QUADRATICS
4.2 Practicing Sneaky Algebra: The Factor Theorem

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SETTING THE SCENE

We saw last lesson that

$$x^2 + 5x + 6 = (x+2)(x+3),$$

showing that $x^2 + 5x + 6$ is a straightforward multiple of $x+2$: it is $x+3$ copies of $x+2$.

This means that if we divide $x^2 + 5x + 6$ by $x+2$ we should get a nice answer, namely, $x+3$.

The goal of this essay is to play with $\frac{x^2+5x+6}{x+2}$, and other algebraic expressions, in the way of the previous essay: work to make each term in the numerator a multiple of the denominator and adjust for errors we introduce as we go along.

This algebra will lead to a deep mathematical result.

PRACTICING SNEAKY ALGEBRA

PROBLEM: *Compute* $\frac{x^2+5x+6}{x+2}$.

Consider first the first term x^2 in the numerator. This is certainly a multiple of x

$$\frac{x(x) \quad +5x+6}{x+2}$$

but we want to see it as a multiple of $x+2$. So, let's make that happen! But we have to introduce $-2x$ to compensate.

$$\frac{x(x+2)-2x \quad +5x+6}{x+2}$$

Now we are playing with

$$\frac{x(x+2)+3x+6}{x+2}.$$

Focus on the second term, the $3x$. This is certainly a multiple of x

$$\frac{x(x+2)+3(x) \quad +6}{x+2}$$

but let's make it a multiple of $x+2$, introducing -6 to compensate for our action.

$$\frac{x(x+2)+3(x+2)-6 \quad +6}{x+2}$$

But this reads

$$\frac{x(x+2)+3(x+2)}{x+2},$$

which we can see as

$$\frac{x(x+2)}{x+2}+\frac{3(x+2)}{x+2},$$

which is $x+3$. That is, we have

$$\frac{x^2+5x+6}{x+2}=x+3$$

just as we expected!

Let's keep playing this way.

PROBLEM: *Is* $2x^2+7x+3$ *a multiple of* $x-1$*? If not, how close is it to being one?*

This seems like an unusual question. But let's just march through each term of $2x^2+7x+3$, making it a multiple of $x-1$ and adjusting for any errors we introduce along the way.

We certainly have $2x^2$ as a multiple of x,

$$2x(x) \quad +7x+3$$

but let's make it a multiple of $x-1$.

$$2x(x-1)+2x \quad +7x+3.$$

This is $2x(x-1)+9x+3$.

Now let's rewrite $9x$ as a multiple of $x-1$. We get

$$\begin{aligned}&2x(x-1)+9(x)+3\\&\quad=2x(x-1)+9(x-1)+9 \quad +3\end{aligned}$$

and this equals

$$2x(x-1)+9(x-1)+12.$$

And it seems that this is as far as we can go. We're stuck with a final number 12.

So $2x^2+7x+3$ is close to being multiple of $x-1$, it is simply "off" by 12.

$$2x^2+7x+3=(x-1)(\mathit{something})+12$$

(And if you want the detail, the "something" is $2x+9$.)

There is no need to stick with quadratic expressions in this game.

PROBLEM: *How close is* x^3-2x+1 *to being a multiple of* $x-5$ *?*

Answer: Let's go through the first few terms of x^3-2x-1, each currently a multiple of x, and adjust them to each be a multiple of $x-5$. Here goes!

$$\begin{aligned}&x^2(x) \quad -2x+1\\&\quad=x^2(x-5)+5x^2 \quad -2x+1\end{aligned}$$

Now

$$\begin{aligned}&x^2(x-5)+5x(x) \quad -2x+1\\&\quad=x^2(x-5)+5x(x-5)+25x \quad -2x+1\\&\quad=x^2(x-5)+5x(x-5)+23x+1\end{aligned}$$

Now

$$x^2(x-5)+5x(x-5)+23(x) \quad +1$$
$$=x^2(x-5)+5x(x-5)+23(x-5)+115 \ +1$$
$$=x^2(x-5)+5x(x-5)+3(x-5)+116$$

We have then that

$$x^3-2x+1=(x-5)(something)+116.$$

It's a multiple of $x-5$ with an "error term" of 116. But what is that 116 really? It must mean something!

Stare at

$$x^3-2x+1=(x-5)(something)+116.$$

It seems irresistible to put $x=5$ into it!

On the right side we get

$$0\cdot(something)+116,$$

which is just $0+116=116$.

On the left side we get $5^3-2\cdot 5+1$, which is the value of the expression at $x=5$ (and this does equal 116).

We have that the error term upon dividing x^3-2x+1 by $x-5$ is the value of the expression at $x=5$.

The analogous statement is true for our previous example. We had

$$2x^2+7x+3=(x-1)(something)+12.$$

Here is seems irresistible to put in $x-1$. The right-hand side then becomes $0+12=12$, our error term, and the left-hand side becomes $2\cdot 1^2+7\cdot 1+3$, the value of the expression at $x=1$.

In general, we have:

We can always divide an expression of the form ax^2+bx+c (or any expression involving higher powers of x) by a term of the form $x-h$ and obtain

$$ax^2+bx+c=(x-h)(something)+E$$

where E is a number, the "error" term. Putting in $x=h$ shows that E is the value of the expression at $x=h$.

PRACTICE 1:

a) *Show that* x^2+x+1 *has value* 111 *at* $x=10$. *Now show that*

$$x^2+x+1=(x-10)(something)+111.$$

b) *Show that* $2x^3-3x^2-5x+1$ *has value* 1 *at* $x=-1$. *Now show that*

$$2x^3-3x^2-5x+1=(x+1)(something)+1.$$

c) *Show that* $2x^2-5x-3$ *has value zero at* $x=3$. *Explain how you now know that* $2x^2-5x-3$ *must be a perfect multiple of* $x-3$ *with no error term.*

THE FACTOR THEOREM

Part c) of the practice problem suggests something profound:

If putting in $x=h$ into an expression

$$ax^2+bx+c$$

gives zero, then ax^2+bx+c is a perfect multiple of $x-h$ with zero error term. We have

$$ax^2+bx+c=(x-h)(something).$$

For example, if $x = 7$ makes an expression equal to zero, then we know $x - 7$ is a factor of that expression with no error.

If $x = -3$ makes an expression equal to zero, then we know $x - (-3)$, that is, $x + 3$, is a factor of that expression.

If $x = 0$ makes an expression equal to zero, then we know $x - 0$, that is, x is a factor of the expression. (Is this example already obvious?)

This result is known as the **Factor Theorem**. Most people call a value $x = h$ that makes an expression $ax^2 + bx + c$ equal to zero, a **zero** of the expression. They will cite the Factor Theorem swiftly by saying:

If h is a zero of an expression, then $x - h$ is a factor of the expression.

PRACTICE 2:

a) *Show that* $x^6 - 1$ *must be a multiple of* $x - 1$ *and also a multiple of* $x + 1$.

b) *Compute* $\frac{x^6 - 1}{x - 1}$.

c) *Compute* $\frac{x^6 - 1}{x + 1}$.

OPTIONAL PRACTICE 3 (Hard): *Show that* $x^3 - 7x + 6$ *equals zero for* $x = 1$, $x = 2$, *and for* $x = -3$. *Now completely factorise the expression, explaining your rationale in careful detail.*

Three Incidental Comments:

1. In this set of lessons we are primarily dealing with quadratic expressions $ax^2 + bx + c$, sums of the first three non-negative whole-number powers of x with numbers as coefficients. But the Factor Theorem applies to any expression that is a sum of any collection of non-negative whole-number powers of x with number coefficients. (Such expressions are called *polynomials*.)

2. Our "sneaky algebra" trick for dividing an expression by a term $x - h$ is the basis of a technique some curricula call *synthetic division*.

3. For a purely visual and extra fun way to divide polynomials, see *Exploding Dots* at www.gdaymath.com/courses.

SOLUTIONS

PRACTICE 1:

a) *Show that* $x^2 + x + 1$ *has value* 111 *at* $x = 10$. *Now show that*

$$x^2 + x + 1 = (x-10)(\textit{something}) + 111.$$

b) *Show that* $2x^3 - 3x^2 - 5x + 1$ *has value* 1 *at* $x = -1$. *Now show that*

$$2x^3 - 3x^2 - 5x + 1 = (x+1)(\textit{something}) + 1.$$

c) *Show that* $2x^2 - 5x - 3$ *has value zero at* $x = 3$. *Explain how you now know that* $2x^2 - 5x - 3$ *must be a perfect multiple of* $x - 3$ *with no error term.*

Answer:

a) We have $10^2 + 10 + 1 = 111$.

$$\begin{aligned} x^2 + x + 1 &= x(x-10) + 10x + x + 1 \\ &= x(x-10) + 11x + 1 \\ &= x(x-10) + 11(x-10) + 110 + 1 \\ &= x(x-10) + 11(x-10) + 111 \\ &= (x-10)(\textit{something}) + 111 \end{aligned}$$

b) We have

$$\begin{aligned} 2(-1)^3 - 3(-1)^2 - 5(-1) + 1 & \\ &= -2 - 3 + 5 + 1 = 1 \end{aligned}$$

Notice: $x - (-1) = x + 1$.

$$\begin{aligned} 2x^3 - 3x^2 - 5x + 1 &= 2x^2(x+1) - 5x^2 - 5x + 1 \\ &= 2x^2(x+1) - 5x(x+1) + 1 \\ &= (x+1)(\textit{something}) + 1 \end{aligned}$$

c) We have $2(3)^2 - 5(3) - 3 = 0$. So the value of the error term when we try to write $2x^2 - 5x - 3$ as a multiple of $x - 3$ is 0.

That is,

$$2x^2 - 5x - 3 = (x-3)(\textit{something}) + 0.$$

That is, $2x^2 - 5x - 3$ as a perfect multiple of $x - 3$

PRACTICE 2:

a) *Show that* $x^6 - 1$ *must be a multiple of* $x - 1$ *and also a multiple of* $x + 1$.

b) *Compute* $\dfrac{x^6 - 1}{x - 1}$.

c) *Compute* $\dfrac{x^6 - 1}{x + 1}$.

Brief Answer:

a) Putting in $x = 1$ gives $(1)^6 - 1 = 0$. So $x - 1$ must be a factor of $x^6 - 1$.

Putting in $x = -1$ gives $(-1)^6 - 1 = 0$. So $x - (-1)$, that is, $x + 1$ must be a factor of $x^6 - 1$.

b)

$$\begin{aligned} \frac{x^6 - 1}{x - 1} &= \frac{x^5(x-1) + x^4(x-1) + x^3(x-1) + x^2(x-1) + x(x-1) + (x-1)}{x - 1} \\ &= x^5 + x^4 + x^3 + x^2 + x + 1 \end{aligned}$$

c) $\dfrac{x^6 - 1}{x + 1} = x^5 - x^4 + x^3 - x^2 + x - 1.$

OPTIONAL PRACTICE 3: *Show that* $x^3 - 7x + 6$ *equals zero for* $x = 1$, $x = 2$*, and for* $x = -3$*. Now completely factorise the expression, explaining your rationale in careful detail.*

Answer:
We have

$$(1)^3 - 7(1) + 6 = 1 - 7 + 6 = 0$$
$$(2)^3 - 7(2) + 6 = 8 - 14 + 6 = 0$$
$$(-3)^3 - 7(-3) + 6 = -27 + 21 + 6 = 0$$

Since $x = 1$ is a zero, we have

$$x^3 - 7x + 6 = (x-1)(\mathit{something}).$$

But $x = 2$ is also a zero and so $x - 2$ must be a factor of the expression as well. That factor must be part of the "something."

$$x^3 - 7x + 6 = (x-1)(x-2)(\mathit{something\ else})$$

But $x = -3$ is also a zero and so $x + 3$ must be a factor of the expression as well. That factor must be part of the "something else."

$$x^3 - 7x + 6 = (x-1)(x-2)(x+3)(\mathit{still\ \ something\ else})$$

Now the "still something else" cannot have any x terms in it – if it did and we expanded everything out on the right, we'd have an expression involving x^4 or a higher power. But there are no such terms in $x^3 - 7x + 6$. So "still something else" must simply be a number. Call it k.

$$x^3 - 7x + 6 = k(x-1)(x-2)(x+3)$$

Now, again, if we expand the right side, we'd have a term kx^3. But the left side has just x^3. It must be that $k = 1$.

So $x^3 - 7x + 6 = (x-1)(x-2)(x+3)$ and we have completely factored the expression.