QUADRATICS
3.1 Mersenne Primes

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SETTING THE SCENE

The story of solving quadratic equations is really the story of area and the power of using symmetry in that story. Every quadratic equation can be solved by drawing a symmetrical square – even the abstract equation $ax^2 + bx + c = 0$. (This led to the famous quadratic formula.)

This lesson deviates from the story of symmetry. The topic discussed doesn't actually belong here. But most curricula want students to attend to the ONE SPECIAL CASE when one can solve a quadratic equation by using an unsymmetrical technique. This technique rarely works in most examples, it relies on intelligent guessing and on luck, and it assumes the numbers involved are easy to work with. The techniques is called *factoring*.

Let me start our discussion on factoring with a historical piece of mathematics. It will be our opening puzzle and when we return to it at the end of the lesson, we'll see why mathematicians are interested in the technique of factoring equations—and, surprisingly, it is not for solving quadratics! (Can you see why I, James, am uncomfortable having this lesson in the middle of our quadratics story? It really does not belong here! Oh well!)

MERSENNE PRIMES

The list of the powers of two begins

2, 4, 8, 16, 32, 64, 128, 256, … .

The n th number in the list is 2^n. For example, the fifth number is $32 = 2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

Now let's subtract 1 from each value.

1	**3**	**7**	**15**	**31**	**63**	**127**	**255**	**...**
$2^1 - 1$	$2^2 - 1$	$2^3 - 1$	$2^4 - 1$	$2^5 - 1$	$2^6 - 1$	$2^7 - 1$	$2^8 - 1$	$\cdots$

In the 1500s and 1600s scholars noticed something curious about this list of numbers. Choose a prime number, say 5, and look at

$$2^5 - 1 = 31$$

It's also prime.

Choose another prime, like 7, or 3 or 2.

$$2^7 - 1 = 127$$
$$2^3 - 1 = 7$$
$$2^2 - 1 = 3$$

and it is again prime! (Check that 127 is indeed prime!)

Is it true that $2^{prime} - 1$ is again sure to be prime?

The answer to that question turns out to be no! This is a false pattern. For example,

$$2^{11}-1=2047$$
$$=23\times 89$$

is not prime. But people started wondering:

Is $2^{prime}-1$ often a prime number?

They started looking for numbers of this form that are prime and had a hard time finding more. The answer seems to be no!

A French monk and mathematician (1588-1618) took particular interest in this problem, worked on it, and today primes of the form $2^{prime}-1$ are called *Mersenne primes.* People wanted to know how many Mersenne primes there are. Are there infinitely many examples to be found, or only finitely many examples?

FAMOUS UNSOLVED MATHEMATICS

To this day no one on this planet knows the answer to that question! At present (at the time of writing this lesson) only 51 examples of Mersenne primes are known with the 51st one being discovered in December of 2018. It is the number

$$2^{82589933}-1$$

(and yes, 82589933 is a prime number). This number, one less than a power of two, is over 24 million digits long!

OPTIONAL EXERCISE: *For world fame, determine how many more Mersenne primes there are to be found. Are there any more? Finitely many more? Infinitely many more?*

But what about numbers of the form $2^{composite}-1$? Could they ever be prime?

The three examples we have in our list are composite.

$2^4-1=15$, composite
$2^6-1=63$, composite
$2^8-1=255$, composite

PRACTICE 1: *Evaluate each of* 2^9-1, $2^{10}-1$, *and* $2^{12}-1$ *and show they too are each composite.*

Mersenne proved that a number of the form $2^{composite}-1$ is never prime, that it is sure to be composite. So, we have then from Mersenne that

$2^{prime}-1$ is sometimes prime and sometimes composite

$2^{composite}-1$ is never prime.

PRACTICE 2: *Is* $2^{13}-1$ *prime or composite?*

OUR PUZZLE FOR THIS LECTURE:

How might one prove that $2^{composite}-1$ is never prime?

Let's be specific. Consider the composite number 300.

How could you convince someone that $2^{300}-1$ *is not prime?*

(By the way, the number $2^{300}-1$ is 91 digits long. It is too big for a calculator to handle!)

SOLUTIONS

PRACTICE 1: *Evaluate each of* $2^9 - 1$, $2^{10} - 1$, *and* $2^{12} - 1$ *and show they too are each composite.*

Answer: We have

$$2^9 - 1 = 511 = 7 \times 73$$
$$2^{10} - 1 = 1023 = 3 \times 341$$
$$2^{12} - 1 = 4095 = 5 \times 819 \ .$$

None are prime.

PRACTICE 2: *Is* $2^{13} - 1$ *prime or composite?*

Answer: It turns out that $2^{13} - 1 = 8191$ is prime! (Is $2^{17} - 1$ prime? Is $2^{19} - 1$?)