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QUADRATICS
4.1 An Opening Puzzler to Set the Scene

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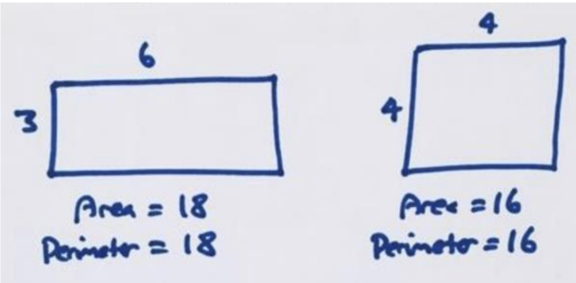
SETTING THE SCENE

In this lesson we finish up our work on the algebra of quadratics. (We'll move to their graphing next.) The topics covered here are more advanced and might, or might not, be part of your curriculum. We'll study them here just in case they are and, well, because they are interesting! There is some good clever thinking to try, and thinking is always fun. (But do feel free to skip content herein if doesn't feel quite relevant to your interests and needs.)

We'll start the lesson with a puzzle whose solution illustrates the algebra we'll need today.

A RECTANGLE PUZZLE

Consider these two rectangles: a three-by-six rectangle and a four-by-four rectangle (that is, square). They each have a mighty curious property.



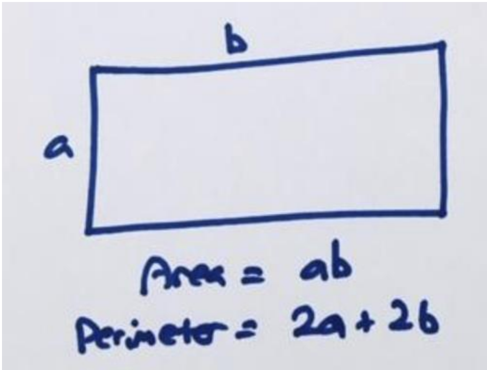
The first has area $3 \times 6 = 18$ square units and the perimeter $3+6+3+6=18$ units. The second has area $4 \times 4 = 16$ square units and the perimeter $4+4+4+4=16$ units. That is, each figure has area and perimeter the same numerical value!

Your puzzle: *Find, if you can, another example of a rectangle with positive integer side lengths whose area and perimeter have the same numerical value. In fact, find ALL examples of such rectangles!*

SOLVING THE PUZZLE

Don't read on if you want to try the puzzle first. I am about to give it away!

Consider an a-by-b rectangle where a and b are positive integers.



To find rectangles with the desired property we need to find positive integers that satisfy the equation

$$ab = 2a + 2b.$$

Let's solve for one variable, say b, to see if we can learn anything about what values it could take.

Subtract $2b$ from both sides of the equation

$$ab - 2b = 2a$$

and see a common factor of b on the left.

$$(a-2)b = 2a$$

This then gives

$$b = \frac{2a}{a-2} \ .$$

And this looks problematic! We want b to be an integer, but it seems that b is going to be a fraction. This is worrisome.

For this expression not to be a fraction we need the numerator to be a multiple of the denominator.

PRINCIPLE: If there is something you want in math (or in life!), MAKE IT HAPPEN! (And deal with the consequences!)

We have a numerator of $2a$ and we want this to be a multiple of the denominator, which is $a-2$.

Well, "$2a$" is certainly a multiple of a. Let's just make it a multiple of $a-2$!

We have $b = \dfrac{2(a)}{a-2}$

Let's make this $\dfrac{2(a-2)}{a-2}$ *instead!*

But we can't just change the numerator. There will be consequences!

We've introduced a $2 \times (-2) = -4$ into the numerator and we need to compensate for this by also introducing a $+4$. So let's write

$$b = \frac{2(a-2)+4}{a-2}.$$

This is still a valid formula for b.

We almost have what we wished for. We see that

$$b = \frac{2(a-2)}{a-2} + \frac{4}{a-2},$$

that is, that

$$b = 2 + \frac{4}{a-2}.$$

We have the integer 2, but there is a fractional term that just won't go away.

But we see that $\dfrac{4}{a-2}$ will also be an integer if $a-2$ is a factor of four, that is, if $a-2$ equals 1, 2, or 4. So, let's consider these three cases.

- If $a-2=1$, then we have $a=3$ and $b = 2 + \frac{4}{1} = 6$, and this is the 3-by-6 rectangle we already know.

- If $a-2=2$, then we have $a=4$ and $b = 2 + \frac{4}{2} = 4$, and this is the 4-by-4 square we already know.

We have one more factor to check and this might lead to a new example!

- If $a-2=4$, then we have $a=6$ and $b = 2 + \frac{4}{4} = 3$ and we have a 6-by-3 rectangle. But this is an example we already know: it is the first rectangle just rotated 90 degrees.

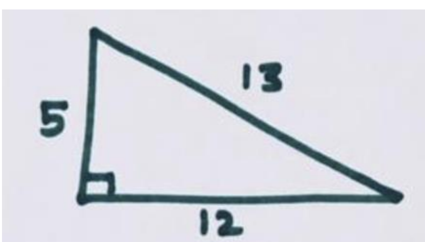
The two examples we already know are the only two positive integer side rectangles with the property that their areas and perimeters have the same numerical values.

PRACTICE 1: *The integer* 4 *actually has three more factors, namely,* -1, -2, *and* -4. *Does setting* $a-2$ *to equal each of these factors give us any valid examples?*

I find this rectangle puzzle just charming. And I love the algebraic technique of simply making the numerator of an algebraic fraction be as close as possible to a multiple of the denominator. And as we shall see precisely this technique will lead us to some profound results later on.

PRACTICE 2: *Find all positive integer solutions to* $ab = a + b + 9$.

PRACTICE 3: *The integer right triangle with sides of length* 5, 12, *and* 13 *also has the property that its area and perimeter have the same numerical value.*



Find another integer right triangle with this property.

SOLUTIONS

PRACTICE 1: *The integer 4 actually has three more factors, namely, -1, -2, and -4. Does setting $a-2$ to equal each of these factors give us any valid examples?*

Answer: Remember we have $b = 2 + \frac{4}{a-2}$.

For $a-2=-1$, we get $a=1$ and $b=-2$, which does not lead to a valid geometric rectangle.

For $a-2=-2$, we get $a=0$ and $b=0$, the degenerate rectangle of zero area and zero perimeter.

For $a-2=-4$, we get $a=-2$ and $b=1$, which does not lead to a valid geometric rectangle.

PRACTICE 2: *Find all positive integer solutions to $ab = a+b+9$.*

Answer: We have $(a-1)b = a+9$ and so

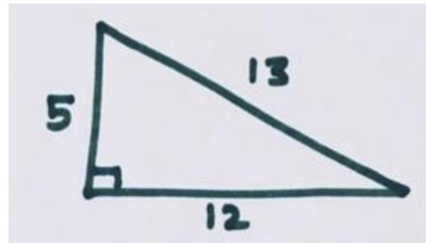
$$b = \frac{a+9}{a-1} = \frac{(a-1)+10}{a-1} = 1 + \frac{10}{a-1}$$

and so, for integer solutions, we need $a-1=1$, 2, 5, or 10. This gives the solutions

$$\begin{aligned} &a=2,\ b=11,\\ &a=3,\ b=6,\\ &a=6,\ b=3,\\ &a=11,\ b=2. \end{aligned}$$

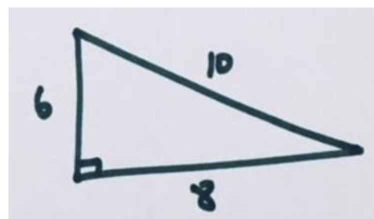
(And one checks that the negative factors of 10 yield no positive solutions.)

PRACTICE 3: *The integer right triangle with sides of length 5, 12, and 13 also has the property that its area and perimeter have the same numerical value.*



Find another integer right triangle with this property.

Answer: The 6-8-10 integer right triangle has this property too.



EXTRA CHALLENGE: *Prove that the 5-12-13 and 6-8-10 right triangles are the only integer right triangles with area and perimeter the same numerical value.*