



Are All U-Shaped Graphs Quadratic?

STUDENT HANDOUTS

Use these handouts for students who would like to practice the problems presented throughout the experience. These are entirely optional, are NOT homework, and are to be savored and mulled on just for mathematical joy.

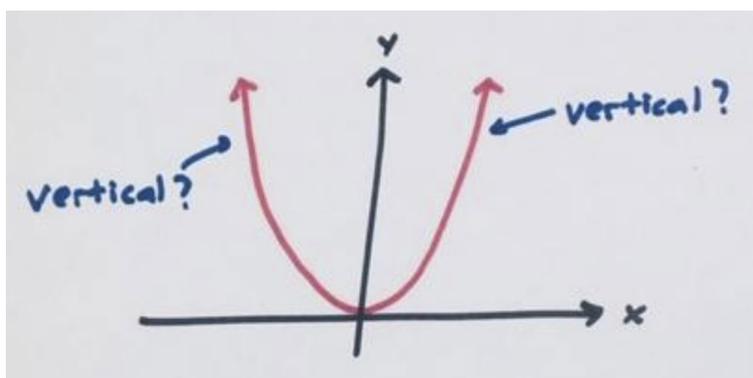
Video Resource

Access videos of *Garden Paths* lessons at <https://gdaymath.com/courses/gmp/>.

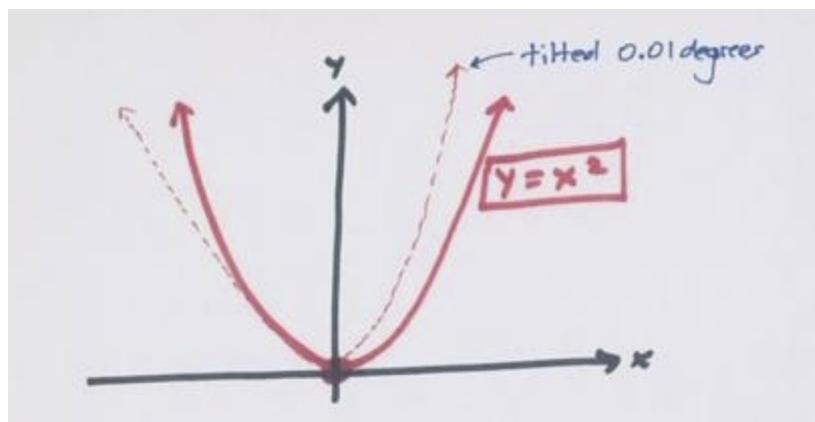
U-SHAPED CURVES
Lesson 1 Handout: Some U-Shaped Graphs

Here are some curious questions. Might you have some thoughts about possibly answering them?

Practice 1: Is the term “U-shaped” actually correct for the shape of the $y = x^2$ graph? After all, the sides of the letter U are vertical. Is the graph of $y = x^2$ ever vertical?

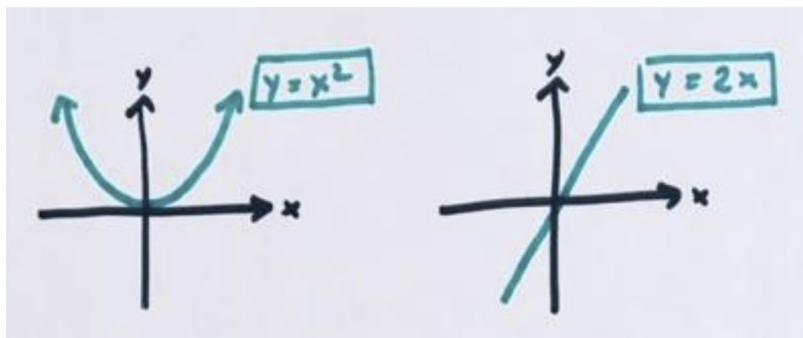


Practice 2: Suppose we tilt the graph of $y = x^2$ (well, rotate it, actually) counterclockwise about the origin just 0.01° . Does the y -axis intercept this tilted graph at some large non-zero value?

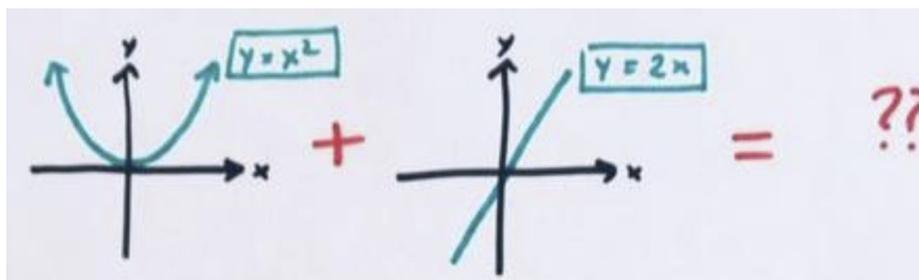


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Practice 3: The graph of $y = x^2$ is a symmetrical U-shaped curve. The graph of $y = 2x$, on the other hand, is not! It is a straight line through the origin with anti-symmetry if you like—the data plot rises to the right as it decreases to the left.



Now for a weird question: *What picture would be obtain if we “added” these two graphs?*



Use the idea suggested in the lesson to make sense of this question. What shape graph results?

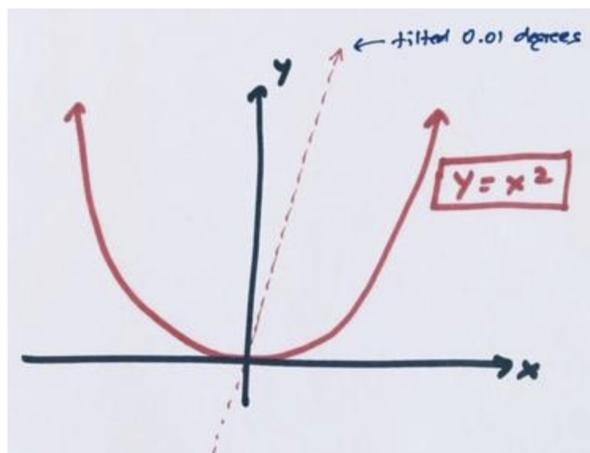
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Solutions to Lesson 1 Handout

1. Suppose the U-shaped graph is vertical at $x = 100$, say. This would mean that there are no points on the graph with x -value larger than 100. But $x = 101$, $y = 10201$ is a point on the graph!

In fact, this reasoning shows that there can be no x -value beyond which there are no points on the graph of $y = x^2$. That is, there can be no vertical “wall” to the graph.

2. It is not at all clear how one mathematically tilts a U-shaped graph some small angle to see if it intercepts the y -axis. But what if we tilt the vertical axis instead? That is, let’s keep the graph of $y = x^2$ the same and tilt the y -axis axis 0.01° clockwise and ask: does this titled line intersect the U-shape graph? This a question equivalent to the original problem.

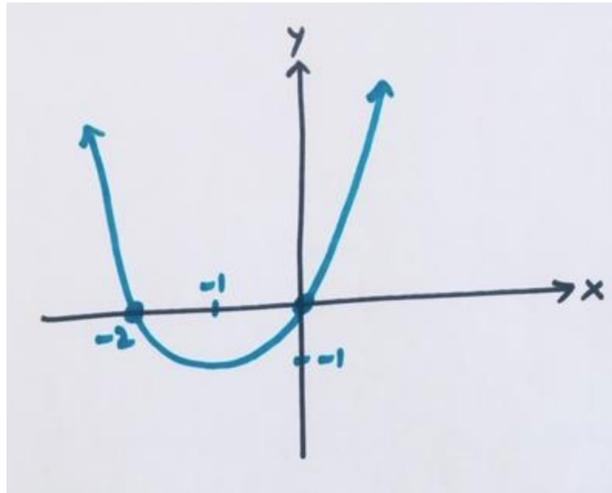


Any line through the origin of some slope $m \neq 0$, given by $y = mx$, intercepts $y = x^2$ at the point (m, m^2) .

So the answer to the original question is: YES!

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3. We get another symmetrical U-shaped graph shifted to a different position in the plane!



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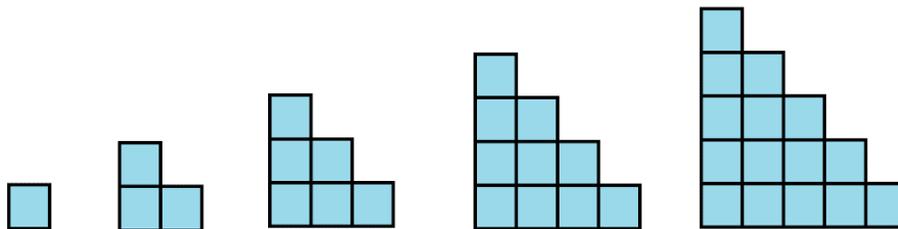
**Lesson 3 Handout:
Sequences – for when you trust patterns**

Try as many of these as are fun and interesting to you.

Practice 1: Make an intelligent guess as to the next number in the sequence

2 3 6 11 18 27 38 ___.

Practice 2: Consider the following sequence of diagrams each made of squares 1 unit wide.



If the implied geometric pattern from these first five figures continues ...

- What would the perimeter of the tenth figure likely be?
- What would the area of the tenth figure likely be?

Practice 3:

- Show that for the following sequence it seems that the third differences are constant. Make a prediction for the next number in the sequence.

0 2 20 72 176 350 612 ...

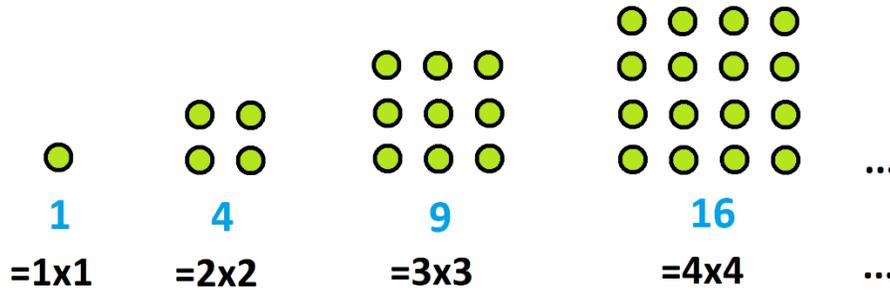
- How many differences must one complete in the sequence below to see a row of constant differences? (The sequence is the powers of two.)

1 2 4 8 16 32 64 128 256 ...

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Practice 4:

a) The sequence of square numbers begins 1, 4, 9, 16, 25, 36, 49, 64, (The n th number in this sequence is n^2 .)



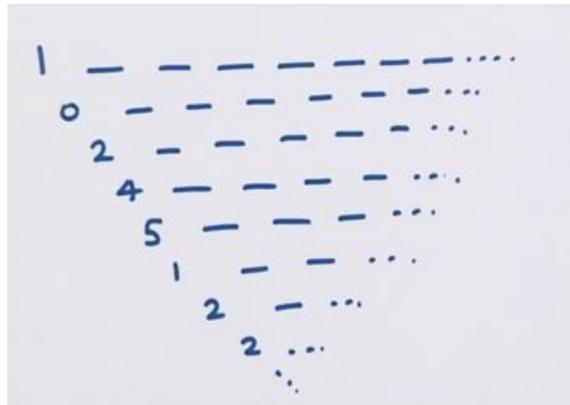
Is there a row in the difference table of the square numbers that is constant?

b) The sequence of cube numbers begins 1, 8, 27, 64, 125, 216, 343, 512, (The n th number in this sequence is n^3 .) Is there a row in the difference table of the cube numbers that is constant?

Practice 5: Use differences to make an intelligent guess as to the next element of this sequence.

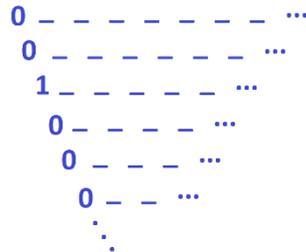
-1 4 7 8 7 4 -1 ___

Practice 6: Show that you can fill in the entire set of blanks in this difference table just from knowing the table's leading diagonal.



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Practice 7: What sequence has $0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ \dots$ as its leading diagonal?



Practice 8: Use difference methods to find a formula for the sequence of numbers

2, 2, 4, 8, 14, 22, 32, ...

(Just so you know, the answer is $n^2 - 3n + 4$. Can you see get this from looking at the leading diagonal for the sequence?)

Practice 9: Find a formula that fits the sequence **0, 2, 10, 30, 68, 130, 222,**

(The answer is $n^3 - 3n^2 + 4n - 1$.)

Practice 10:

- Find a formula that fits the sequence **5, 8, 11, 14, 17, 20, 23,**
- Find a formula that fits the sequence **3, 3, 3, 3, 3, 3, 3,**
- Find a formula that fits the sequence **1, 3, 15, 43, 93, 171, 283,**
- Find a formula that fits the sequence **1, 0, 1, 10, 33, 76, 145, 246, 385,**
- Find formulae for as many of these sequences you feel like doing!

3 3 7 21 51 103 183 297 ...

0 9 24 45 72 105 144 189 240 ...

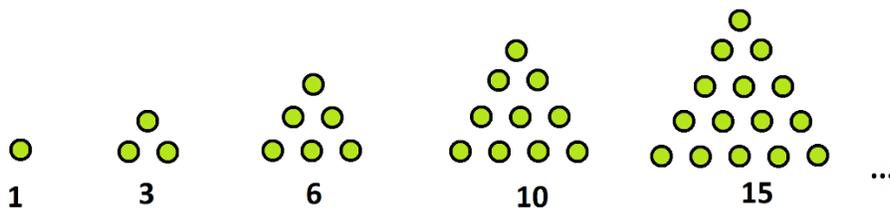
6 24 60 120 210 336 ...

230 275 324 377 434 495 ...

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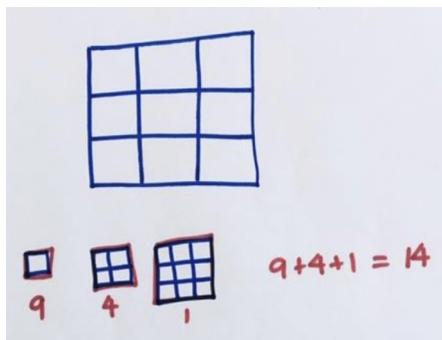
Practice 11: Find a general formula for the n th triangular number: **1, 3, 6, 10, 15, 21, 28, 36,**

...



(Don't be afraid of fractions!)

Practice 12: Let $S(n)$ be the total number of squares, of any size, one can find in an $n \times n$ grid of squares. For example, $S(3) = 14$ because one can find nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of $9 + 4 + 1 = 14$ squares in a 3×3 grid.



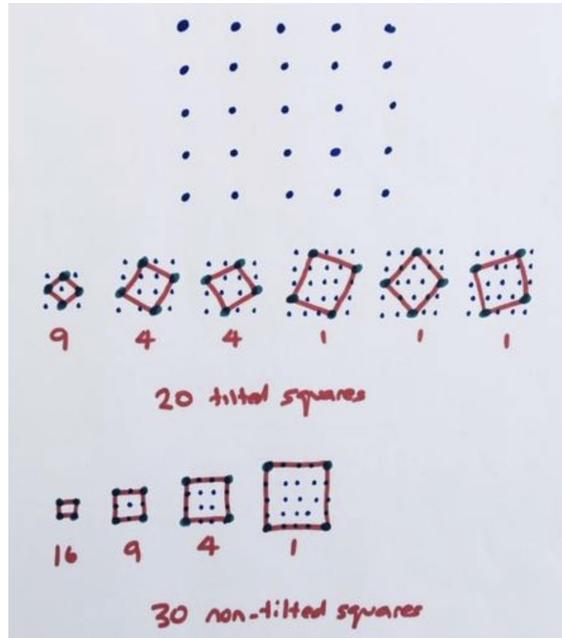
a) Find $S(1)$, $S(2)$, $S(4)$, and $S(5)$.

b) What do our general difference methods suggest for a general formula for $S(n)$?

c) **OPTIONAL CHALLENGE:** What is the value of $1^2 + 2^2 + 3^2 + \dots + 99^2 + 100^2$?

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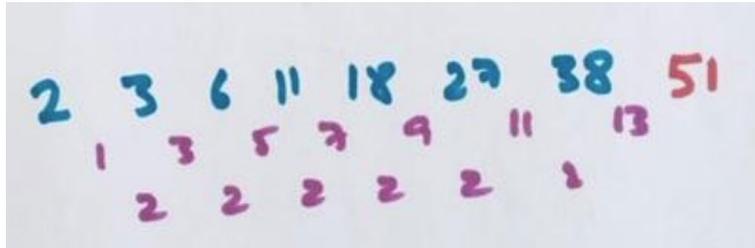
d) **OPTIONAL CHALLENGE:** Care to count tilted and non-tilted squares on arrays? For instance, on a five-by-five array of dots on can draw 30 non-tilted squares and 20 titled squares giving 50 squares in total.



U-SHAPED CURVES

Solutions to Lesson 3 Handout

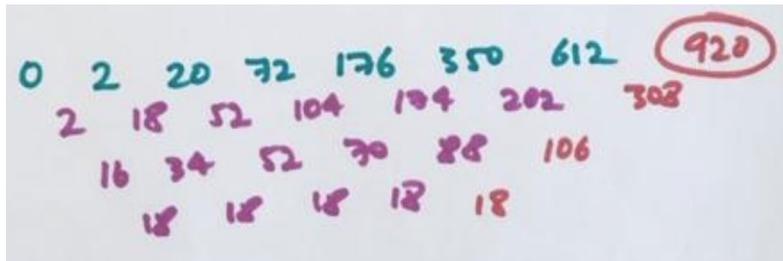
1. One might guess 51.



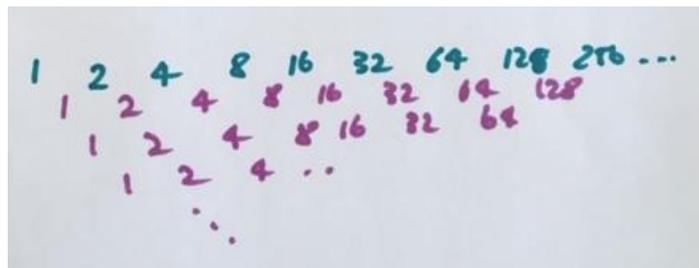
2. a) It looks the perimeters follow the sequence 4, 8, 12, 16, 20, ..., which are the multiples of four. (Can you explain why the n th figure is sure to have perimeter $4n$?) The tenth figure will have perimeter **40**.

b) It looks like areas follow the sequence 1, 3, 6, 10, 15, ... with differences the odd numbers. This would mean the next few areas are 21, 28, 36, 45, and **55** for the tenth one.

3. a)

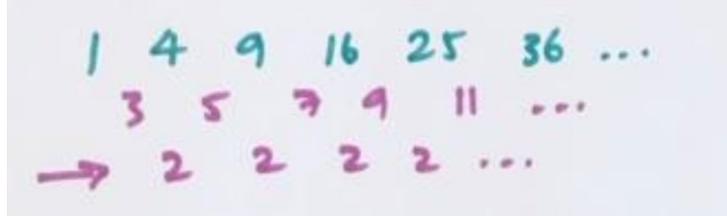


b) We will never obtain a row of constant differences!



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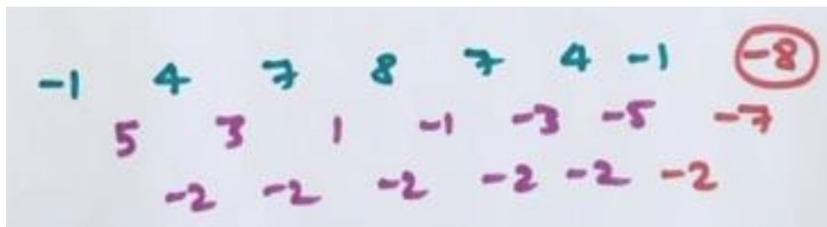
4. a) Yes. The second differences are constant.



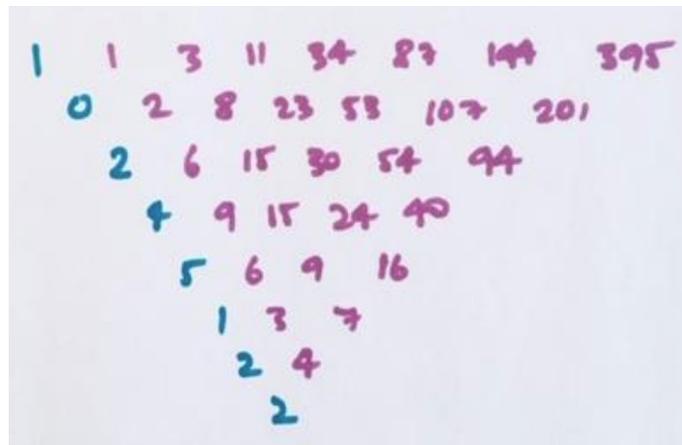
b) Yes. The third differences are constant.



5.

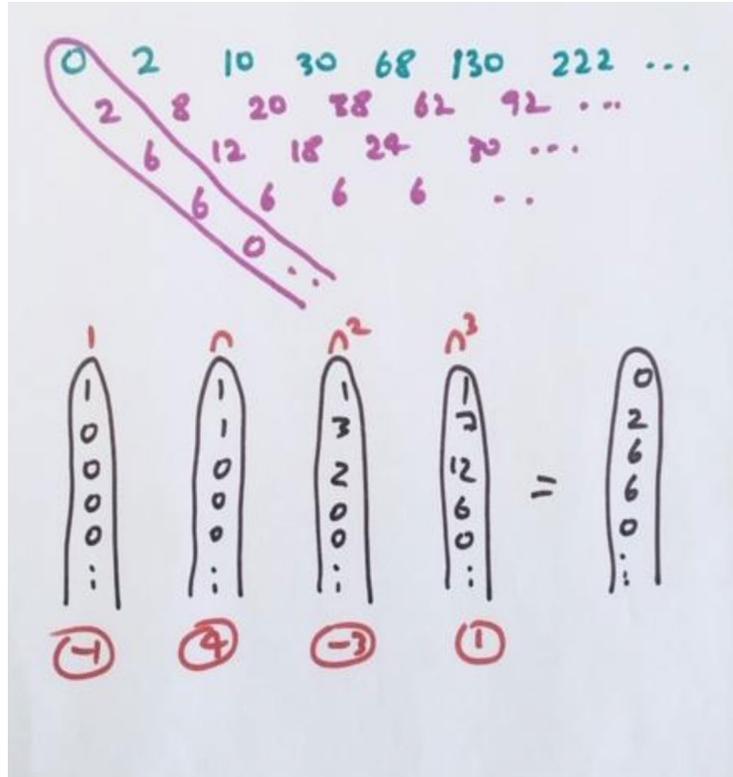


6.



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9.



10. a) $3n + 2$ b) $3n$ c) $n^3 - n^2 - 2n + 3$ d) $n^3 - 5n^2 + 7n - 2$

e) $n^3 - 4n^2 + 5n + 1$; $3n^2 - 3$; $n^3 + 3n^2 + 2n$; $2n^2 + 39n + 189$

11. $\frac{1}{2}n^2 + \frac{1}{2}n$

12. a) $S(1) = 1$, $S(2) = 5$, $S(3) = 14$, $S(4) = 20$, and $S(5) = 45$.

b) $S(n) = \frac{1}{3}n^2 + \frac{1}{2}n^2 + \frac{1}{6}n$.

c) Did you notice that $S(1) = 1^2$, $S(2) = 1^2 + 2^2$, $S(3) = 1^2 + 2^2 + 3^2$, $S(4) = 1^2 + 2^2 + 3^2 + 4^2$, and $S(5) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$?

U-SHAPED CURVES

If we are trusting all the patterns we see, then

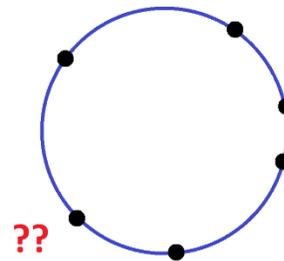
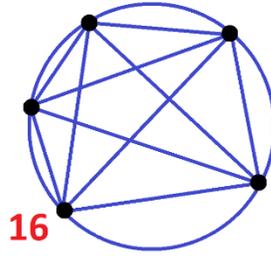
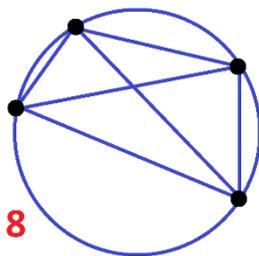
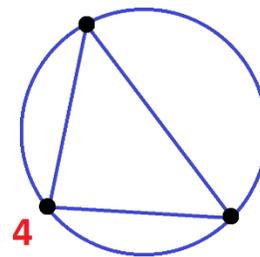
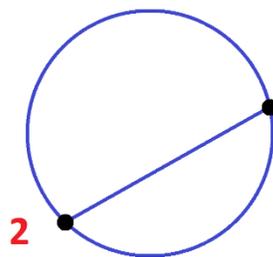
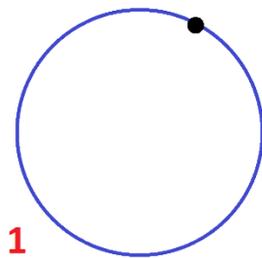
$$\begin{aligned} 1^2 + 2^2 + \cdots + 100^2 &= S(100) \\ &= \frac{1000000}{3} + \frac{10000}{2} + \frac{100}{6} = 338350. \end{aligned}$$

d) This is a real challenge!

Lesson 3 Additional Handout
DOTS ON A CIRCLE

Draw some circles with dots on their boundaries: one with 1 dot on its boundary, one with 2 dots on its boundary, one with 3 dots on its boundary, and so on.

Next, for each circle, connect each and every pair of boundary dots with a line segment. This divides each circle into a number of regions. Count the regions.



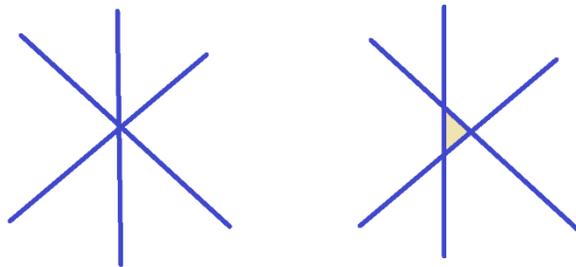
The circle with one dot has **1** region. The circle with two dots, **2** regions. The circle with three dots, **4** regions. The circle with four dots, **8** regions. The circle with five dots, **16** regions.

How many pieces do you expect to see from six boundary dots?

If we trust patterns, we expect to see **32** regions. The count seems to double every time.

U-SHAPED CURVES

- a) Let's agree to arrange dots so around the boundary of the circle so that no three lines pass through the same intersection point. That is, let's assume no regions are "masked" and that we can see the maximal possible count of regions that could result.



Show that **6** dots on the boundary of the circle actually yield just **31** regions.

- b) Suppose 7 dots are placed on the boundary of a circle in such a manner that the maximal number of regions result when one connects pairs of dots with line segments. How many regions is that?

What is the maximal number of regions that can result with 8 dots? With 9 dots?

- c) Care to keep computing more and more terms of our sequence?

1 2 4 8 16 31 _ _ _ _ _ ...

ULTRA-CHALLENGE: Might there still be a formula for the numbers in this sequence?

(See chapter 4 of MATH GALORE! (MAA, 2012). High-school students found such a formula and proved it works!)

Solutions to Lesson 3 Additional Handout

We obtain the sequence of region counts

1, 2, 4, 8, 16, 31, 57, 99, 163, 256, 386, 562, 794, ...

If you apply our difference method to this sequence, it seems these numbers follow the formula

$$\frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1.$$

This formula turns out to be correct and can be obtained by sound logical methods (see the reference given). It matches the formula $1 + C_2^n + C_4^n$.

U-SHAPED CURVES

Lesson 4 Handout

Parabolas and Non-Parabolas

How to Fold a Parabola

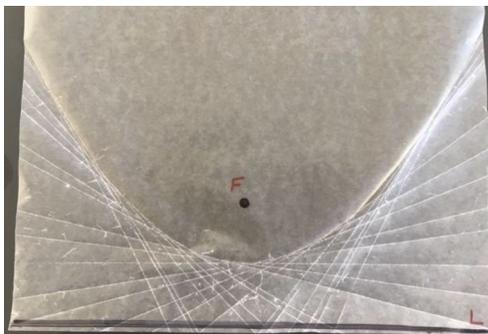
Draw a dot a couple of inches up from the bottom edge of the page for the focus F and imagine the bottom edge line as the directrix L .



Now lift up the bottom edge and align one point on it with the point F . Make a crease and unfold.



Do this another 50 times or so, lifting different points along the bottom edge up to the point F and making a crease line each and every time.



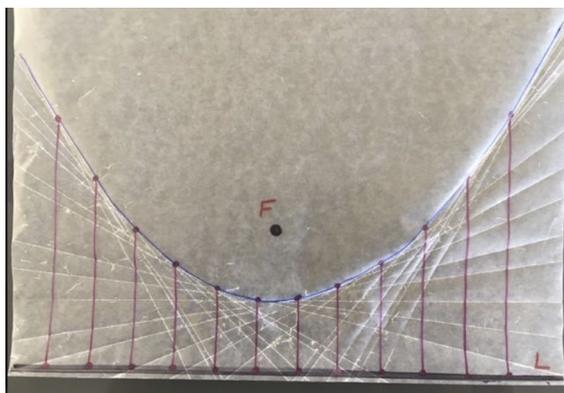
Those crease lines outline a curve, and that curve is a parabola! (See the lecture notes for a proof of this claim.)

U-SHAPED CURVES

ACTIVITY 1

High school textbooks make the claim that a parabola is the same shape curve as the graph of a quadratic equation $y = ax^2 + bx + c$. Does this seem feasible?

- Make a parabola by folding paper.
- Mark off regular intervals along the bottom edge and measure heights from the bottom edge to the curve as shown. This gives a sequence of values.



- Compute the difference table for this data.
- Within human error, does it seem reasonable to say that you have constant double differences?

CHALLENGE: Suppose we situate matters in the coordinate plane so that the directrix L is a horizontal line k units below the x -axis (and so has equation $y = -k$) and the focus F is a point k units high on the vertical axis (and so has coordinates $(0, k)$).

Translate the geometric condition for a point $P = (x, y)$ to be the parabola with focus F and directrix L into an algebraic condition. Is that algebraic condition a quadratic equation?

HARD CHALLENGE: Conversely, prove the graph of a quadratic equation $y = ax^2 + bx + c$ is sure to be a parabola. What is its focus? What is its directrix?

U-SHAPED CURVES

ACTIVITY 2

a) Hang a piece of light-weight chain on a white-board.



b) With a ruler, draw a horizontal line and mark off regular intervals along it. Measure the horizontal heights shown and collect a sequence of data values.



c) Use difference methods to find a quadratic formula for the heights along the chain. Be honest about what you notice.

U-SHAPED CURVES

EXTENDED ACTIVITY 3: *Is the St. Louis Gateway Arch in the shape of a quadratic curve? Find out by making measurements on a photograph of the arch.*

EXTENDED ACTIVITY 4: *Is a semi-circle given by a quadratic expression? Find out by tracing a pot lid on a piece of paper and making measurements with a ruler.*

EXTENDED ACTIVITY 5: *Is the path of a projectile truly quadratic (or does air resistance have a significant effect on the shape of paths)? Find a time-lapse series of photographs of a basketball or some other tossed object and measure the height of the ball on the photographs at regular time intervals.*

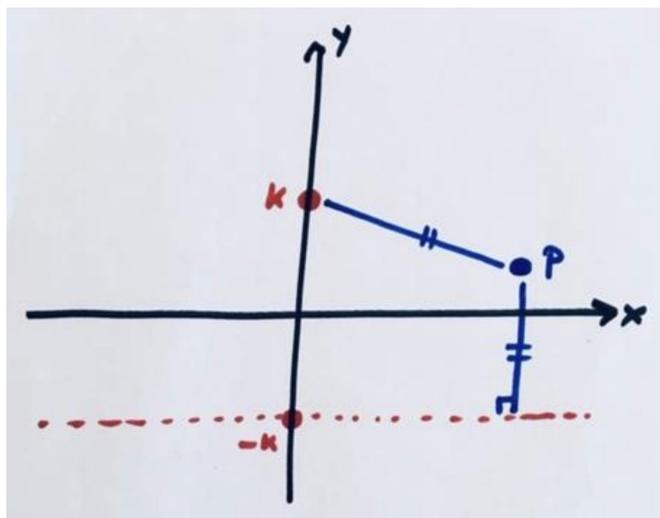
U-SHAPED CURVES

Solutions to Lesson 4 Handout

1. There will be human error involved, but there should indeed be constant double differences as the formula for the curve is indeed quadratic. We will now prove this.

Situate matters so that the focus F is at the point $(0, k)$ in the plane and the directrix L is given by $y = -k$.

As is proved in the lesson notes, for a point $P = (x, y)$ to be on the folded curve, we need its distance from F to match its distance from L .



This gives the equation

$$\sqrt{x^2 + (y - k)^2} = y + k .$$

Algebra now gives

$$x^2 + (y - k)^2 = (y + k)^2$$

$$x^2 - 2yk = 2yk$$

$$y = \frac{1}{4k} x^2$$

We have a quadratic equation for the curve.

U-SHAPED CURVES

We have that $y = \frac{1}{4k}x^2$ has focus $(0, k)$ and directrix $y = -k$.

So $y = ax^2$ (with $\frac{1}{4k} = a$) has focus $\left(0, \frac{1}{4a}\right)$ and directrix $y = -\frac{1}{4a}$.

Thus $y = ax^2 + c$ has focus $\left(0, \frac{1}{4a} + c\right)$ and directrix $y = -\frac{1}{4a} + c$.

And $y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a^2}$ has focus $\left(-\frac{b}{2a}, \frac{1}{4a} + c - \frac{b^2}{4a^2}\right)$ and directrix

$$y = -\frac{1}{4a} + c - \frac{b^2}{4a^2}.$$

2, 3, 4, and 5. Continue on with this experience to determine the answers to these question.

U-SHAPED CURVES
Lesson 5 Handout: Debunking Patterns

Practice 1: Convince yourself that the formula

$$8 \frac{(n-10)(n-20)}{(-4)(-14)} + 122 \frac{(n-6)(n-20)}{(4)(-10)} + 4600 \frac{(n-6)(n-10)}{(14)(10)}$$

gives the value 8 for $n = 6$, the value 122 for $n = 10$, and the value 4600 for $n = 20$.

Practice 2: Find a formula that gives the value 9000 for $n = 3$, the value -45 for $n = 5$, and the value $\frac{2}{3}$ for $n = 8$. (Don't bother simplifying your formula.)

Practice 3: Write down a formula that fits this data set.

x	1	2	7	15
y	5	10	8	-3

Practice 4: You are asked to write the next number in this sequence:

1 2 3 4 _

You decide to write $\sqrt{\pi}$. Write down a formula that justifies your answer.

Practice 5: Find an expression $an^2 + bn + c$ that fits the sequence **5 6 10**. (That is, find a formula that gives 5 for $n = 1$, 6 for $n = 2$, and 10 for $n = 3$.)

U-SHAPED CURVES
Solutions to Lesson 5 Handout

1. Do indeed check this.

2. This does the trick

$$9000 \frac{(n-5)(n-8)}{(-2)(-5)} - 45 \frac{(n-3)(n-8)}{(2)(-3)} + \frac{2}{3} \frac{(n-3)(n-5)}{(5)(3)}$$

3. This does the trick

$$5 \frac{(n-2)(n-7)(n-15)}{(-1)(-6)(-14)} + 10 \frac{(n-1)(n-7)(n-15)}{(1)(-5)(-13)} + 8 \frac{(n-1)(n-2)(n-15)}{(6)(5)(-8)} - 3 \frac{(n-1)(n-2)(n-7)}{(14)(13)(8)}$$

4. This does the trick

$$1 \frac{(n-2)(n-3)(n-4)(n-5)}{(-1)(-2)(-3)(-4)} + 2 \frac{(n-1)(n-3)(n-4)(n-5)}{(1)(-1)(-2)(-3)} + 3 \frac{(n-1)(n-2)(n-4)(n-5)}{(2)(1)(-1)(-2)}$$

$$+ 4 \frac{(n-1)(n-2)(n-3)(n-5)}{(3)(2)(1)(-1)} + \sqrt{\pi} \frac{(n-1)(n-2)(n-3)(n-4)}{(4)(3)(2)(1)}$$

5. This works: $5 \frac{(n-2)(n-3)}{(-1)(-2)} + 6 \frac{(n-1)(n-3)}{(1)(-1)} + 10 \frac{(n-1)(n-2)}{(2)(1)}$.

U-SHAPED CURVES
Lesson 6 Handout
Fitting Quadratics to Data

Practice 1: a) Show that $y = 7 \frac{(x-4)(x-5)}{(-5)(-6)} + \frac{(x+1)(x-5)}{(5)(-1)} + 10 \frac{(x-1)(x-4)}{(4)(1)}$ is a quadratic equation in disguise. Show that the graph of this equation passes through the data points $(-1, 7)$, $(4, 1)$, and $(5, 10)$.

b) Find a quadratic formula that fits the data $(2, 5)$, $(-1, 6)$, and $(5, 46)$ and make your answer look as friendly as possible.

x	2	-1	5
y	5	6	46

c) Find a quadratic formula that fits the data $(2, 5)$, $(3, 8)$, and $(5, 14)$ and make your answer look as friendly as possible. Explain what happens!

d) There is no quadratic formula $y = ax^2 + bx + c$ that fits the data $(2, 5)$, $(10, 6)$, and $(10, 100)$. (The same input value of $x = 10$ cannot give two different output values.) So then, how does Lagrange's Interpolation method fail when you try to use it?

Of course, from the previous lesson we know how to find a polynomial that fits any number of data points, not just three.

Practice 2: Use Lagrange's Interpolation Formula to find the equation of the line between two points (p, m) and (q, n) with $p \neq q$. Is your equation equivalent to " $y = mx + b$ " where m is the slope of the line segment between the two points?

U-SHAPED CURVES
Solutions to Lesson 6 Handout

1. a) We have $y = \frac{7}{30}(x-4)(x-5) - \frac{1}{5}(x+1)(x-5) + \frac{10}{4}(x-1)(x-4)$. If we were to expand the product in each term, we see that we would obtain an expression of the form $y = Ax^2 + Bx + C$.

One can see that substituting $x = -1$ into the original equation really does give $y = 7$, and so on.

b) We get $y = 5 \frac{(x+1)(x-5)}{(3)(-3)} + 6 \frac{(x-2)(x-5)}{(-3)(-6)} + 46 \frac{(x-2)(x+1)}{(3)(6)}$, which simplifies to

$$\begin{aligned} y &= -\frac{5}{9}(x^2 - 4x - 5) + \frac{1}{3}(x^2 - 7x + 10) + \frac{23}{9}(x^2 - x - 2) \\ &= \left(-\frac{5}{9} + \frac{1}{3} + \frac{23}{9}\right)x^2 + \left(\frac{20}{9} - \frac{7}{3} - \frac{23}{9}\right)x + \left(\frac{25}{9} + \frac{10}{3} - \frac{46}{9}\right) = \frac{7}{3}x^2 - \frac{8}{3}x + 1 \end{aligned}$$

c) We get an expression that simplifies to $y = 3x - 1$. The data turns out to be linear.

d) You find that the Lagrange's Interpolation formula has a denominator equal to zero in one of its terms. It breaks down.

2. We have $y = m \frac{(x-q)}{(p-q)} + n \frac{(x-p)}{(q-p)}$, which simplifies to

$$y = \frac{m(x-q) - n(x-p)}{p-q} = \frac{m-n}{p-q}x + \frac{np-mq}{p-q}$$

And $\frac{m-n}{p-q}$, the coefficient of x in this equation, is indeed the slope of the line.

U-SHAPED CURVES

Lesson 7 Handout: Personal Polynomials

My proper name is JAMES and I am particularly fond of the polynomial formula

$$p(x) = \frac{83}{24}x^4 - \frac{497}{12}x^3 + \frac{4141}{24}x^2 - \frac{3463}{12}x + 164.$$

(We know from the story of Exploding Dots that a combination of the non-negative powers of x is called a polynomial.)

It has

- $p(1) = 10$ and the 10th letter of the alphabet is J,
- $p(2) = 1$ and the 1st letter of the alphabet is A,
- $p(3) = 13$ and the 13th letter of the alphabet is M,
- $p(4) = 5$ and the 5th letter of the alphabet is E,
- $p(5) = 19$ and the 19th letter of the alphabet is S.

Practice 1: Find your own personal polynomial!

Practice 2: What is the expression that “spells” JIM? Is the graph of lesson 1 a quadratic graph?

Comment: Go to www.globalmathproject.org/personal-polynomial/ for a really cool web app that finds—and graphs—your personal polynomial for you! (Also see more videos explaining the mathematics.)

U-SHAPED CURVES

Solutions to Lesson 7 Handout

1. Work it out!

2. The polynomial for JIM is $y = 10 \frac{(x-2)(x-3)}{(-1)(-2)} + 9 \frac{(x-1)(x-3)}{(1)(-1)} + 13 \frac{(x-1)(x-2)}{(2)(1)}$, which

simplifies to $y = \frac{5}{2}x^2 - \frac{17}{2}x + 16$. It's graph is indeed the one shown in lesson 1.

U-SHAPED CURVES
Lesson 8 Handout:
Sequences – for when you really do trust patterns

Practice 1: *Trusting patterns, find a formula for the sequence*

3 7 29 99 247 503 897 1459

Practice 2: *Does Newton's approach give the formula n^2 for the sequence of square numbers 1, 4, 9, 16, 25, 36, ...? Does it give the correct formula for the cube numbers?*

OPTIONAL:

- Consider the sequence of the powers of three: 1 3 9 27 81 243 729 Show that its leading diagonal seems to be the powers of two.*
- Consider the sequence of the powers of four: 1 4 16 64 256 1024 4096 Show that its leading diagonal seems to be the powers of three.*
- Prove that the leading diagonal of the powers of a*

$$1 \quad a \quad a^2 \quad a^3 \quad a^4 \quad a^5 \quad a^6 \quad \dots$$

is sure to be the powers of $a - 1$.

Practice 3: *Consider the sequence of numbers*

0 9 108 891 5832 32805 166212 780759 ...

Draw the difference table for this sequence.

Compute the difference table for its leading diagonal.

Compute the difference table for the leading diagonal of this difference table.

Keep doing this for a while and see that the sequence is exhibiting behavior analogous to that of the powers of three.

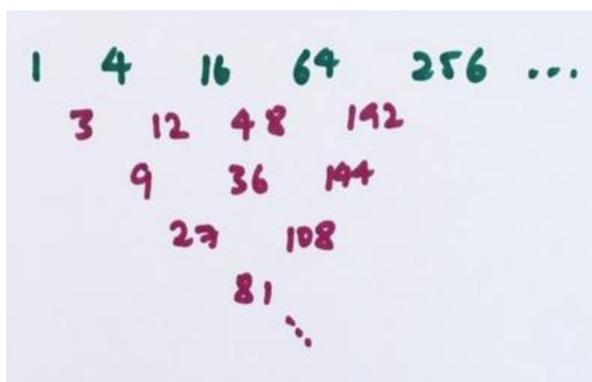
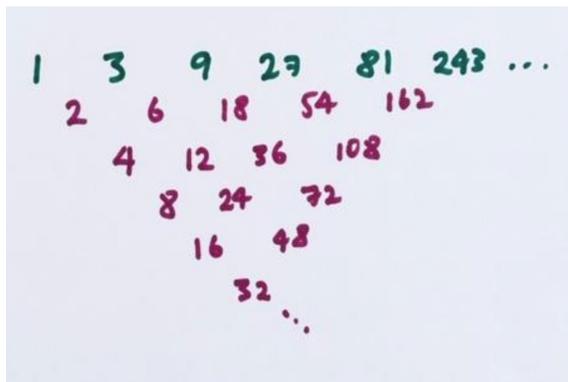
Find a formula for the n th term of this sequence.

U-SHAPED CURVES
Solutions to Lesson 8 Handout

1. We get $3 + 4 \cdot (n-1) + 18 \cdot \frac{(n-1)(n-2)}{2} + 30 \cdot \frac{(n-1)(n-2)(n-3)}{6}$.

2. For the square numbers we get $1 + 3(n-1) + 2 \frac{(n-1)(n-2)}{2}$, which does indeed simplify to n^2 .

For the cube numbers we get $1 + 7(n-1) + 12 \frac{(n-1)(n-2)}{2} + 6 \frac{(n-1)(n-2)(n-3)}{6}$, which does indeed simplify to n^3 .

OPTIONAL:


U-SHAPED CURVES

1	a	a^2	a^3	a^4	...
$a-1$	$a(a-1)$	$a^2(a-1)$	$a^3(a-1)$...	
$(a-1)^2$	$a(a-1)^2$	$a^2(a-1)^2$...		
$(a-1)^3$	$a(a-1)^3$...			

If one line of the table is $(a-1)^k$, $a(a-1)^k$, $a^2(a-1)^k$, $a^3(a-1)^k$, ... for some value k , then we can see that the next line of the table is sure to be $(a-1)^{k+1}$, $a(a-1)^{k+1}$, $a^2(a-1)^{k+1}$, $a^3(a-1)^{k+1}$, This means the powers of $(a-1)$ do appear on the leading diagonal.

3. This is awfully tedious, but one does see that the leading diagonals do suggest that there are powers of three involved.

If we divide the n the term of the given sequence by 3^n , we get the sequence

$$0, 3, 12, 33, 72, 135, 228, 397, \dots$$

which follows the formula $n^3 - 3n^2 + 5n - 3$, suggesting the original sequence is given by

$$3^n \cdot (n^3 - 3n^2 + 5n - 3).$$

U-SHAPED CURVES

Lesson 8 Additional Handout

THE SLIDE PUZZLE

This classic puzzle is usually phrased in terms of frogs and toads leap-frogging over each other. Here I'll just use counters.

Let's start with two black and two white counters arranged in a row of five boxes as shown.

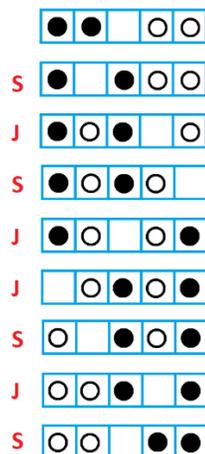


White counters can only move left. Black counters can only move right.

Each move is either a slide (S) in which a counter moves one place over to an empty cell or a jump (J) in which a counter leap-frogs over an adjacent counter (of any color) into an empty cell two places over.

The goal is to have the black and white counters switch positions.

With $N = 2$ counters of each color the puzzle can be solved in 8 moves.



Let $P(N)$ be the number of moves required to solve an analogous puzzle with N black counters and N white counters arranged in a row of $2N + 1$ boxes.

U-SHAPED CURVES

We have shown that $P(2) = 8$.

- a) Find $P(3)$ by solving the $N = 3$ version of the puzzle.



- b) Find $P(1)$, $P(4)$, and $P(5)$.
 c) Assuming we can trust patterns, find a possible formula for $P(N)$.

Solving the $N = 2$ puzzle followed the pattern of moves

S J S J J S J S.

Here we have four slides and four jumps.

- d) What patterns of moves solves the $N = 1$, $N = 3$, $N = 4$, and $N = 5$ puzzles? How many slides appear in each? How many jumps?
 Any conjectures as to how many slides and how many jumps there will be in solving a general puzzle with N black and N white counters?

- e) With N black and N white counters explain why a solution must involve N^2 jumps. (Use a logical argument this time and don't rely on patterns.)
 f) With N black and N white counters how many places to the left must each white counter move and how many places to the right must each black counter move? Explain why we must have $P(N) = 2N(N + 1) - N^2$. (Does this match your answer to part c?)
 g) Solve the puzzle again for the case $N = 3$. This time watch the location of the blank space. Is each of the seven boxes empty at some point of play? Must this be the case?

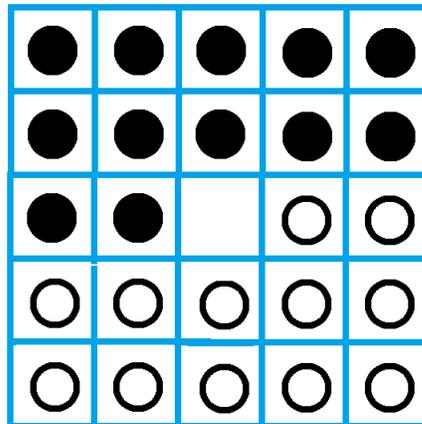
U-SHAPED CURVES

Challenge 1: Suppose there are M black counters and N white counters.



Is it always possible to rearrange the counters so that the black ones sit to the right and the white ones sit to the left? Is there a pattern to the moves required? Is there a general formula for the number of moves required?

Challenge 2: Is it possible to interchange the black and white counters in this two-dimensional array? Here counters may move in any direction – left, right, up, down – via slides and jumps.



Solutions to lesson 8 SLIDE PUZZLE

a) Check that that the puzzle can be solved in 15 moves.

b) $P(1) = 3$, $P(2) = 8$, $P(3) = 15$, $P(4) = 24$, $P(5) = 35$.

c) Difference methods suggest $P(N) = N^2 + 2N$.

d)

For $N = 1$ we have the pattern **S J S** with 2 slides and 1 jump.

For $N = 2$ we have the pattern **S J S J J S J S** with 4 slides and 4 jumps.

For $N = 3$ we have the pattern **S J S J J S J J J S J S** with 6 slides and 9 jumps.

For $N = 4$ we have the pattern **S J S J J S J J J S J J J S J S** with 8 slides and 16 jumps.

For $N = 5$ we have the pattern **S J S J J S J J J S J J J J S J J J S J S** with 10 slides and 25 jumps.

In general it looks like there are $2N$ slides and N^2 jumps.

e) Each of the N black counters must move past each of the N white counters. This can only be done with jumps. Thus each black counter must make N jumps and so there are $N \times N$ jumps in total. (And note, each white counter must jump past each black counter too. But this will be automatically accomplished by these same $N \times N$ jumps too.)

f) Each counter must move $N + 1$ places either left or right, so it looks like we need a total of $2N \times (N + 1)$ slides. But some of these "slides" are jumps. Each jump moves a counter two places instead of one but still only counts as one move. So the number $2N(N + 1)$ for the number of moves is off by 1 for each jump. As there are N^2 jumps, the total number of moves must be $2N(N + 1) - N^2$, which does indeed match our work above.

g) Every cell of the board (except the center) has to be occupied by a new counter. In order for a counter to move to that position, that cell must be empty at some point. Thus each cell is indeed empty at some point of play.

U-SHAPED CURVES

Challenge 1: Yes! There are total of MN jumps and $N(M + 1) + M(N + 1) - MN = MN + M + N$ moves in total.

Challenge 2: Yes! Just solve the central row of the puzzle as though it's its own one-dimensional game but, between moves, quickly swap the counters in the column with the blank space. As the blank space does occur in every cell of the central row (by part g), all the counters in each column will be switched, as will be the counter in the central row.