QUADRATICS
4.4 Roots

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SETTING THE SCENE

Earlier, we used the technique of factoring to find the zeros of a quadratic expression. We can also do the reverse: knowing the zeros of a quadratic can lead to its factorisation.

PROBLEM: *Consider the quadratic expression* $3x^2+x-2$. *It has value zero for* $x=-1$ *and for* $x=\frac{2}{3}$. *Use these observations to completely factorise* $3x^2+x-2$.

Answer: One can indeed check that $3(-1)^2+(-1)-2$ and $3\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)-2$ are each zero.

That $x=-1$ is a zero of the expression means that $x+1$ must be a factor of the expression. Thus

$$3x^2+x-2=(x+1)(\mathit{something}).$$

That $x=\frac{2}{3}$ is a zero of the expression means that $x-\frac{2}{3}$ is also a factor of the expression. This additional factor must be sitting in the "something" part of what we have so far.

$$3x^2+x-2=(x+1)\left(x-\frac{2}{3}\right)(\mathit{something\ else})$$

Now, what could the "something else" be?

If we expand what we have so far

$$\begin{aligned}3x^2+x-2&=(x+1)\left(x-\frac{2}{3}\right)(\mathit{something\ else})\\&=\left(x^2+\frac{1}{3}x-\frac{2}{3}\right)(\mathit{something\ else})\end{aligned}$$

we see that our "something else" cannot contain any terms with the variable x, because in expanding the right-hand side all the way we'd find $x^2\cdot x=x^3$, or a higher power of x, appearing on the right. There are no such high powers of x on the left. This means our "something else" can only be a number.

We must have

$$\begin{aligned}3x^2+x-2&=(x+1)\left(x-\frac{2}{3}\right)(k)\\&=k(x+1)\left(x-\frac{2}{3}\right)\end{aligned}$$

where k is a number.

Next question: What must that number be?

If we expand the right side again, we'd see we get the term kx^2. The left side has $3x^2$ and so k must be 3.

We have

$$3x^2+x-2=3(x+1)\left(x-\frac{2}{3}\right).$$

We have factorized the quadratic!

Comment: Some people might prefer to avoid the mention of fractions and rewrite this factorization as

$$\begin{aligned} 3x^2 + x - 2 &= (x+1) \times 3\left(x - \frac{2}{3}\right) \\ &= (x+1)(3x-2). \end{aligned}$$

This just a matter of choice and style and not a matter of mathematics.

THE FACTOR THEOREM - PUSHED

In general, we see that if we can find two different zeros $x = p$ and $x = q$ of a quadratic expression $ax^2 + bx + c$, then we have two factors of the expression.

$$ax^2 + bx + c = (x-p)(x-q)(something)$$

The "something" that remains in the factorization can only be a number, and that number must be a, the coefficient of the x^2 term in the expression.

In summary:

> If $ax^2 + bx + c$ is zero for $x = p$ and $x = q$, then
> $$ax^2 + bx + c = a(x-p)(x-q).$$

PRACTICE 1: *Solve $2x^2 - 5x + 3 = 0$ via the square method or via the quadratic formula. Use your result to then factorise $2x^2 - 5x + 3$.*

New Terminology

A value that makes an expression equal to zero is called a "zero of the expression." For example -1 and $\frac{2}{3}$ are each zeros of $3x^2 + x - 2$.

But people also use the word *root* for a zero of an expression: -1 and $\frac{2}{3}$ are each roots of $3x^2 + x - 2$.

The root is connected to the term "square root." After all, $\sqrt{2}$ is a root of the expression $x^2 - 2$.

PRACTICE 2:
What are the two roots of $x^2 - a^2$?
Derive, yet again, the difference of two squares formula.

PRACTICE 3: *If $x = \frac{1}{2}$ and $x = -1$ are the roots of $ax^2 + 3x + b$, what are a and b?*

PRACTICE 4: *One of the roots of $ax^2 + bx + c$ is double the other. Show that*

$$b^2 : ac = 9 : 2.$$

PRACTICE 5: *If 3 is a root of $2x^2 + rx - 3$ and $rx^2 + x + k$ has two identical roots, what is k?*

PRACTICE 6:
a) *Show that $4x^2 - 4x + 1$ has only one root.*
b) *Does $4x^2 - 4x + 1 = 4(x-p)(x-q)$ for some values p and q?*
[Can you see what people say that $4x^2 - 4x + 1$ has a "repeated root"?]

PRACTICE 7: *Explain the following result:*

> If $ax^2 + bx + c$ is zero only for $x = p$, then
> $$ax^2 + bx + c = a(x-p)^2.$$

RATIONALISING DENOMINATORS AND NUMERATORS

Look at the difference of two squares formula.

$$x^2 - a^2 = (x-a)(x+a)$$

We can apply this to non-square numbers too. For instance, we see

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

$$x^2 - 17 = \left(x - \sqrt{17}\right)\left(x + \sqrt{17}\right)$$

and so on.

Such expressions were handy for scientists and engineers in the days before the invention of calculators. In doing a mathematical calculation, it is possible to obtain a fractional answer with a square root in an awkward location. For example, one might obtain the answer

$$\frac{1}{\sqrt{2}}.$$

There is nothing mathematically wrong with this answer and mathematicians will leave this expression as it is. But if you are an engineer and need to know the decimal approximation of this number you would want to go further.

Today, we'd just use our calculators and see that $\frac{1}{\sqrt{2}} \approx 0.707$. But in the early days, one would have to refer to booklets of mathematical values, read that $\sqrt{2} \approx 1.414$, and the attempt the long division calculation

$$1.0000000 \div 1.414$$

by hand. Not fun!

But scholars realized that if you multiply the numerator and denominator of the expression each by $\sqrt{2}$, effectively removing the awkward square root term from the denominator to get

$$\frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

then the pencil-and-paper calculation is much easier!

$$\frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = 0.707\ .$$

So comes the piece of advice: **IF** you need to know the decimal approximation of a numerical expression and **IF** you have to compute this by hand, try rewriting the expression with no square roots in the denominator.

Again, the mathematics does not care how you express a fractional quantity. This was just a piece of practical advice for scientists and engineers from the early 1900s.

PROBLEM: *Rationalise the denominator of* $\frac{1}{3-\sqrt{2}}$*. That is, rewrite the number so that no square roots appear in its denominator.*

Answer: Here we can use the difference of two squares

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

with $x = 3$.

Let's multiply the numerator and the denominator of the expression each by $3 + \sqrt{2}$. This won't change the value of the quantity.

We get

$$\frac{1}{3-\sqrt{2}}=\frac{3+\sqrt{2}}{\left(3-\sqrt{2}\right)\left(3+\sqrt{2}\right)}$$
$$=\frac{3+\sqrt{2}}{9-2}$$
$$=\frac{3+\sqrt{2}}{7}.$$

PRACTICE 8: *Rationalise the numerator of* $\frac{1+\sqrt{3}}{5}$.

PRACTICE 9: *Write* $\frac{60-\sqrt{2}}{1+\sqrt{2}}$ *in the form* $a+b\sqrt{2}$ *with* a *and* b *integers.*

SOLUTIONS

PRACTICE 1: *Solve* $2x^2 - 5x + 3 = 0$ *via the square method or via the quadratic formula. Use your result to then factorise* $2x^2 - 5x + 3$.

Answer: We get that the zeros of $2x^2 - 5x + 3$ $x = 1$ and $x = \frac{3}{2}$. Thus

$$2x^2 - 5x + 3 = 2(x-1)\left(x - \frac{3}{2}\right).$$

This equals $(x-1)(2x-3)$, if you prefer.

PRACTICE 2:

What are the two roots of $x^2 - a^2$*?*
Derive, yet again, the difference of two squares formula.

Answer: Both $x = a$ and $x = -a$ make $x^2 - a^2$ equal to zero. So

$$\begin{aligned} x^2 - a^2 &= 1 \cdot (x-a)(x+a) \\ &= (x-a)(x+a). \end{aligned}$$

PRACTICE 3: *If* $x = \frac{1}{2}$ *and* $x = -1$ *are the roots of* $ax^2 + 3x + b$*, what are* a *and* b*?*

Answer: We must have

$$ax^2 + 3x + b = a\left(x - \frac{1}{2}\right)(x+1).$$

Expanding the right side gives

$$ax^2 + \frac{a}{2}x - \frac{a}{2}.$$

Comparing this with the left side shows that $\frac{a}{2} = 3$ and $-\frac{a}{2} = b$, giving $a = 6$ and $b = -3$.

PRACTICE 4: *One of the roots of* $ax^2 + bx + c$ *is double the other. Show that*

$$b^2 : ac = 9 : 2.$$

Answer: Call one of the roots p. Then the other is $2p$ and we have

$$ax^2 + bx + c = a(x-p)(x-2p).$$

Expanding the right side gives

$$ax^2 + bx + c = ax^2 - 3apx + 2ap^2$$

showing that $b = -3ap$ and $c = 2ap^2$.

Thus

$$b^2 = 9a^2p^2$$

$$ac = 2a^2p^2$$

and so $\frac{b^2}{ac} = \frac{9}{2}$.

PRACTICE 5: *If* 3 *is a root of* $2x^2 + rx - 3$ *and* $rx^2 + x + k$ *has two identical roots, what is* k*?*

Answer: We have that

$$2(3)^2 + r(3) - 3 = 0$$

showing that $r = 5$.

That $5x^2 + x + k$ has two identical roots means that the discriminant is zero. This mean

$$1^2 - 4 \cdot 5 \cdot k = 0$$

giving $k = \frac{1}{20}$.

PRACTICE 6:

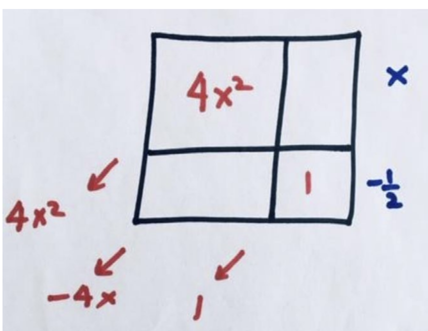
a) *Show that* $4x^2 - 4x + 1$ *has only one root.*

b) *Does* $4x^2 - 4x + 1 = 4(x-p)(x-q)$ *for some values* p *and* q*?*

[Can you see what people say that $4x^2 - 4x + 1$ *has a "repeated root"?]*

Answer: a) Its discriminant is $16 - 4 \cdot 4 \cdot 1 = 0$, and so it has only one root.

b) Solving the quadratic gives the one solution $x = \frac{1}{2}$. So by the (ordinary) Factor Theorem, $x - \frac{1}{2}$ is a factor of $4x^2 - 4x + 1$. A quick sketch reveals that what the other factor must be. It's $4x - 2$.



So we have

$$4x^2 - 4x + 1 = \left(x - \frac{1}{2}\right)(4x - 2).$$

Let's "pull out" a factor of 4 from the second term to rewrite this as

$$4x^2 - 4x + 1 = 4\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right).$$

So the answer to part b is YES! The factor $x - \frac{1}{2}$ is repeated.

PRACTICE 7: *Explain the following result:*

> If $ax^2 + bx + c$ is zero only for $x = p$, then
> $$ax^2 + bx + c = a(x-p)^2.$$

Answer: By the Factor Theorem, $x - p$ is a factor of $ax^2 + bx + c$.

$$ax^2 + bx + c = (x-p)(something)$$

Now the "something" can only be a term involving numbers and x. (There can be no x^2 or higher terms.) So

$$ax^2 + bx + c = (x-p)(mx+n)$$

for some numbers m and n. If we expand the right side we'll see the term mx^2 appear. We conclude that m must be the number a.

$$ax^2 + bx + c = (x-p)(ax+n)$$

Let's "pull out" a factor a.

$$ax^2 + bx + c = a(x-p)\left(x + \frac{n}{a}\right)$$

We see now that $x = -\frac{n}{a}$ makes the expression equal to zero. But we are told that p is the only zero of this quadratic expression. So it must be then that $-\frac{n}{a}$ equals p. This means we have

$$ax^2 + bx + c = a(x-p)(x-p)$$

and the claim is true.

PRACTICE 8: *Rationalise the numerator of* $\frac{1+\sqrt{3}}{5}$.

Answer:

$$\frac{1+\sqrt{3}}{5}=\frac{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}{5\left(1-\sqrt{3}\right)}$$

$$=\frac{1-3}{5\left(1-\sqrt{3}\right)}=\frac{-2}{5\left(1-\sqrt{3}\right)}.$$

PRACTICE 9: *Write* $\frac{60-\sqrt{2}}{1+\sqrt{2}}$ *in the form* $a+b\sqrt{2}$ *with* a *and* b *integers.*

Answer:

$$\frac{60-\sqrt{2}}{1+\sqrt{2}}=\frac{\left(60-\sqrt{2}\right)\left(1-\sqrt{2}\right)}{\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)}$$

$$=\frac{60-\sqrt{2}-60\sqrt{2}+2}{1-2}$$

$$=\frac{62-61\sqrt{2}}{-1}$$

$$=-62+61\sqrt{2}.$$