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QUADRATICS
4.3 Practicing the Factor Theorem; The Difference of Two squares; The Difference (and Sum) of Two Cubes.

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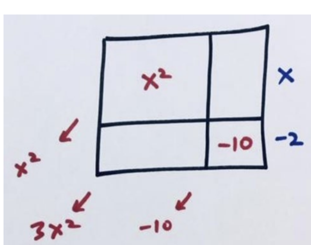
SETTING THE SCENE

Recall the practice of the Factor Theorem.

PROBLEM: *Consider the quadratic expression* $x^2 + 3x - 10$.
a) *Show that putting in* $x = 2$ *makes this expression zero.*
b) *Show that* $x - 2$ *is a factor of* $x^2 + 3x - 10$.

Answer: a) We have $(2)^2 + 3(2) - 10 = 0$, as hoped.
b) By the Factor Theorem, it must be the case then that $x - 2$ is a factor of the quadratic expression.

If we wish to actually find the other factor, we can draw an unsymmetrical rectangle with the information we have and deduce its other side length to be $x + 5$.



Or we can perform our sneaky algebra.

$$\begin{aligned} x^2 + 3x - 10 &= x(x-2) + 2x + 3x - 10 \\ &= x(x-2) + 5x - 10 \\ &= x(x-2) + 5(x-2) + 10 - 10 \\ &= x(x-2) + 5(x-2) \end{aligned}$$

We see $x^2 + 3x - 10 = (x+5)(x-2)$.

Let's now apply the Factor Theorem to deduce classic mathematics formulas.

THE DIFFERENCE OF TWO SQUARES

We've seen the Difference of Two Squares Formula in a previous lesson. But we can derive the formula again with the thinking of the Factor Theorem.

PRACTICE 1:
a) *Consider the expression* $x^2 - 81$. *Explain why* $x - 9$ *must be a factor of this expression. What must its other factor be?*
b) *Consider the expression* $x^2 - a^2$ *for some number* a. *Explain why* $x - a$ *must be a factor of this expression. Deduce that* $x^2 - a^2 = (x-a)(x+a)$.

THE DIFFERENCE (and sum) OF TWO CUBES

PROBLEM: *Show that* $x^3 - 27$ *is a multiple of* $x-3$.

The answer to this is swift: All we have to notice is that $(3)^3 - 27$ equals zero. That is, that $x = 3$ is a zero of the expression. By the Factor Theorem, $x-3$ is indeed a factor of the expression.

So we have

$$x^3 - 27 = (x-3)(something).$$

PROBLEM: *Show that* $2^{300} - 27$ *is not prime.*

This is question we left unanswered from a previous lesson. We can answer it now!

Answer: The number 2^{300} is a cube number: it is $\left(2^{100}\right)^3$. So by the what we've just established, we have

$$2^{300} - 27 = \left(2^{100} - 3\right)(something).$$

This number factors and so is <u>not</u> prime!

In general, for the difference of two cubes

$$x^3 - a^3$$

with a some number, we see that putting $x = a$ into the expression gives zero: $a^3 - a^3 = 0$. So this means

$$x^3 - a^3 = (x-a)(something).$$

PRACTICE 2: *Explain why we can be sure that the expression for a sum of two cubes* $x^3 + a^3$ *has* $x+a$ *as a factor giving*

$$x^3 + a^3 = (x+a)(something).$$

Some curricula want students to memorise what the other factors are in each of these equations. We can work them out.

Let's write $x^3 - a^3$ as a multiple of $x-a$.

We have

$$\begin{aligned} x^3 - a^3 &= x^2(x) - a^3 \\ &= x^2(x-a) + ax^2 - a^3 \\ &= x^2(x-a) + ax(x) - a^3 \\ &= x^2(x-a) + ax(x-a) + a^2x - a^3 \\ &= x^2(x-a) + ax(x-a) + a^2(x-a) + a^3 - a^3 \\ &= x^2(x-a) + ax(x-a) + a^2(x-a) \end{aligned}$$

That is,

$$x^3 - a^3 = (x-a)\left(x^2 + ax + a^2\right).$$

PRACTICE 3: *What is the unidentified factor in* $x^3 + a^3 = (x+a)(something)$*?*

PRACTICE 4:

a) *Write* $x^4 - 16$ *as the product of three factors.*

b) *Write* $x^6 - 64$ *as a product of four factors.*

PRACTICE 5: *Show that* $2^{90} + 1$ *is divisible by* 1025.

SOLUTIONS

PRACTICE 1:

a) *Consider the expression* $x^2 - 81$. *Explain why* $x-9$ *must be a factor of this expression. What must its other factor be?*

b) *Consider the expression* $x^2 - a^2$ *for some number* a. *Explain why* $x-a$ *must be a factor of this expression. Deduce that* $x^2 - a^2 = (x-a)(x+a)$.

Answer:

a) Putting $x = 9$ into the expression gives $9^2 - 81 = 0$. So by the Factor Theorem, $x-9$ is a factor of $x^2 - 81$.

We have

$$
\begin{aligned}
x^2 - 81 &= x(x) - 81 \\
&= x(x-9) + 9x - 81 \\
&= x(x-9) + 9(x-9) + 81 - 81 \\
&= x(x-9) + 9(x-9)
\end{aligned}
$$

showing that $x^2 - 81 = (x-9)(x+9)$.

b) This is identical work. Putting $x = a$ into the expression gives zero and so $x-a$ must be a factor of the expression.

And

$$
\begin{aligned}
x^2 - a^2 &= x(x-a) + a(x-a) + a^2 - a^2 \\
&= x(x-a) + a(x-a) \\
&= (x-a)(x+a).
\end{aligned}
$$

PRACTICE 2: *Explain why we can be sure that the expression for a sum of two cubes* $x^3 + a^3$ *has* $x+a$ *as a factor giving*

$$x^3 + a^3 = (x+a)(\mathit{something}).$$

Answer: Putting $x = -a$ into $x^3 + a^3$ gives $(-a)^3 + a^3 = 0$. So by the Factor Theorem, $x-(-a)$, that is, $x+a$, is a factor of $x^3 + a^3$.

PRACTICE 3: *What is the unidentified factor in* $x^3 + a^3 = (x+a)(\mathit{something})$*?*

Answer: We have

$$
\begin{aligned}
x^3 + a^3 &= x^2(x) + a^3 \\
&= x^2(x+a) - ax^2 + a^3 \\
&= x^2(x+a) - ax(x+a) + a^2x + a^3 \\
&= x^2(x+a) - ax(x+a) + a^2(x+a) - a^3 + a^3 \\
&= (x+a)(x^2 - ax + a^2).
\end{aligned}
$$

PRACTICE 4:

a) *Write* $x^4 - 16$ *as the product of three factors.*

b) *Write* $x^6 - 64$ *as a product of four factors.*

Answer:

a)

$$
\begin{aligned}
x^4 - 16 = (x^2)^2 - 4^2 &= (x^2-4)(x^2+4) \\
&= (x-2)(x+2)(x^2+4)
\end{aligned}
$$

b)

$$
\begin{aligned}
x^6 - 64 &= (x^3)^2 - 8^2 \\
&= (x^3 - 8)(x^3 + 8) \\
&= (x-2)(x^2+2x+4)(x+2)(x^2-2x+4)
\end{aligned}
$$

PRACTICE 5: *Show that* $2^{90}+1$ *is divisible by* 1025.

Answer: Let's look at $x^{90}+1$.

This is

$$\begin{aligned}\left(x^{30}\right)^3+1^3 &= \left(x^{30}+1\right)\left(x^{60}-x^{30}+1\right)\\ &= \left(\left(x^{10}\right)^3+1^3\right)\left(x^{60}-x^{30}+1\right)\\ &= \left(x^{10}+1\right)\left(x^{20}-x^{10}+1\right)\left(x^{60}-x^{30}+1\right)\end{aligned}$$

For $x=2$, this reads

$$2^{90}+1=\left(2^{10}+1\right)\left(something\right)$$

and $2^{10}+1=1024+1=1025$.