edfinity®

QUADRATICS
2.3 Lots of Practice; The Opening Puzzle

James Tanton

SETTING THE SCENE

Your textbook is full of many problems to practice solving quadratic equations. Many of the problems will look like the ones we've presented in our previous essays, to simply examine and solve a stated equation. But some might look quite different and, on first appearance, might seem to have nothing to do with quadratic equations. In this essay we'll present some unusual-looking quadratic equation practice questions.

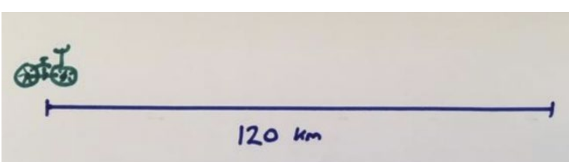
SETTING UP EQUATIONS

Here's a question.

PRACTICE 1: *I ride my bike along a straight stretch of road. The road is 120 km long and I ride at a constant speed.*

I then ride my bike back along the same stretch of road, again at a constant speed, but this time 10 km/hr faster than I did before.

I was one hour quicker on my return journey. What were my two riding speeds?



Do you remember the two key steps to problem solving?

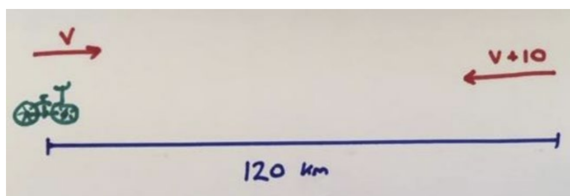
<u>Step 1</u>: *Have an emotional reaction.*

Take a deep breath and then

<u>Step 2</u>: *DO SOMETHING! Anything!*

This question does look a bit scary. Where are the numbers? Where's the equation? All I see is the distance 120 km and a time difference of 1 hour.

Just to DO SOMETHING, we could indicate on the picture one constant speed of v km/hr, say, and then a second constant speed of $v+10$ km/hr for the way back. That's a start!



What next?

We can get some times!

Going 120 km at a constant speed of v km/hr will take $\frac{120}{v}$ hours. The return journey, 120 km at a speed of $v+10$ km/hr, will take $\frac{120}{v+10}$ hours.

This is 1 hour quicker.

So we must have

$$\frac{120}{v+10}=\frac{120}{v}-1.$$

I bet we can rework this equation to get a quadratic equation in v which we can then solve!

Your job: *Finish this question.*

PRACTICE 2: *Xavier, in a speed banana-eating contest, ate his first set of six bananas in t seconds, but took 5 seconds longer eating his second set of six bananas. His banana-eating rate was $\frac{6}{t}$ bananas-per-second for his first set. His rate was 0.1 bananas-per-second less for his second set.*

How long did Xavier spend eating his first six bananas?

WHAT IS THE VARIABLE? CHANGING PERSPECTIVE

Equations that don't look quadratic at first just might be, if your change your perspective. But there might be issues!

PROBLEM: *Solve* $\sqrt{x}+2=x$.

Right away there is something curious to note about this equation. In order for the quantity $\sqrt{x}$ to make sense, x cannot be negative. This question must then have the hidden assumption that x *is greater than or equal to zero.* That assumption not explicitly stated, but it must be there.

Next: Is it a quadratic equation? If we regard "x" as the variable, then certainly not: it is not an equation of the form $ax^2+bx+c=0$. But if we write $x=\left(\sqrt{x}\right)^2$ (remember, we must be assuming x is a non-negative number), then the equation is

$$\sqrt{x}+2=\left(\sqrt{x}\right)^2$$

which <u>is</u> a quadratic equation in the variable $\sqrt{x}$. It's

$$\left(\sqrt{x}\right)^2-\left(\sqrt{x}\right)-2=0.$$

Challenge: *What does the quadratic formula give for the solutions of this equation? Are they actually solutions?*

Let me attempt to solve the original equation $\sqrt{x}+2=x$ a different way. Let's write

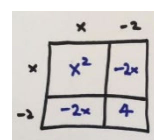
$$\sqrt{x}=x-2$$

and then square both sides of the equation to get

$$x=(x-2)^2.$$

The right-hand side is an $x-2$ by $x-2$ square, which expands as x^2-4x+4, so we have the equation

$$x=x^2-4x+4.$$



This is the quadratic equation

$$x^2-5x+4=0.$$

To practice the quadratic formula, we see that this has solutions

$$x=\frac{5\pm\sqrt{25-16}}{2}=\frac{5\pm3}{2}=4 \text{ or } 1.$$

So it looks like $x=4$ and $x=1$ are the solutions to the equation $\sqrt{x}+2=x$.

But look what happens when we check!
For $x = 4$ we have

$$\sqrt{4} + 2 = 4$$

which is true and correct. So yes, $x = 4$ really is a solution.

For $x = 1$ we have

$$\sqrt{1} + 2 = 1$$

which is FALSE! We see that $x = 1$ is <u>not</u> actually a solution!

What's going on?

We mentioned in a previous essay that the $\sqrt{}$ symbol is a symbol from geometry and is an asymmetrical symmetry: it only allows the positive square root of a number. We started with the equation $\sqrt{x} = x - 2$, which is a restricted equation as it will allow only for one type of square root, the positive one. And then we squared the equation to play with $x = x^2 - 4x + 4$ instead. But this new equation has no $\sqrt{}$ symbol mentioned in it and so allows for <u>all</u> possible roots of numbers. It is not surprising then that it might give more "solutions" than the original equation will allow.

UPSHOT: Watch out if you manipulate an unusual equation focused on an unusual variable (such as $\sqrt{x}$) to obtain a more familiar type of equation: you might obtain candidate solutions that are not actually allowed for the original, more restrictive equation. Always check which of your final candidate solutions are actually solutions.

PRACTICE 3: *Solve* $w + 3\sqrt{w} + 2 = 0$.

PRACTICE 4: *Find at least two solutions to* $x^4 - 3x^2 - 4 = 0$.

PRACTICE 5: *Find all possible solutions to* $\frac{4}{x^2} + \frac{4}{x} + 1 = 0$.

THE OPENING PUZZLE

Recall the opening puzzle essay 2.1 asks for two numbers which sum to 10 and have product 26.

$$sum = 10$$
$$product = 26$$

Finding such a pair of numbers is tricky.

Let's call the numbers we're looking for p and q. They must satisfy

$$p + q = 10$$
$$pq = 26 \ .$$

Actually, we see that once we find one number, say p, then the first equation gives us q: it must be $10 - p$. So, let's focus on the number p.

The second equation, with this focus on p, thus reads

$$p(10 - p) = 26 .$$

Expanding the left side, we get $10p - p^2 = 26$, which can be written in the familiar form of a quadratic equation

$$p^2 - 10p + 26 = 0 .$$

Ooh! And this is interesting! I've become pretty adept at the square method and I can see I want to rewrite this as

$$p^2 - 10p + 25 = -1 .$$

(Do you see why my brain wanted this? If not, quickly draw the square.)

This now reads as

$$\left(p-5\right)^2=-1,$$

which has no solutions!

Exercise: *If you prefer, use the quadratic formula to show that* $p^2-10p+26=0$ *has no solutions.*

There are no two numbers that sum to 10 and have product 26. This was a trick question!

PRACTICE 6: *Find all values* k *for which there is a pair of numbers that sum to* 10 *and have product* k.

$$sum=10$$
$$product=k$$

(For instance, we saw that $k=24$ *and* $k=25$ *are both possible values for* k*, and* $k=26$ *is not.)*

PRACTICE 7: *Let* S *be a fixed number. Find, in terms of* S*, the largest possible number* k *for which there is a pair of numbers that sum to* S *and have product* k.

$$sum=S$$
$$product=k$$

PRACTICE 8: *Find the smallest possible value* k *for which the equation* $x+\frac{1}{x}=k$ *has a solution for a positive value* x.

PRACTICE 9: *a) Find two numbers that differ by* 100 *and have product 5069.*

b) When Anu thought about this question she first said to herself: “Symmetry is my friend. A symmetrical solution would have the two numbers the same. So let me represent the numbers by how different they are from being the same.” She decided to write the two numbers as $n-50$ *and* $n+50$*. How do you think she then proceeded with the problem?*

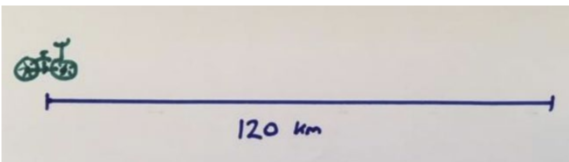
c) *Find two numbers whose sum is* 100 *and whose product is 2491. (What is a “symmetrical” way to set up this problem?)*

SOLUTIONS

PRACTICE 1: *I ride my bike along a straight stretch of road. The road is* 120 *km long and I ride at a constant speed.*

I then ride my bike back along the same stretch of road, again at a constant speed, but this time 10 *km/hr faster than I did before.*

I was one hour quicker on my return journey. What were my two riding speeds?



Answer: Following on from the start of the solution in the essay, we have

$$\frac{120}{v+10} = \frac{120}{v} - 1.$$

Let's avoid fractions! Let's multiply through by v,

$$\frac{120v}{v+10} = 120 - v\ ,$$

and then through by $v+10$,

$$120v = 120(v+10) - v(v+10).$$

This simplifies to

$$0 = 1200 - v^2 - 10v,$$

that is, to

$$v^2 + 10v - 1200 = 0.$$

Solving gives

$$v^2 + 10v + 25 = 1225$$
$$(v+5)^2 = 1225$$
$$v+5 = 35 \text{ or } -35$$
$$v = 30 \text{ or } -40$$

It seems only the positive number for a speed is relevant for this problem, so my two speeds were

$v = 30$ km/hr

and

$v + 10 = 40$ m/hr.

PRACTICE 2: *Xavier, in a speed banana-eating contest, ate his first set of six bananas in* t *seconds, but took 5 seconds longer eating his second set of six bananas. His banana-eating rate was* $\frac{6}{t}$ *bananas-per-second for his first set. His rate was* 0.1 *bananas-per-second less for his second set.*

How long did Xavier spend eating his first six bananas?

Answer: We are being told that Xavier's second banana-eating rate, $\frac{6}{t+5}$, is 0.1 less than $\frac{6}{t}$. So

$$\frac{6}{t+5} = \frac{6}{t} - 0.1.$$

Multiplying through by t and then $t+5$ and then 10 eventually gives the equation

$$t^2 + 5t - 300 = 0.$$

This has solutions $t = 15$ or -20. Only the first solution is relevant to the context.

Xavier ate his first six bananas in 15 seconds.

PRACTICE 3: *Solve* $w+3\sqrt{w}+2=0$.

Answer: The quadratic formula gives

$$\sqrt{w}=\frac{-3\pm 1}{2}=-1 \text{ or } -2.$$

But $\sqrt{w}$ is a positive quantity, and so no proposed solution is valid. This equation has no solutions.

PRACTICE 4: *Find at least two solutions to* $x^4-3x^2-4=0$.

Answer: The quadratic formula gives

$$x^2=\frac{3\pm 5}{2}=4 \text{ or } -1.$$

As x^2 cannot be negative, we can only entertain $x^2=4$, that is, $x=2$ or $x=-2$. One checks that these are indeed valid solutions to the original equation.

PRACTICE 5: *Find all possible solutions to* $\frac{4}{x^2}+\frac{4}{x}+1=0$.

Answer: This question tacitly assumes that x is not zero. We'll see if this might be an issue later on.

Let's multiply through by x^2 to avoid fractions.

$$4+4x+x^2=0$$

The quadratic formula gives

$$x=\frac{-4\pm 0}{2}=-2.$$

There is only one potential solution (and it is non-zero) and one checks that $x=-2$ really is a solution!

PRACTICE 6: *Find all values* k *for which there is a pair of numbers that sum to* 10 *and have product* k.

$$\begin{aligned} sum &= 10 \\ product &= k \end{aligned}$$

(For instance, we saw that $k=24$ *and* $k=25$ *are both possible values for* k*, and* $k=26$ *is not.)*

Answer: Following the ideas in this essay, we need two numbers p and q with

$$\begin{aligned} p+q &= 10 \\ pq &= k. \end{aligned}$$

The first equation gives $q=10-p$ so we can read the second equation as

$$p(10-p)=k$$

that is, as

$$p^2-10p+k=0.$$

The square method has as look at

$$(p-5)^2=25-k,$$

which has solution(s) only if $25-k\geq 0$, that is, if k is less than or equal to 25.

PRACTICE 7: *Let S be a fixed number. Find, in terms of S, the largest possible number k for which there is a pair of numbers that sum to S and have product k.*

$$sum = S$$
$$product = k$$

Answer: Call the two numbers p and q. Then we want

$$p + q = S$$
$$pq = k$$

to have a solution with k as large as possible.

Writing $q = S - p$, the second equation gives

$$p^2 - Sp + k = 0.$$

We want this to have a solution with k as large as possible.

One can use the square method here. But let's use this quadratic formula this time.

The discriminant of this equation is $S^2 - 4k$, and this must be ≥ 0 for this equation to have solutions. That is, we need $k \leq \frac{S^2}{4}$.

The largest possible value of k possible then is $\frac{S^2}{4}$.

PRACTICE 8: *Find the smallest possible value k for which the equation $x + \frac{1}{x} = k$ has a solution for a positive value x.*

Answer: The question wants us to look at the equation $x + \frac{1}{x} = k$ only for positive values of x. It follows then that the value of k we seek will be positive.

Let's multiply the equation through by x to avoid fractions. We need to solve

$$x^2 - kx + 1 = 0.$$

This has a solution if the discriminant is ≥ 0, that is, if

$$k^2 - 4 \geq 0.$$

Since we are considering only positive values of k, this means we need $k \geq 2$.

The smallest value of k that yields a solution to the equation for a positive x is $k = 2$. (And $x = 1$ works in this case.)

PRACTICE 9: *a) Find two numbers that differ by 100 and have product 5069.*

b) When Anu thought about this question she first said to herself: "Symmetry is my friend. A symmetrical solution would have the two numbers the same. So let me represent the numbers by how different they are from being the same." She decided to write the two numbers as $n - 50$ and $n + 50$. How do you think she then proceeded with the problem?

c) *Find two numbers whose sum is* 100 *and whose product is 2491. (What is a "symmetrical" way to set up this problem?)*

Answer:
For a) and b):
Two numbers of the form $n - 50$ and $n + 50$ differ by 100 and have product $n^2 - 2500$. We thus need $n^2 = 5069 + 2500 = 7569$, giving $n = 87$ or -87. Our two numbers could thus be 37 and 137, or -137 and -37.

c) The “symmetrical” answer would have the two numbers be 50 and 50. Let’s represent two general numbers then as $50-n$ and $50+n$. (These sum to 100.) Then their product is $2500-n^2$ and we want this to equal 2491. So we need $n^2=9$ giving $n=3$ or -3. Either way, the two numbers are 47 and 53.