QUADRATICS
2.1 Deriving the Quadratic Formula

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SETTING THE SCENE

You may already know that there is famous formula one can use to solve all quadratic equations. Most curricula want students to know this formula, memorise it, and use the formula with speed. Personally, as a mathematician, and I don't feel the need to do anything with speed! Plus, I find memorisation joyless and unnecessary. So what if I take an extra twenty seconds to draw a square on the side of my page to solve a quadratic equation?

But here's the secret: We have actually been doing the quadratic formula all along! Our square method, the *quadrus* method, is the quadratic formula in disguise.

The goal of this essay is to explain this.

But before we start, here is a little puzzle we'll come back to later.

PUZZLE 1: *I am thinking of two numbers. Their sum is 10 and their product is 24. What are my two numbers?*

Are you thinking 4 and 6? This puzzle is too easy. It's not my real puzzle.

PUZZLE 2: *I am thinking of two numbers. Their sum is 10 and their product is 25. What are my two numbers?*

This has a trick to it. Are you now thinking 5 and 5? (No one said the two numbers were different!)

But that too wasn't my real puzzle. Here's the real puzzle.

PUZZLE 3: *I am thinking of two numbers. Their sum is 10 and their product is 26. What are my two numbers?*

Now matters are not so easy! What could my two numbers be?

THE QUADRATIC FORMULA

Recall how we solve a level 6 quadratic equation such as

$$5x^2 - 3x + 2 = 4.$$

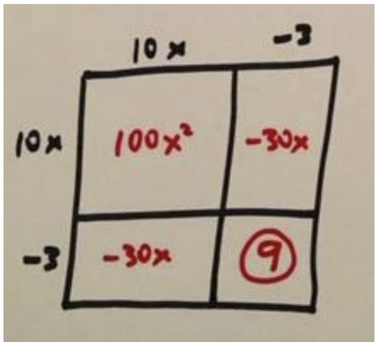
We start by multiplying through by 5 to make the first term a nice square.

$$25x^2 - 15x + 10 = 20$$

Now we notice an odd middle term, so we multiply through by 4 to make that term even and, at the same time, ensure we still have a perfect square out front.

$$100x^2 - 60x + 40 = 80$$

The numbers are large, but they are the perfect numbers to create a lovely square.



But we see we need to adjust the number 40 and make a 9. So we subract 31 from both sides of the equation.

$$100x^2 - 60x + 9 = 49$$

And we like this equation because the left side is a perfect $10x-3$ by $10x-3$ square. We have

$$(10x-3)^2 = 49$$

which we can readily solve.

$$10x - 3 = 7 \text{ or } -7$$

$$10x = 10 \text{ or } -4$$

$$x = 1 \text{ or } -\frac{2}{5}$$

Beautiful!

Now let's do the same work again, not on an equation with specific numbers, but on an abstract quadratic equation.

A CURRICULUM CONVENTION

Almost every curriculum insists that quadratic equations be written with some left side set **equal to zero** on the right. For example, most curricula would prefer to rewrite

$$5x^2 - 3x + 2 = 4$$

as

$$5x^2 - 3x - 2 = 0$$

by subtracting 4 from each side.

We've never bothered to do that! With the square method we knew we were likely going to change the constant term in a quadratic expression, so we just waited until we could see what number would be best for it. Most school curricula insist, however, that you make a change right away and rewrite the equation so that it equals zero, even if you decide to change the number again later on.

So let's follow the curriculum convention and assume we are solving a quadratic equation of the form

$$\boxed{ax^2 + bx + c = 0}.$$

Example: For $5x^2 - 3x - 2 = 0$ we have

$$a = 5,$$
$$b = -3,$$
$$c = -2.$$

Let's follow our square method and solve this equation.

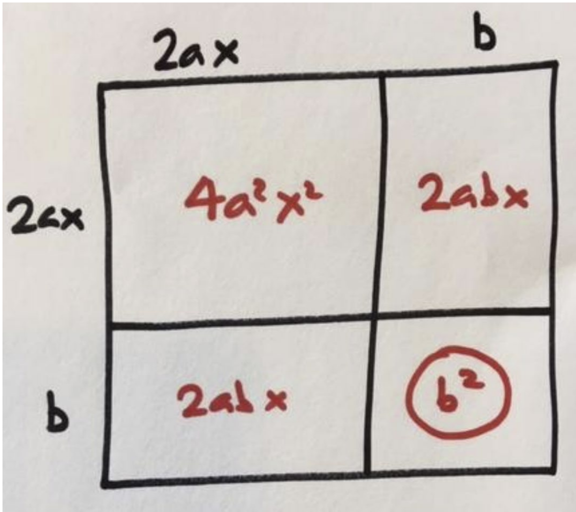
Multiply through by a to make the first term a nice square.

$$a^2x^2 + abx + ac = 0$$

Just in case the middle term is odd (we don't know), let's cover ourselves and multiply through by 4.

$$4a^2x^2 + 4abx + 4ac = 0$$

We're now ready to draw the square.



Splitting "$4abx$" into two gives regions of area $2abx$. If one side length of each is $2ax$, then the remaining side length is b. This means we need a final piece of the square of area b^2. But we have $4ac$ instead of b^2 in our equation $4a^2x^2+4abx+4ac=0$.

Let's subtract $4ac$ from both sides

$$4a^2x^2+4abx=-4ac$$

and then add b^2 to each side

$$4a^2x^2+4abx+b^2=-4ac+b^2.$$

Most people choose to rewrite the right-hand side.

$$4a^2x^2+4abx+b^2=b^2-4ac$$

Now $4a^2x^2+4abx+b^2$ is a perfect $2ax+b$ by $2ax+b$ square. We have

$$(2ax+b)^2=b^2-4ac$$

which is a level 2 problem.

Something squared is b^2-4ac and so we get

$$2ax+b=\sqrt{b^2-4ac} \text{ or } -\sqrt{b^2-4ac}.$$

Adding $-b$ throughout gives

$$2ax=-b+\sqrt{b^2-4ac} \text{ or } -b-\sqrt{b^2-4ac}.$$

Dividing through by $2a$ then gives our final solutions.

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Most people prefer to combine these two solutions by using a funny $\pm$ symbol in the middle to indicate we can have a $+$ sign or a $-$ sign.

The Famous Quadratic Formula

If $ax^2+bx+c=0$,

then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Example: For $5x^2-3x=2=0$ we have $a=5$, $b=-3$, and $c=-2$. The quadratic formula then gives

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4\cdot(5)\cdot(-2)}}{2\cdot 5}.$$

This is

$$x=\frac{3\pm\sqrt{49}}{10}$$

which is

$$x=\frac{3+7}{10} \text{ or } \frac{3-7}{10}$$

giving $x=1$ or $x=-\dfrac{2}{5}$, just as before!

PRACTICE 1: *Solve $3x^2+5x+1=9$ by using the quadratic formula and then again by using the square method. (Of course, you should get the same answers each time!)*

I personally prefer the square method for solving quadratic equations: I understand what to do and there is nothing for me to memorize, I adjust numbers as I go along and don't worry about making my sure the equation "equals zero," and I think it is fun! But it is a slower enterprise.

Many people prefer to use the quadratic formula because it is speedier.

But let's be honest: If your goal really is just to get an answer and to do so as fast as possible for some reason, then the most intelligent thing is not to follow either approach and simply use a free algebra system on the internet instead! (We live in the 21st century after all!)

And this explains why I really like the square method. It is the story of the *THINKING*, and mathematical thinking brings me joy. Getting the answers to specific problems is not the point of the story. It is not the important part.

HONESTY MOMENT: Most every quadratic equation presented to students in a curriculum uses "nice" numbers. But if I had to solve an awkward quadratic equation like

$$1.3x^2 + \frac{\pi}{3}x - \frac{17}{\sqrt{2\frac{1}{2}}} = 0$$

by hand, the square method would be miserable. I'd probably use the quadratic formula. (Though that would likely be miserable too!)

PRACTICE 2 (OPTIONAL): *Solve*

$$1.3x^2 + \frac{\pi}{3}x - \frac{17}{\sqrt{2\frac{1}{2}}} = 0.$$

(Did you notice that this question is optional?)

PRACTICE

PRACTICE 3: *Solve whichever of these quadratic equations you feel like doing, using whatever method you like. (Or solve some twice with two different methods!)*

a) $6x^2 - x + 10 = 11$
b) $30x^2 - 17x = 2$
c) $x^2 - 4x + 4 = 0$
d) $2x^2 + 5 = 11x$
e) $93x^2 - 117 = 0$
f) $x^2 + x + 1 = 2$
g) $x^2 + x + 1 = 1$
h) $x^2 + x + 1 = 0$

PRACTICE 4: *Solve for* x *in terms of* a *and* b.

$$x^2 - (a+b)x + ab = 0.$$

PRACTICE 5: *Find a number* p *so that* $1 + \frac{1}{p}$ *equals* p.

PRACTICE 6: *A quadratic equation* $ax^2 + bx + c = 0$ *has precisely one solution. What is the value of* $b^2 - 4ac$?

PRACTICE 7: *The list of* oblong numbers *begins* $2, 6, 12, 20, 30, 42, 56, \ldots$. *Here each number is the product of two consecutive integers.*

$$2 = 1 \times 2$$
$$6 = 2 \times 3$$
$$12 = 3 \times 4$$

etc.

a) *What is the one-hundredth number in the list?*
b) 5402 *is an oblong number. At which position in the list does it sit?*

SOLUTIONS

PRACTICE 1: *Solve* $3x^2 + 5x + 1 = 9$ *by using the quadratic formula and then again by using the square method. (Of course, you should get the same answers each time!)*

Answer: Rewrite the equation as $3x^2 + 5x - 8 = 0$. The quadratic formula then gives $x = \dfrac{-5 \pm \sqrt{25+96}}{6} = \dfrac{-5 \pm 11}{6}$ and so $x = 1$ or $x = -\dfrac{8}{3}$.

The square method suggests working with

$$36x^2 + 60x + 25 = 121$$

that is,

$$(6x-5)^2 = 121$$

which gives the same solutions.

PRACTICE 2 (OPTIONAL): *Solve*

$$1.3x^2 + \frac{\pi}{3}x - \frac{17}{\sqrt{2\frac{1}{2}}} = 0.$$

(Did you notice that this question is optional?)

Answer:

$x = \dfrac{-\frac{\pi}{3} \pm \sqrt{\frac{\pi^2}{9} - \frac{88.4}{\sqrt{2.5}}}}{2.6}$ which I do not feel like simplifying.

PRACTICE 3: *Solve whichever of these quadratic equations you feel like doing, using whatever method you like. (Or solve some twice with two different methods!)*

a) $6x^2 - x + 10 = 11$
b) $30x^2 - 17x = 2$
c) $x^2 - 4x + 4 = 0$
d) $2x^2 + 5 = 11x$
e) $93x^2 - 117 = 0$
f) $x^2 + x + 1 = 2$
g) $x^2 + x + 1 = 1$
h) $x^2 + x + 1 = 0$

Brief Answers:

a) $x = \frac{1}{2}$ or $-\frac{1}{3}$.
b) $x = \frac{2}{3}$ or $-\frac{1}{10}$.
c) $x = 2$.
d) $x = 5$ or $\frac{1}{2}$.
e) $x = \pm\sqrt{\frac{117}{93}}$.
f) $x = \frac{-1+\sqrt{5}}{2}$ or $\frac{-1-\sqrt{5}}{2}$.
g) $x = 0$ or -1.
h) No solutions.

PRACTICE 4: *Solve for* x *in terms of* a *and* b.

$$x^2 - (a+b)x + ab = 0.$$

Answer: We did this question using the square method in a previous essay. Let's use the quadratic formula this time.

$$x = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab}}{2}$$
$$= \frac{a+b \pm \sqrt{a^2 + b^2 - 2ab}}{2}$$
$$= \frac{a+b \pm \sqrt{(a-b)^2}}{2}$$

So

$$x = \frac{a+b+a-b}{2} \text{ or } \frac{a+b-(a-b)}{2}$$

giving

$$x = a \text{ or } b.$$

PRACTICE 5: *Find a number* p *so that* $1+\frac{1}{p}$ *equals* p.

Answer: Notice that this question implicitly assumes that p is not zero. (We'll see if this is an issue or not later on, I guess.)

Multiplying the equation through p gives

$$p+1 = p^2$$

that is,

$$p^2 - p - 1 = 0.$$

This has solutions

$$p = \frac{1+\sqrt{5}}{2} \text{ and } p = \frac{1-\sqrt{5}}{2}.$$

Neither of these is zero. They are both valid solutions.

PRACTICE 6: *A quadratic equation* $ax^2 + bx + c = 0$ *has precisely one solution. What is the value of* $b^2 - 4ac$ *?*

Answer: According to the quadratic formula, the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For there to be precisely one solution, we need $b^2 - 4ac$ to be zero. This gives the solutions

$$x = \frac{-b+\sqrt{0}}{2a} \text{ and } \frac{-b-\sqrt{0}}{2a}.$$

And since $\sqrt{0} = 0$, these both equal $\frac{-b}{2a}$ and we have one solution.

PRACTICE 7: *The list of* oblong numbers *begins* $2, 6, 12, 20, 30, 42, 56, \ldots$. *Here each number is the product of two consecutive integers.*

$$2 = 1 \times 2$$
$$6 = 2 \times 3$$
$$12 = 3 \times 4$$

etc.

a) What is the one-hundredth number in the list?

b) 5402 *is an oblong number. At which position in the list does it sit?*

Answer: The n th number in the list is given by $n(n+1)$.

a) The 100th number in the list is 10100.

b) We need to solve $n(n+1) = 5402$, that is,

$$n^2 + n - 5402.$$

We get $n = \frac{-1 \pm \sqrt{21609}}{2} = \frac{-1 \pm 147}{2}$.

So $n = 73$. (The negative option is not relevant to this question.)