sketch graphs of

a) $y = 3(x-5)^2$

b) $y = 3(x-5)^2 + 4$

c) $y = -2(x+4)^2 + 40$.

Practice 38: Which of the following equations could have the graph shown?



a) $y = \frac{4}{3}(x+1)^2 + 1$

b) $y = -\frac{4}{3}(x+1)^2 + 1$

c) $y = \frac{4}{3}(x-1)^2 + 1$

d) $y = -\frac{4}{3}(x-1)^2 + 1$

e) $y=\frac{4}{3}(x+1)^2-1$

f) $y=-\frac{4}{3}(x+1)^2-1$

g) $y=\frac{4}{3}(x-1)^2-1$

h) $y=-\frac{4}{3}(x-1)^2-1$

Practice 39: Sketch the graph of $y=-2(x+10)^2-7$.

Practice 40: The graph of a quadratic equation has a vertical line of symmetry at $x=3$, and has highest value $y=17$. Which of the following could be an equation for that quadratic?

a) $y=200(x-3)^2+17$

b) $y=-200(x-3)^2+17$

c) $y=200(x-3)^2-17$

d) $y=-200(x-3)^2-17$

Practice 41: Sketch a graph for each of the following equations.

a) $y=2-x^2$

b) $y=\frac{1}{3}\left(x-\frac{1}{2}\right)^2-4$

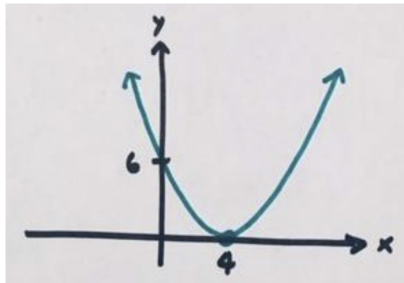
c) $y=0.0034(x+0.276)^2+0.778$

d) $y=200000(x-200000)^2-200000$

Practice 42: If $y=a(x+b)^2+c$ has a graph passing through the origin and with $(2,3)$ as the vertex, then what is the value of $a+b+c$?

a) $\frac{1}{4}$ b) $1\frac{3}{4}$ c) $4\frac{1}{4}$ d) $5\frac{1}{4}$

Practice 43: Write a quadratic equation that fits this graph.



Practice 44: Write down a quadratic equation whose graph passes through the points $(3,18)$ and $(17,18)$ and has lowest value 5.

Practice 45: Write down a quadratic equation whose graph passes through the x axis at $x=-2$ and at $x=10$ and passes through the y axis at $y=-6$.

Practice 46: Write down quadratic equations with symmetrical U-shaped graphs possessing the following properties:

a) Crosses the x-axis at 3 and 5 and the y-axis at 1000.

b) Passes through $(4,10)$, $(6,10)$ and $(8,13)$.

c) Has vertex $(5,5)$ and passes through $(4,4)$.

d) Has vertex the origin and passes through the point $\left(\sqrt{2},\pi\right)$.

Practice 47: The graph of a quadratic equation passes through the points $(3,81)$, $(4,9)$, and $(-10,9)$. What is the x-coordinate of its vertex?

Practice 48: Sketch a graph of $y=2(x-3)(x-23)+200$.

Practice 49: Sketch a graph of $y=-(x-3)(x+5)+6$.

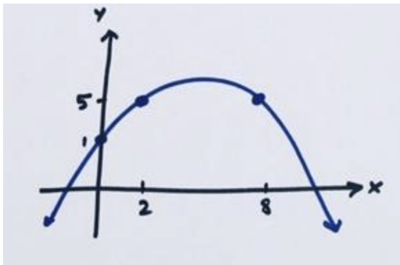
Practice 50: Sketch a graph of $y=-2x(x-80)+3$.

Practice 51: Which value of r forces the graph of $y=3x^2+6x+r$ have 5 as the smallest possible y value?

Practice 52: Find a negative value for a so that $y=x^2+ax+a$ has smallest possible value -3.

Practice 53: Find a formula for the location of the line of symmetry of a general quadratic equation $y=ax^2+bx+c$.

Practice 54: Find the quadratic equation whose graph appears as shown.



Practice 55: Write down a quadratic equation whose graph has x intercepts $x=-3$ and $x=11$ and y intercept 10.

Practice 56: Sketch the graph of $y=-3x^2-18x+5$ and then use the graph to rewrite the equation in "vertex form."

Practice 57: Consider the equation $y=3x^2-6x+20$.

a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x intercepts of the graph.
f) Find the y intercept of the graph.

Practice 58: Consider the equation $y=5x^2-10x$.

a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x intercepts of the graph.
f) Find the y intercept of the graph.

Practice 59: Solve the following quadratic equations.

a) $(x-3)(x+5)=1$

b) $x^2=(2x-1)(2x+1)-5$

c) $(x-10)(x+1)+5=12$

Practice 60: Find, in terms of c, the value k so that $y=(x+c)(x-c)+k$ gives -2 as the smallest possible y value.

Practice 61:

a) Solve $x^2+10x+30=0$ and see what happens when you try.

b) Sketch the graph of $y=x^2+10x+30$.

c) Use the graph to explain what happened in part a)

d) Will $x^2+10x+30=11$ have a solution? If so, how many solutions?

e) For which value(s) k does $x^2+10x+30=k$ have only one solution?

Practice 62: How many solutions does $-x^2+4x-5=0$ have? Answer this question not by algebra, but by graphing.

Practice 63: a) Find a value k so that the graph of

$$y=5x^2-10x+k$$

just touches the x axis.

b) Find a value m so that $y=-2x^2-18x+m$ gives the highest output value of 100.

c) Find a value p so that $y=(x-p)(x-3p)$ has smallest output value of -10.

Practice 64: A rectangle has side lengths $7-r$ and $3+r$ for some value r. What value for r gives a rectangle of maximal area?

Practice 65: Here are three quadratic equations:

(A) $y=3(x-3)(x+5)$

(B) $y=2x^2+6x+8$

(C) $y=2(x-4)^2+7$

i) For which of these three expressions is it very easy to answer the question: "What is the smallest y-value the expression can produce?"

ii) For which of these three expressions is it very easy to answer the question: "Where does the graph of the equation cross the x-axis?"

iii) For which of these three expressions is it very easy to answer the question: "Where does the graph of the quadratic cross the y axis?"

iv) For which of these three expressions is it very easy to answer the question: "What are the coordinates of the vertex in this equation's graph?"

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SOLUTIONS

Practice 1: 51

Practice 2: a) 24 b) 21

Practice 3: a) The third differences seem to be the constant value 18. If so, the next number in the sequence would be 890.
b) The second differences seem to be the constant value -2. If so, the next number in the sequence would be -8.
c) You never will. The powers of two, as differences, keep giving the powers of two!

Practice 4: 0, 0, 1, 3, 6, 10, 15, 21, 28, …

Practice 5: Did you see the answers given?

Practice 6: a) $3n+2$.
b) 3.
c) n^3-n^2-2n+3. (Third differences are constant value 6.)
d) n^3-5n^2+7n-2. (Third differences are constant value 6.)
e) n^3-4n^2+5n+1. (Third differences are constant value 6.)

Practice 7: $\frac{1}{2}n^2+\frac{1}{2}n$.

Practice 8:
$S(1)=1, S(2)=5, S(3)=14, S(4)=30, S(5)=55$
Difference methods suggest

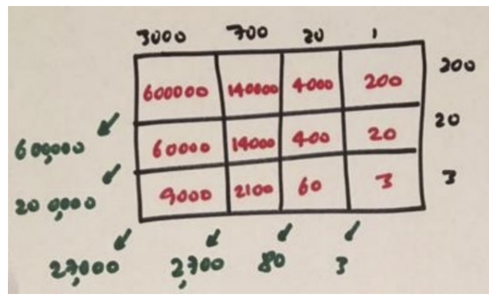
$$S(n)=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n.$$

Did you notice in counting that
$S(3)=1+4+9$ and $S(4)=1+4+9+16$,
and so on? In which case,

$$\begin{aligned}1^2+2^2+\cdots+100^2 &= S(100)\\ &=\frac{1}{3}100^3+\frac{1}{2}100^2+\frac{1}{6}100\\ &=338350\end{aligned}$$

OPTIONAL: You get that the fourth differences are constant. Can you write done the fourth degree polynomial that results?

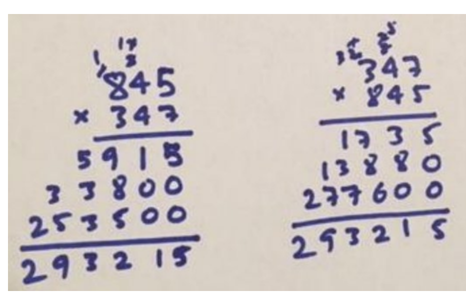
Practice 9:
One might divide a rectangle into twelve pieces by thinking of 3721 as 3000 + 700 + 20 + 1 and 223 as 200 + 20 + 3. The areas of the individual pieces then add to 829783.



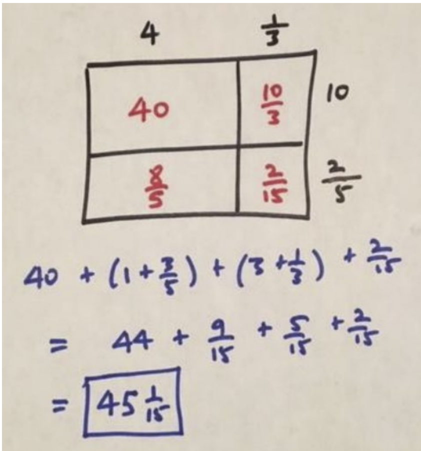
Of course, one need not divide the rectangle this way. One could think of 3721 as 2000 + 1500 + 10 + 10 + 1 and 223 as 100+100+10+10+1+1+1, for instance.

All ways of subdividing the rectangle should, of course, yield the same final answer of 829783.

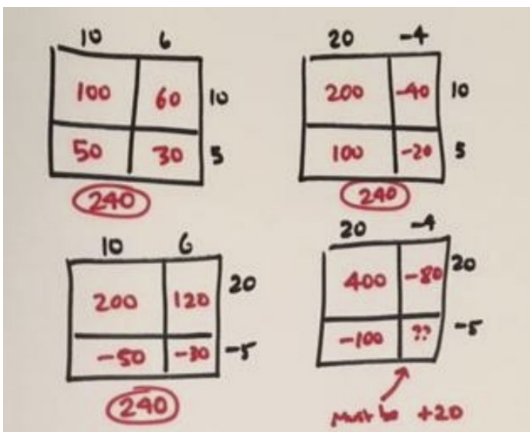
Practice 10: It is not at all obvious to me why at face value the traditional long multiplication algorithm naturally gives the same answers whether you compute $a\times b$ or $b\times a$. What you write on the page looks different. It is actually somewhat surprising that the final lines after each summation are the same.



Practice 11:

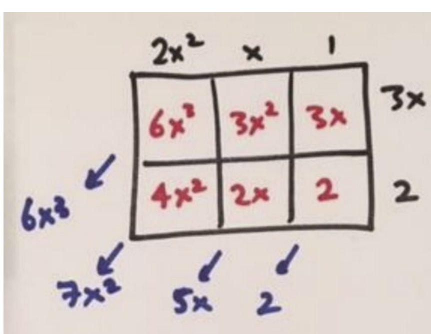


Practice 12:



Practice 13:

a) $(2x^2 + x + 1)(3x + 2) = 6x^3 + 7x^2 + 5x + 2$



b) Put $x = 10$ to see that this reads:

$(200 + 10 + 1)(30 + 2)$ equals

$6000 + 700 + 50 + 2 = 6752$.

c) We have $(200 - 10 + 1)(-30 + 2)$, that is, $191 \times (-28)$, equals $-6000 + 700 - 50 + 2$, which is -5348.

Practice 14:

This problem is not-symmetrical as the pen has two full vertical sides and only one horizontal side. The problem we discussed in the essay just before this was symmetrical with two sides of each type.

But if one looks into the mirrored wall, it will look as though we have a rectangular pen, double the area and double the perimeter (80 meters), with FOUR sides of fencing.

We know the answer to the symmetrical problem is a symmetrical square, so the mirrored reflection must be a 20 meter-by-20 meter square pen to get maximal area. But the actual pen without the reflection is half of this, a 10 meter-by-20 meter pen!

She should build a 10-by-20 pen.

SNEAKY!

Practice 15:

a) $x^2 = 25$ so $x = 5$ or -5.

b) $y^2 = 9$, so $y = 3$ or -3.

c) $x^2 = \frac{1}{100}$, so $x = \frac{1}{10}$ or $-\frac{1}{10}$.

d) $p^2 = \frac{1}{16}$, so $p = \frac{1}{4}$ or $-\frac{1}{4}$.

e) $a^2 = 0$, so $a = 0$.

f) $x = \sqrt{20}$ or $-\sqrt{20}$.

g) Has no solutions.

Practice 16:

a) No solutions. (No quantity squared can be negative.)

b) $4x-6=0$, so $x=\frac{3}{2}$.

c) $4x-6=2$ or -2, so $x=2$ or 1.

d) $4x-6=\sqrt{5}$ or $-\sqrt{5}$,

so $x=\frac{6+\sqrt{5}}{4}$ or $\frac{6-\sqrt{5}}{4}$.

Practice 17:

a) $(y+1)^2=25$

$y+1=5$ or -5

$y=4$ or -6

b) $(p-2)^2=4$

$p-2=2$ or -2

$p=4$ or 0

c) $\left(34x-77\frac{1}{2}\right)^2=-9$ has no solutions.

d) $x-\sqrt{2}=\sqrt{5}$ or $-\sqrt{5}$

$x=\sqrt{2}+\sqrt{5}$ or $\sqrt{2}-\sqrt{5}$.

Practice 18:

a) $(p-3)^2=9$

$p-3=3$ or -3

$p=6$ or 0

b) $(x-2)^2=1$

$x-2=1$ or -1

$x=3$ or 1

c) $(x-10)^2=7$

$x=10+\sqrt{7}$ or $10-\sqrt{7}$

d) $(r-8)^2=-2$ has no solutions.

e) $\left(x+\sqrt{5}\right)^2=36$

$x=\sqrt{5}+6$ or $\sqrt{5}-6$

f) $\left(x-\sqrt{2}\right)^2=19$

$x=\sqrt{2}+\sqrt{19}$ or $\sqrt{2}-\sqrt{19}$.

Practice 19:

$(x+A)^2=B^2$

$x+A=B$ or $-B$

$x=B-A$ or $-(A+B)$

Practice 20:

a) $f^2+8f+16=81$

$(f+4)^2=81$

$f=5$ or -13

b)

$w^2 - 22w + 90 = -31$

$w^2 - 22w + 121 = 0$

$(w-11)^2 = 0$

$w = 11$

c) $x^2 - 6x + 9 = 12$

c)

$(x-3)^2 = 12$

$x = 3 + \sqrt{12}$ or $3 - \sqrt{12}$

Practice 21:

a) $w = 1$ or 4

b) $x = 1$ or -10

c) $p = -0.5$

d) $x = 2$ or -5

e) $x = \frac{3}{2}$ or $-\frac{5}{2}$

f) $x = \sqrt{6}$ or $-\sqrt{6}$

Practice 22:

a) $x = \frac{3}{\sqrt{2}}$ or $-\frac{3}{\sqrt{2}}$.

(Some curriculum prefer to write this as $\frac{3\sqrt{2}}{2}$ and $-\frac{3\sqrt{2}}{2}$?)

b) $x = 1$ or $-\frac{2}{3}$.

c) $\alpha = \frac{3}{2}$ or $-\frac{1}{2}$.

d) $x = 2$ or -3.

e) $x = \frac{3+\sqrt{249}}{6}$ or $\frac{3-\sqrt{249}}{6}$.

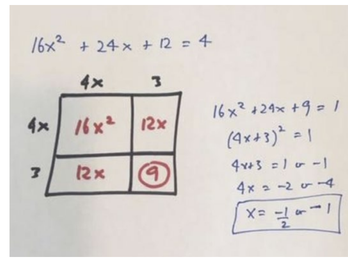
f) $k = \frac{5+\sqrt{35}}{10}$ or $\frac{5-\sqrt{35}}{10}$

Practice 23: We have a perfect square up front and an even middle term. So it looks good!

But if you try to solve it with the square method you find yourself dealing with fractions!

To avoid them, let's try multiplying through by 4. Then all is good!

We get $x = -\frac{1}{2}$ or -1.



Practice 24:

a) Lots of different answer are possible for a). One strategy is to work with $(x+500)^2 = 1$, for instance. This is the quadratic equation

$$x^2 + 1000x + 250000 = 1.$$

b) Work with $(x-4)^2 = 0$. This is the quadratic equation

$$x^2 - 8x + 16 = 0.$$

c) This one is harder. Maybe think

$$x = 6 - 4 \text{ or } 6 + 4$$

to then think

$$(x-6)^2 = 16$$

which is the quadratic equations

$$x^2 - 12x + 36 = 16.$$

Practice 25:

a) If we have a $2x$ by x rectangle, then we need $2x^2 = 30$. This means x must be $\sqrt{15}$. It is a $2\sqrt{15}$ by $\sqrt{15}$ rectangle.

b) If we have an x by $x+4$ rectangle, then we need $x(x+4) = 30$, that is, we need

$$x^2 + 4x = 30.$$

This is $(x+2)^2 = 34$ and so $x = \sqrt{34} - 2$. (We must choose the positive length.) We thus have a $\sqrt{34} - 2$ by $\sqrt{34} + 2$ rectangle.

Practice 26: The quantity $(x-4)^2$ is always postive, or zero if $x = 4$. Thus y has smallest possible value $2 \times 0 + 6 = 6$, occuring when $x = 4$.

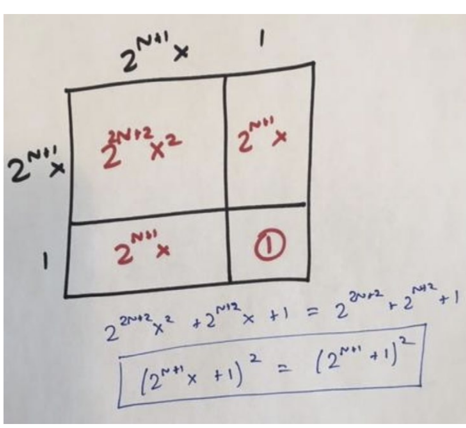
Practice 27: Something cubed is 27. That something could be 3.

So $x + 1 = 3$, that is, $x = 2$, is one solution.

Practice 28: We'll answer f).

Multiply through by 2^{N+2} to examine

$$2^{2N+2}x^2 + 2^{N+2}x = 2^{2N+2} + 2^{N+2}.$$

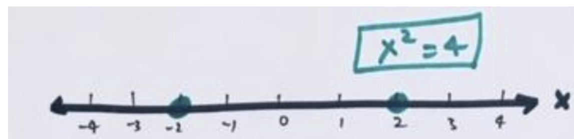


So

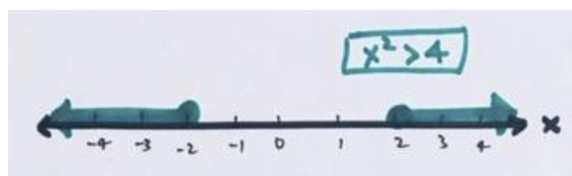
$$2^{N+1}x + 1 = 2^{N+1} + 1 \text{ or } -2^{N+1} - 1$$
$$2^{N+1}x = 2^{N+1} \text{ or } -2^{N+1} - 2$$
$$x = 1 \text{ or } -1 - \frac{1}{2^N} = -\frac{2^N + 1}{2^N}$$

Practice 29:

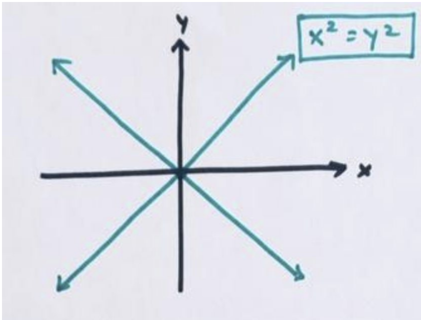
a) Only $x = 2$ and $x = -2$ make $x^2 = 4$ a true number sentence. A natural way to represent this visually might be as follows.



b) Any value greater than or equal to 2, or less than or equal to -2, makes $x^2 \geq 4$ a true number sentence. One might represent this visually as shown.

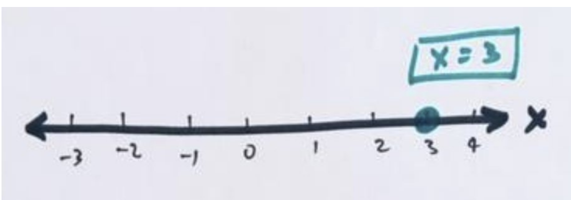


Practice 30: One gets and X-shaped graph.

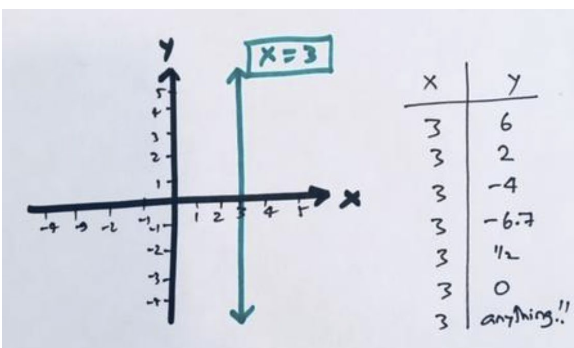


Practice 31:

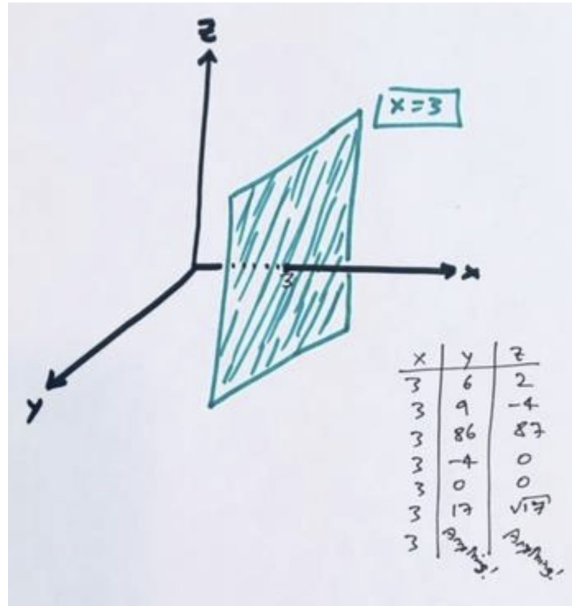
a) There is only one value for x that makes "$x = 3$" a true sentence, namely, x being 3.



b)

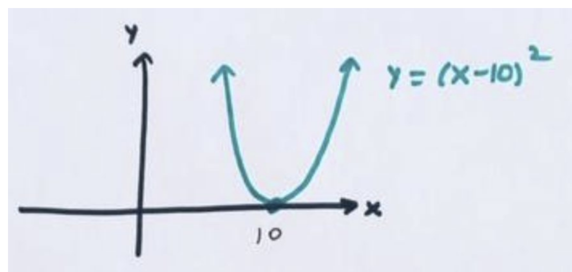


c) Draw three mutually perpendicular number lines. The graph is an entire plane of points.

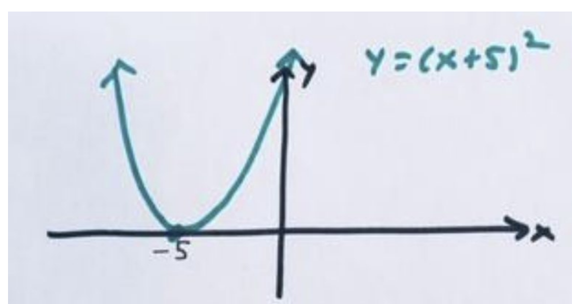


Practice 32:

a) We see $x = 10$ is behaving like zero.

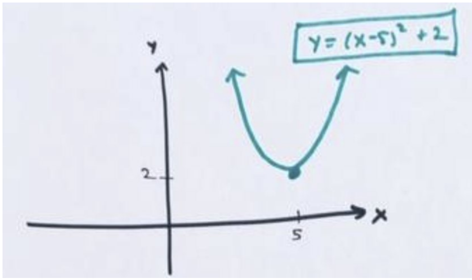


b) We see $x = -5$ is behaving like zero.

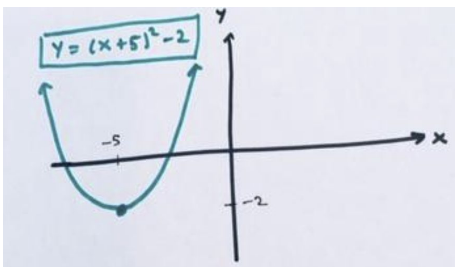


Practice 33:

a) We see $x = 5$ is behaving like zero and everything is shifted upwards 2 units.



b) We see $x = -5$ is behaving like zero and everything is shifted upwards -2 units, that is, down two units.



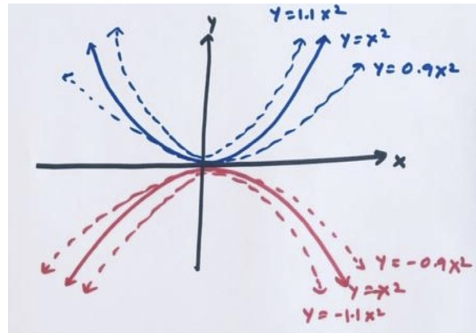
Practice 34: The graph does not possess vertical lines. For instance, if the graph became vertical at say $x = 100$, then the graph never extends to x-values beyond 100, which is absurd, because we know for example that $x = 101$, $y = 10201$ is a data point to be plotted.

Practice 35: We need to take the equation $y = -x^2$ and adjust it so that $x = 4$ behaves like zero and all data points are shifted 6 units higher. $y = -(x-4)^2 + 6$ works.

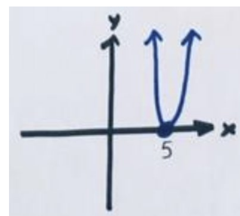
Actually, $y = -2(x-4)^2 + 6$,

$y = -\frac{1}{3}(x-4)^2 + 6$, and $y = -7(x-4)^2 + 6$ work too.

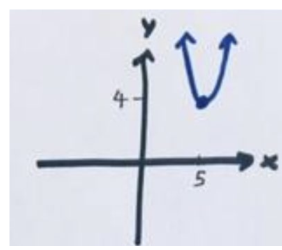
Practice 36: Roughly, we get:



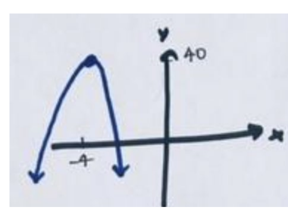
Practice 37: a) A steep U-shaped graph shifted so that $x = 5$ behaves like zero.



b) The same as the previous graph, except all data points are 4 units higher.

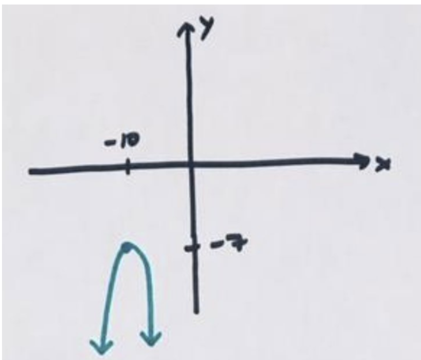


c) This is a steep upside-down U graph, with $x = -4$ behaving like zero, and all data points 40 units up.

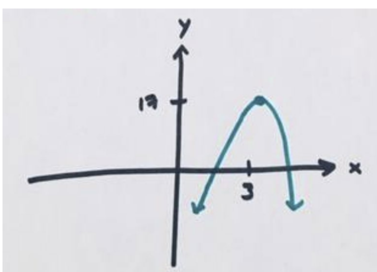


Practice 38: This is an upward-facing U-shaped graph with $x = 1$ behaving like zero and all data points shifted down 1. Only option g) can work.

Practice 39: We see $x = -10$ is behaving like zero, with a steepness factor of -2 at play (so the graph will be a steep downward-facing U-shape), with all data values shifted down 7 units.



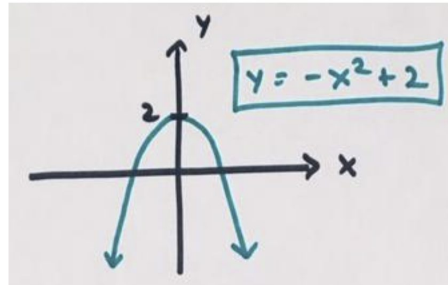
Practice 40: We must have a graph like this:



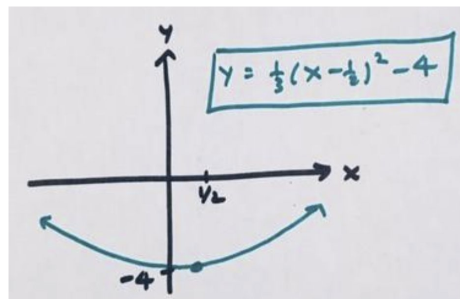
Only option b) can work.

Practice 41:

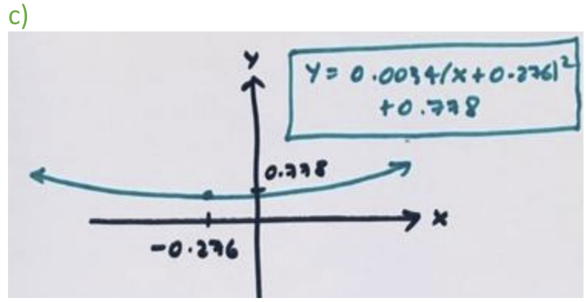
a)



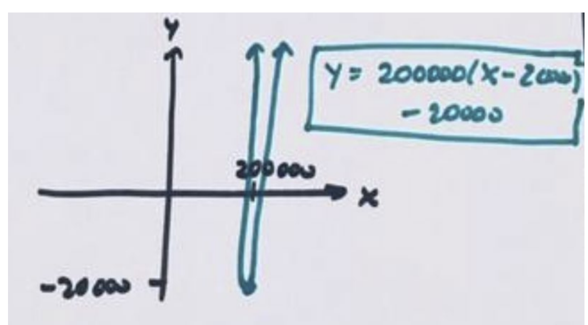
b)



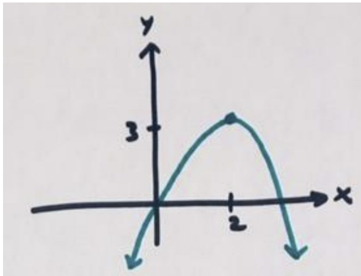
c)



d)



Practice 42: The graph must look like this:



So we see the equation must be of the form $y = a(x-2)^2 + 3$, and so $b = -2$ and $c = 3$.

The graph passes through $x = 0$, $y = 0$ and so

$$0 = a(-2)^2 + 3$$

must be a true sentence about numbers. This forces $a = -\frac{3}{4}$ and so $a + b + c = \frac{1}{4}$, option a).

Practice 43: We see that it is a graph basically coming from $y = x^2$ but with $x = 4$ behaving like zero. So we can try

$$y = (x-4)^2.$$

But we see this is not right: when $x = 0$ we get $y = 16$, not 6. We are missing a steepness factor!

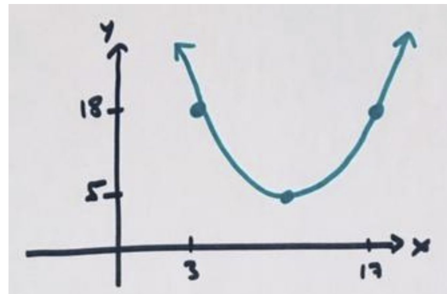
Try $y = a(x-4)^2$.

Now $x = 0$, $y = 6$ should yield a true number sentence, so $6 = 16a$ forcing $a = \frac{3}{8}$.

We have

$$y = \frac{3}{8}(x-4)^2.$$

Practice 44 We must have a graph that looks like this:



So let's just use common sense to figure things out.

We want a symmetrical graph and so the line of symmetry must be at $x = 10$, midway between 3 and 17. So the quadratic equation producing this graph must have the form

$$y = a(x-10)^2 + 5$$

for some steepness a.

When $x = 3$ we should have $y = 18$, showing that

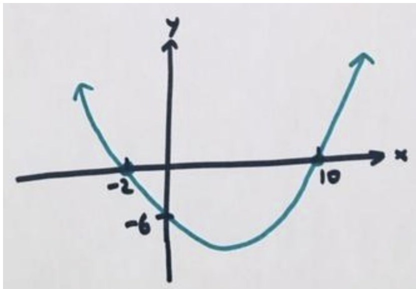
$$18 = a(-7)^2 + 5$$

should be a true number sentence. This forces

$$a = \frac{13}{49}.$$

So $y = \frac{13}{49}(x-10)^2 + 5$ works.

Practice 45: We have this picture.



The equation must be of the form

$$y = a(x-4)^2 + b$$

When $x = 10$, $y = 0$ and so

$$0 = 36a + b$$

must be a true number sentence. We have $b = -36a$.

When $x = 0$, $y = -6$ and so

$$-6 = 16a + b$$

must also be a true number sentence. We see that

$$-6 = 16a - 36a$$

showing that $a = \frac{3}{10}$. Consequently $b = -\frac{54}{5}$.

The equation we need is

$$y = \frac{3}{10}(x-4)^2 - \frac{54}{5}.$$

Practice 46: Sketch a picture of each scenario and use logic to determine which x-value is behaving like zero in each case. Then go from there!

a) $y = \frac{200}{3}(x-4)^2 - \frac{200}{3}$

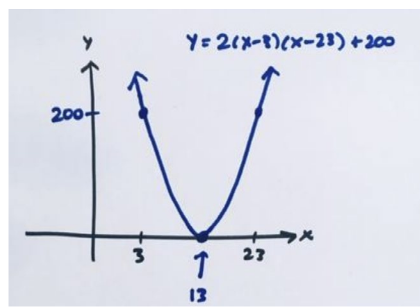
b) $y = \frac{3}{8}(x-5)^2 + \frac{77}{8}$

c) $y = -(x-5)^2 + 5$

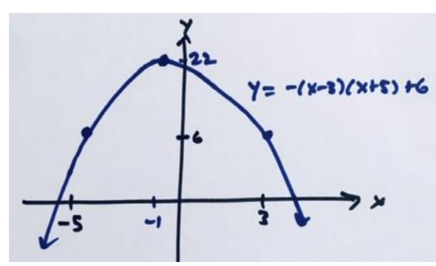
d) $y = \frac{\pi}{2}x^2$

Practice 47: We see that $(4,9)$ and $(-10,9)$ are two symmetrical points on a symmetrical graph, and so its line of symmetry must be halfway between $x = 4$ and $x = -10$, namely, at $x = -3$. And $x = -3$ must be the x-coordinate of the vertex.

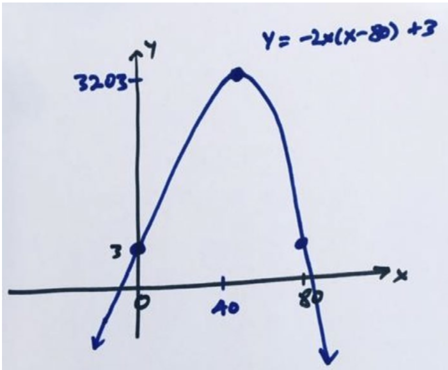
Practice 48:



Practice 49:

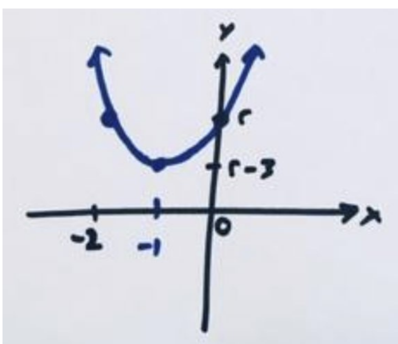


Practice 50:



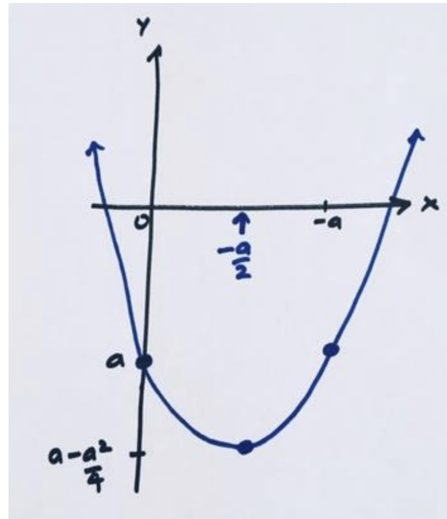
Practice 51: We have $y = 3x(x+2) + r$ showing that the line of symmetry is halfway between $x = 0$ and $x = -2$, namely, at $x = -1$. The vertex is on this line.

At $x = -1$ we have $y = 3 - 6 + r = r - 3$.



We must have $r - 3 = 5$ giving $r = 8$.

Practice 52: We have $y = x(x+a) + a$.



(The picture assumes a is negative.)

We see that the line of symmetry is at $-\frac{a}{2}$ and

for $x = -a/2$, $y = \left(-\frac{a}{2}\right)\left(\frac{a}{2}\right) + a = a - \frac{a^2}{4}$

We see we need

$$a - \frac{a^2}{4} = -3.$$

That is, we need $a^2 - 4a - 12 = 0$.

Solving

$$(a-2)^2 = 16$$

$a - 2 = 4$ or -4

$a = 6$ or -2.

Choose $a = -2$.

Practice 53: We have $y = x(ax+b)+c$ which shows

$$x = 0 \quad \to \quad y = c$$
$$x = -\frac{b}{a} \quad \to \quad y = c$$

The line of symmetry is halfway between $x = 0$ and $x = -\frac{b}{a}$, which is at $x = -\frac{b}{2a}$.

Practice 54: The picture suggests an equation of the form $y = a(x-2)(x-8)+5$. When $x = 0$ we should have $y = 1$, so we need

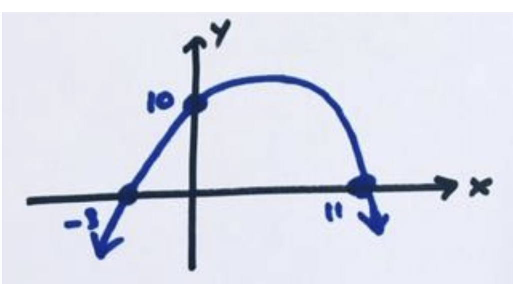
$$1 = a(-2)(-8)+5$$

giving $a = -\frac{1}{4}$. So the equation is

$$y = -\frac{1}{4}(x-2)(x-8)+5.$$

Practice 55: The sketch suggests the equation

$$y = a(x+3)(x-11)+0.$$

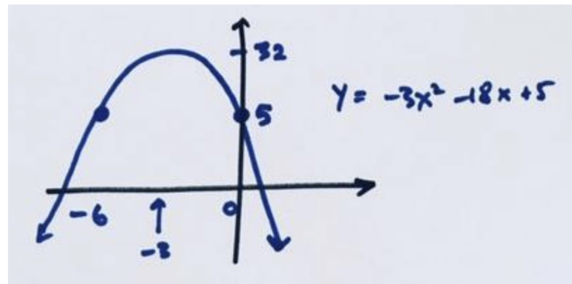


When $x = 0$ we need $y = 10$ giving $a = -\frac{10}{33}$.

The equation is

$$y = -\frac{10}{33}(x+3)(x-11).$$

Practice 56: We have $y = -3x(x+6)+5$ showing the line of symmetry is at $x = -3$. For this x value, we have $y = -3(-3)(3)+5 = 32$. The graph thus appears



In vertex form, we must have

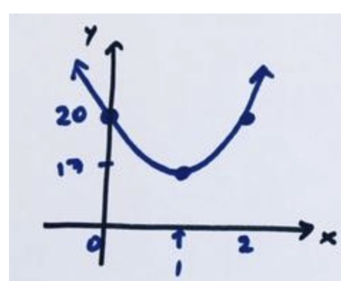
$$y = a(x+3)^2 + 32$$

with $a = -3$ to yield a "$-3x^2$" term when expanded. So the vertex form of the equation is

$$y = -3(x+3)^2 + 32.$$

Practice 57:

a) $(1,17)$
b) At $x = 1$
c) $y = 3(x-1)^2 + 17$
d)



e) There are none
f) $(0,20)$.

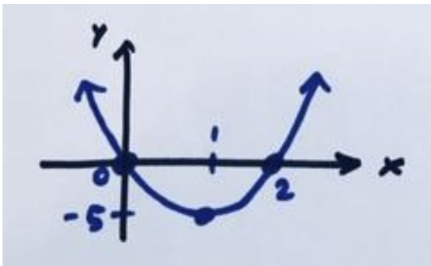
Practice 58:

a) $(1,-5)$

b) At $x=1$

c) $y- = 5(x-1)^2 - 5$

d)



e) $(0,0)$ and $(2,0)$

f) $(0,0)$

Practice 59:

a) Rewrite as $x^2+2x-15=1$. Solving gives $x=-1+\sqrt{17}$ or $x=-1=\sqrt{17}$.

b) Rewrite as $x^2 = 4x^2 - 1 - 5$, that is, as $x^2 = 2$ with solutions $x=\sqrt{2}$ or $x=-\sqrt{2}$.

c) Rewrite as $x^2-9x-5=12$ which has solutions $x=\dfrac{9\pm\sqrt{13}}{2}$.

Practice 60: We have that $x=c$ and $x=-c$ give symmetrical points on a symmetrical graph. The line of symmetry is thus at $x=0$, and this is where the vertex lies.

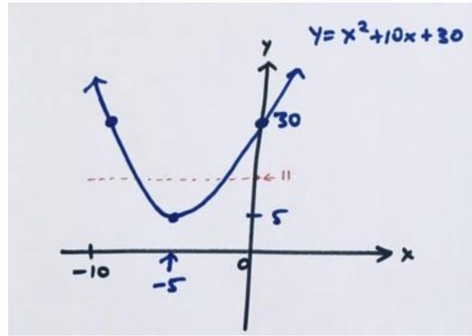
At $x=0$, $y=k-c^2$. We want this to equal -2 and so we must have

$$k=c^2-2.$$

Practice 61:

a) We have $(x+5)^2=-5$. There are no solutions.

b)

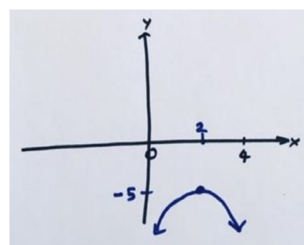


c) We see from the graph that there are no x values that give a y value of zero.

d) We see from the graph the there are two x values that give $y=11$.

e) We see from the graph that the answer is $k=5$.

Practice 62: We see from the graph of $y=-x^2+4x-5$ that there are no solutions to $0=-x^2+4x-5$.



Practice 63:

a) Writing $y=5x(x-2)+k$ shows that $x=1$ is the line of symmetry. We need the vertex of the graph of this equation to have height zero, so we need $0=5(1)(-1)+k$ meaning we need $k=5$.

b) Write $y=-2x(x+9)+m$. We need $\left(-\dfrac{9}{2},100\right)$ to be the vertex, so we need $-2\left(-\dfrac{9}{2}\right)\left(\dfrac{9}{2}\right)+m=100$ giving $m=\dfrac{119}{2}$.

c) The line of symmetry of the equation's graph is $x=2p$. We need $(p)(-p)=-10$ so we need $p=\sqrt{10}$ or $-\sqrt{10}$.

Practice 64: The area is $A=(7-r)(3+r)$.

This is a quadratic equation with $r=7$ and $r=-3$ both interesting. The line of symmetry is at $r=2$ and this is where the vertex of its graph lies. As this is a downward-facing quadratic, $r=2$ gives the maximal value.

Practice 65:

These answers are subjective.

i) (C) might be the easiest. We can see the smallest value of 7 occurs for $x=4$.

ii) (A) It crosses at $x=3$ and $x=-5$.

iii) (B) Putting in $x=0$ gives $y=8$.

iv) (C) The vertex is $(4,7)$.