QUADRATICS
2.2 Counting Solutions and the Discriminant

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SETTING THE SCENE

Consider the quadratic equation

$$x^2 + 6x + 9 = something$$

where the "something" is a number yet to be specified. The square method shows that the left side of this question is actually a perfect square. The equation is really a level 2 problem in disguise.

$$(x+3)^2 = something.$$

And we've seen before that such equations can have with 2, 1, or 0 solutions. For instance,

$(x+3)^2 = 4$ has precisely two solutions since there are two roots of the number 4,

$(x+3)^2 = 0$ has precisely one solution since there is only one square root of zero,

$(x+3)^2 = -5$ has no solutions since there are no square roots of a negative value.

So this means, going back to the original form of the equation,

$$x^2 + 6x + 9 = 4$$
$$x^2 + 6x + 9 = 0$$
$$x^2 + 6x + 9 = -5$$

have, respectively, 2, 1, and 0 solutions.

PRACTICE 1: *Use the quadratic formula to solve each of the equations shown. Ascertain the feature of the formula that determines whether the count of solutions is going to be 2, 1, or 0.*

$$x^2 + 6x + 9 = 4$$
$$x^2 + 6x + 9 = 0$$
$$x^2 + 6x + 9 = -5$$

THE DISCRIMINANT

The quadratic formula says that the solutions to a quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

The only potentially troublesome part of these supposed solutions is the quantity $b^2 - 4ac$ under the root.

Question: *It looks like having $a = 0$ would also be problematic. But if $a = 0$, would we be solving a quadratic equation?*

If $b^2 - 4ac$ turns out to be a positive value, then $\sqrt{b^2-4ac}$ makes sense and both possible solutions are valid. The quadratic equation will have 2 distinct solutions.

If $b^2 - 4ac$ turns out to be zero, then $\sqrt{b^2 - 4ac} = 0$ too and our two potential solutions "collapse" to just one solution.

$$x = \frac{-b+0}{2a} \text{ or } \frac{-b-0}{2a}$$

and these both equal $-\frac{b}{2a}$. The quadratic equation has just 1 solution.

If $b^2 - 4ac$ turns out to be negative, then $\sqrt{b^2 - 4ac}$ is meaningless and the quadratic formula is meaningless. There are 0 solutions to the quadratic equation.

Question: *Did you observe each of these phenomena when doing practice problem 1?*

Some curricula like to emphasize what we just observed.

Definition: For a quadratic equation $ax^2 + bx + c = 0$, the quantity $b^2 - 4ac$ is call the **discriminant** of the equation. (It "discriminates" the counts of solutions to the equation.)

If $b^2 - 4ac > 0$, then the equation has 2 solutions.
If $b^2 - 4ac = 0$, then the equation has 1 solution.
If $b^2 - 4ac < 0$, then the equation has 0 solutions.

Of course, one can determine the number of solutions a quadratic equation simply by solving it! But if speed is important for some reason, then maybe a focus on the discriminant is helpful.

PRACTICE

Here are some strange practice problems designed solely to test whether or not you have discriminant thinking in your head. (Of course, they can each be solved via the square method, but the questions aren't designed to be "nice" that way.)

PRACTICE 2: *For which values of q does $x^2 + qx + 1 = 0$ have precisely 2 solutions?* (Actually: Solve this question using the square method too just to see the quantity $q^2 - 4$ naturally appear.)

PRACTICE 3: *Find all the values b so that*

$$x^2 - 2a^2x + a^4 = b^3$$

has precisely 2 solutions.

PRACTICE 4: *For which values of k does $kx^2 + 2kx + 1 = 0$ have just 1 solution? For which values of k does it have 0 solutions?*

PRACTICE 5: *If $ax^2 + bx = 244\frac{1}{2}$ has 2 solutions, what will their sum?*

FINAL (OPTIONAL) COMMENT

The quadratic formula shows that if a quadratic equation $ax^2 + bx + c = 0$ has two solutions, then those two solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

That is, the solutions are

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

and

$$x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Just to make things visually clearer, let's write D for the number $\frac{\sqrt{b^2 - 4ac}}{2a}$. Then the two solutions to the quadratic equation are

$$x = -\frac{b}{2a} + D \text{ and } x = -\frac{b}{2a} - D.$$

That is, the two solutions are each a distance D either side of the value $-\frac{b}{2a}$ on the number line.

We also see that the sum of the two solutions is

$$\left(-\frac{b}{2a} + D\right) + \left(-\frac{b}{2a} - D\right) = -\frac{b}{a}.$$

None of this is really that important, but sometimes people like to explore properties of the quadratic formula and make observations like these.

SOLUTIONS

PRACTICE 1: *Use the quadratic formula to solve each of the equations shown. Ascertain the feature of the formula that determines whether the count of solutions is going to be 2, 1, or 0.*

$$x^2+6x+9=4$$
$$x^2+6x+9=0$$
$$x^2+6x+9=-5$$

Answer: For $x^2+6x+5=0$ we have $x=\frac{-6\pm\sqrt{16}}{2}$. Now 16 has two square roots in arithmetic, namely 4 and -4. The quadratic formula captures this by using the positive root of 16, namely $\sqrt{16}=4$, and inserting the $\pm$ sign. It gives two distinct solutions then because 4 and -4 are distinct values.

In general, if the quantity under the square root sign is a positive number, then the quadratic equation will have two distinct solutions.

For $x^2+6x+9=0$ we have $x=\frac{-6\pm\sqrt{0}}{2}$. Now $\sqrt{0}=0$ so the formula gives the solutions $x=\frac{-6+0}{2}$ and $x=\frac{-6-0}{2}$. These are both equal to the number $\frac{-6}{2}=-3$ and we have one solution.

In general, if the quantity under the square root sign is zero, then we'll have one (repeated?) solution.

For $x^2+6x+14=0$ we have $x=\frac{-6\pm\sqrt{-20}}{2}$. There square root of a negative quantity does not exist, and so there are no solutions.

In general, if the quantity under the square root sign is a negative number, then the quadratic equation has no solutions.

PRACTICE 2: *For which values of q does $x^2+qx+1=0$ have precisely* 2 *solutions?* (Actually: Solve this question using the square method too just to see the quantity q^2-4 naturally appear.)

Answer:
The discriminant is "b^2-4ac" $=q^2-4$. This is a positive number when $q^2>4$, that is when

$$q>2 \text{ or } q<-2.$$

The square method has us work with

$$4x^2+4qx+4=0$$

and, drawing the picture, we see we want the number q^2, not 4, as the final term.

$$4x^2+4qx+q^2=q^2-4$$
$$(2x+q)^2=q^2-4$$

and this has two solutions when $q^2>4$, just as before.

PRACTICE 3: *Find all the values b so that*

$$x^2-2a^2x+a^4=b^3$$

has precisely 2 *solutions.*

Answer: Rewrite the equation as

$$x^2 - 2a^2x + a^4 - b^3 = 0.$$

The discriminant is

$$\left(-2a^2\right)^2 - 4\cdot 1\cdot\left(a^4 - b^3\right)$$

which equals

$$4a^4 - 4a^4 + 4b^3 = 4b^3.$$

This is positive when $b^3 > 0$, that is, precisely when b is positive.

PRACTICE 4: *For which values of* k *does* $kx^2 + 2kx + 1 = 0$ *have just* 1 *solution? For which values of* k *does it have* 0 *solutions?*

Answer: The discriminant is

$$4k^2 - 4k.$$

This equals zero when $k^2 - k = 0$. This is its own quadratic equation, and solving it gives $k = 0$ or $k = 1$. (Alternatively: Look at $k^2 = k$. Clearly $k = 0$ is a solution. And if $k \neq 0$ we are permitted to divide by sides by k to get $k = 1$.)

So $kx^2 + 2kx + 1 = 0$ has just one solution for $k = 0$ and $k = 1$.

One can check that the discriminant

$$4k^2 - 4k = 4k(k-1)$$

is negative if $0 < k < 1$; zero, as we have seen, for $k = 0, 1$; and positive otherwise.

So the equation has no solutions if $0 < k < 1$.

PRACTICE 5: *If* $ax^2 + bx = 244\frac{1}{2}$ *has* 2 *solutions, what will be their sum?*

Answer: The two solutions of any quadratic equation $ax^2 + bx + c = 0$ sum to

$$\left(\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) + \left(\frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$
$$= -\frac{b}{2a} + -\frac{b}{2a}$$
$$= -\frac{b}{a}.$$