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QUADRATICS
5.1 Introduction to Graphing, and a Graphing Puzzler

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SETTING THE SCENE

So far we've talked about the algebra of quadratics and saw the power of symmetry throughout that story. Now we'll discuss the graphing of quadratics and see the power of symmetry at play yet again. Symmetry is such a mighty good friend!

Before we move to the opening puzzle, let's have a brief discussion about what mathematics is.

MATHEMATICS AS A LANGUAGE

Many people say that mathematics is a language. And this is true. Since this essay and the video accompanying it are being portrayed in English, the language of mathematics is, right now, English! (And if I were writing in Hindi or in Korean, the language of mathematics would be Hindi or Korean.)

Every mathematical statement is a sentence. For example, the statement

$$5 = 2 + 3$$

has a noun (the quantity "5"), a verb ("equals"), and an object (the quantity "$2+3$"). As such the sentence should come with proper English punctation: It needs a full stop (period) at its end.

$$5 = 2 + 3.$$

The statement $7 > 4 + 9$ is also a sentence.

Our first sentence happened to be a true sentence about numbers and this second one a false sentence about numbers. As mathematics tends to focus on truth, it is interested in sentences that represent true statements about numbers.

But matters are a little curious in an algebra class. We write sentences about numbers all the time, but the numbers are not specified. For example

$$w^2 = p$$

is a sentence. It has noun "an unspecified quantity w squared," verb "equals," and object "an unspecified number p". But without knowing what specific numbers one has in mind for w and p we cannot determine whether this a true or a false sentence. (It is like saying: "Karthik is over 6 feet tall." Without knowing which particular Karthik of the world the speaker is referring to, we do not know whether this sentence is true or false.)

Since mathematics tends to focus on truth it seems natural to collect from an equation, like $w^2 = p$, all the data values that make this a true sentence about numbers. People usually collate such data in a table.

w	p
2	4
1	1
0	0
-3	9
$\frac{1}{2}$	$\frac{1}{4}$
$\sqrt{\pi}$	π

For instance, $w = 2$, $p = 4$ is a data pair that leads to a true number sentence, namely $(2)^2 = 4$, and so could appear in the table. The data pair $w = 5$, $p = 16$ does not yield a true number sentence, $(5)^2 = 16$ is a false number sentence, and so would not appear in the table.

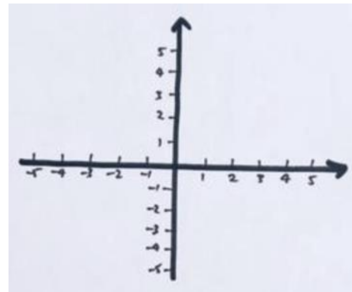
VISUALISING DATA

After four lectures with me you can tell that I am a very visual person. In fact, most people find the visual representation of ideas and data compelling, useful, and insightful. In particular, it is good to give a visual representation to data.

A *graph* is a visual representation of data values.

From our equation $w^2 = p$ it was natural to collect data values that yield true number sentences.

In our early grades we represented numbers visually on a number line. We here have two sets of possible number values and so it seems we'll need two number lines: one for the w values and one for the p values. Mathematics does not care how and where we place these two number lines on the page, but it has become the social convention to place one number line horizontally, with the positive values heading to the right, the other vertically, with positive values heading upwards, having these two lines cross each other at their zeros.



Now comes the question: Which number line should be for the w values and which for the p values? Again, mathematics does not care which number line is labeled which. But society does!

Most people find me writing $w^2 = p$ strange, and would prefer me to write instead

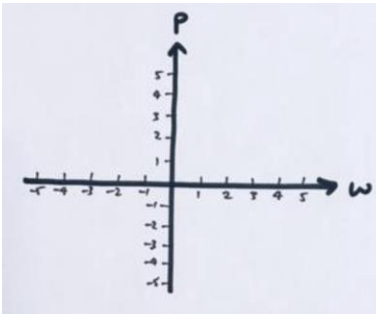
$$p = w^2.$$

Here it looks like we have a formula for the p-value that matches any given w value: when $w = 8$, we have $p = 8^2 = 64$; when $w = -0.2$, we have $p = (-0.2)^2 = 0.04$; and so on. These are data pairs giving true number sentences.

So it feels like the chosen value for w is the "driving force" here, with the matching value of p depending on the w-value chosen. People might say that w is the independent variable here with p the dependent variable.

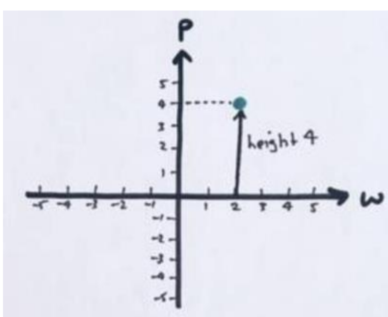
If you happen to be thinking this way, then social convention (society again!) wants to set the horizontal number line for the independent variable and the vertical one for the dependent variable.

Jargon: People tend to call the two number lines in our picture <u>axes</u>. The point where the two axes cross is called the <u>origin</u> of this system of "coordinate axes."

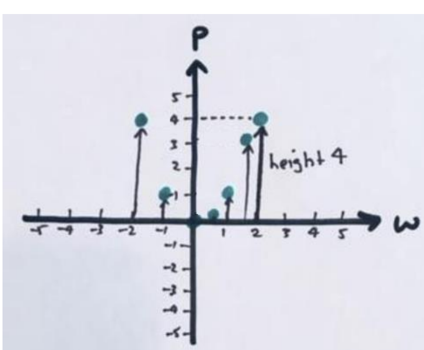


Now: how does one represent data on this set of crossing number lines?

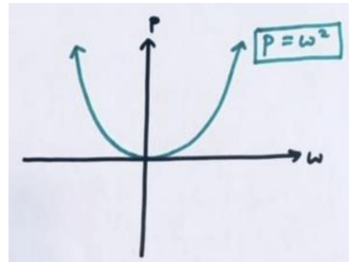
To represent the first data pair in our table, $w = 2$, $p = 4$, find the number 2 on the w axis and move upwards 4 units. Mark that point at height 4. Because the p axis is vertical, we can see that the height of this point lines up with the value $p = 4$ on the vertical axis.



In this way we can depict each data pair in our table with a point in the picture.



And if we could plot every single integer and fractional and irrational pair of data values possible it seems the set of all points depicting our data trace out a single U-shaped curve.



People will label this curve with the equation that generated the data values for that curve.

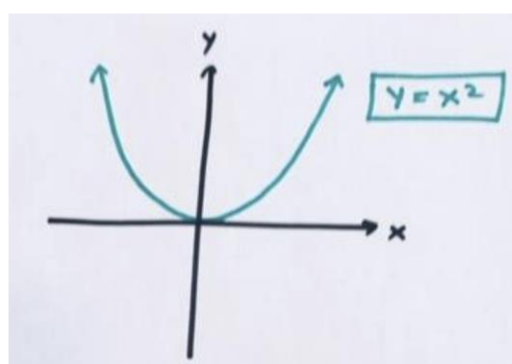
And notice that in this special example we have a lovely symmetrical curve!

A CHANGE OF NOTATION

In upper-school mathematics people tend to use the same letters of the alphabet over and over again for unknown values in algebra class. Instead of calling the independent variable w and the dependent variable p it has become convention to use:

x for the independent variable
y for the dependent variable.

So to follow convention, let's relabel our graph as shown. We have the graph of the equation $y = x^2$.



And this graph really is symmetrical about the vertical axis. For instance,

- for $x = 2$ we have $y = (2)^2 = 4$ and for $x = -2$ we have $y = (-2)^2 = 4$, the same.
- for $x = 10$ we have $y = (10)^2 = 100$ and for $x = -10$ we have $y = (-10)^2 = 100$, the same.

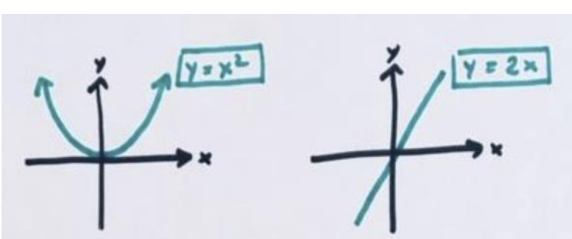
And so on.

As we progress with our story of graphing, we'll make powerful use of this symmetry.

THE OPENING PUZZLER

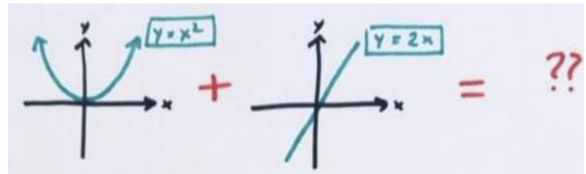
We're now finally ready for the opening puzzle.

I am sure you have graphed equations before. (But hopefully my remarks in this essay have helped deepen your understanding as to what graphing actually means.) And we just graphed the equation $y = x^2$ and saw a lovely symmetrical U-shaped graph. And if you graph the equation $y = 2x$ (make a table of all the data values that give a true number sentence and then plot those data values), you'll see the picture of a line of slope (gradient) 2 through the origin.



Here's a weird question: *What picture would we obtain if we "added" these two graphs?*

What could I mean by that?



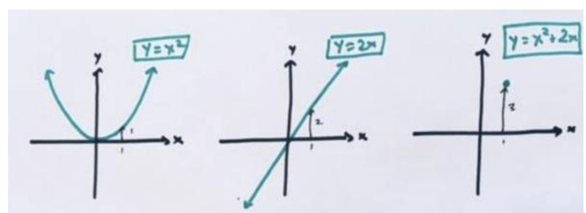
Let's look at each x value and add their matching y values. For example:

For $x = 1$, we'll add $(1)^2$ and $2(1)$ and get 3.

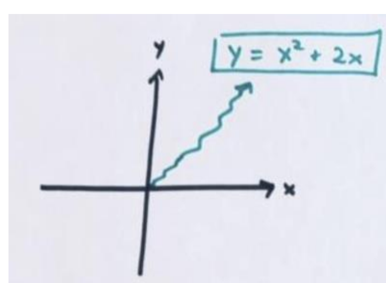
For $x = 2$, we'll add $(2)^2$ and $2(2)$ and get 8.

For $x = 10$ we'll add $(10)^2$ and $2(10)$ and get 120.

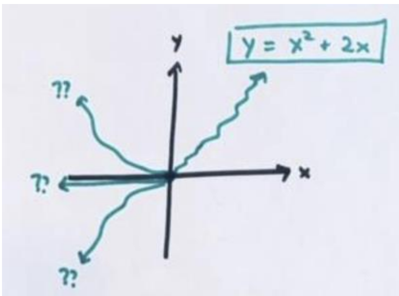
That that is, we are plotting the graph of the equation $y = x^2 + 2x$.



It is clear as that as we put in larger and larger positive x values, we are getting higher and higher points. The graph rises upwards as we move to the right. (Maybe in a straight line? Maybe in some curved way?)



Things are more interesting to the left for negative x values: the graph of $y = x^2$ has data points with positive heights in this region, but the graph of $y = 2x$ has points of negative heights here. We'll be adding together positive and negative heights. Will they cancel out and give zero heights? Will the positives heights "beat" the negative ones and give an overall graph of positive height? Or will the negative heights "win"?



Do plot the graph of $y = x^2 + 2x$. Make a table of data values that give true number sentences and plot those data points. Consider enough of them to get a good sense of the shape of the graph. You are in for a surprise!

Insight: The graph of $y = x^2$ has beautiful vertical symmetry. The graph of $y = 2x$ has no vertical symmetry. In fact, it has "anti-symmetry": the heights to the left of the vertical axis are the exact opposites of the heights on the right. So there is no reason to believe that "adding" these two graphs will give a symmetrical image.

PRACTICE 1:

a) Sketch a graph of the one-variable equation $x^2 = 4$. *(So it's graph will require only one number line, one for* x *values.)*

b) Sketch a graph of the one-variable inequality $x^2 \geq 4$.

PRACTICE 2: *Sketch a graph of the two-variable equation* $x^2 = y^2$.

PRACTICE 3:

a) *Sketch a graph of the one-variable equation* $x = 3$.

b) *Sketch a graph of* $x = 3$ *thinking of it as a two-variable equation. (Imagine it as* $x + 0 \cdot y = 3$ *if you like.)*

c) *Sketch a graph of* $x = 3$ *thinking of it as a three-variable equation. (Imagine it as* $x + 0 \cdot y + 0 \cdot z = 3$ *if you like.) How will you draw your three number lines?*

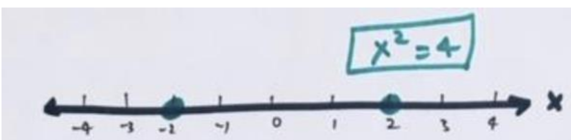
SOLUTIONS

PRACTICE 1:

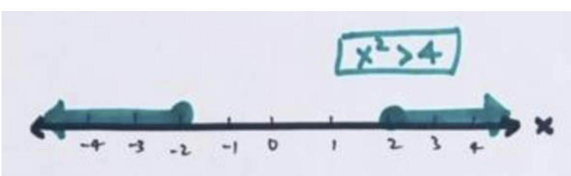
a) Sketch a graph of the one-variable equation $x^2 = 4$*. (So it's graph will require only one number line, one for* x *values.)*

b) Sketch a graph of the one-variable inequality $x^2 \geq 4$*.*

Answer: a) Only $x = 2$ and $x = -2$ make $x^2 = 4$ a true number sentence. A natural way to represent this visually might be as follows.

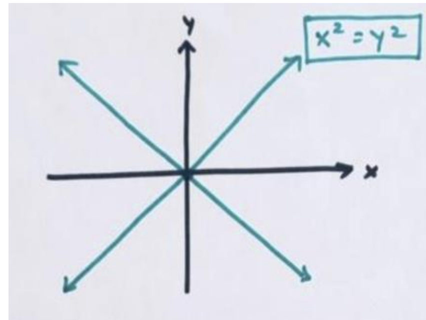


b) Any value greater than or equal to 2, or less than or equal to -2, makes $x^2 \geq 4$ a true number sentence. One might represent this visually as shown.



PRACTICE 2: *Sketch a graph of the two-variable equation* $x^2 = y^2$*.*

Answer: One gets and X-shaped graph.



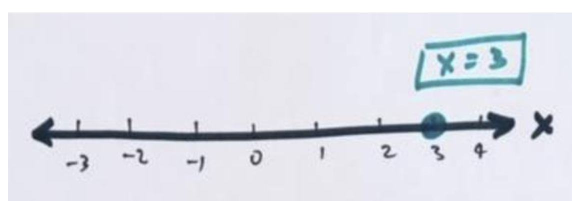
PRACTICE 3:

a) *Sketch a graph of the one-variable equation* $x = 3$*.*

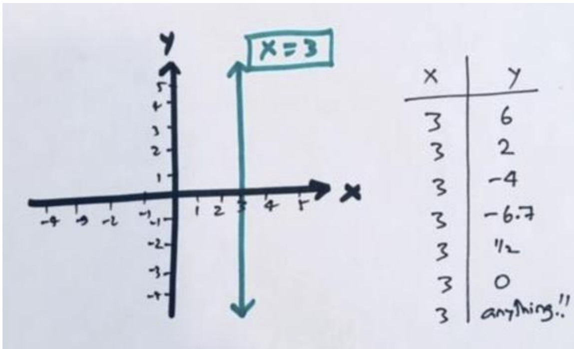
b) *Sketch a graph of* $x = 3$ *thinking of it as a two-variable equation. (Imagine it as* $x + 0 \cdot y = 3$ *if you like.)*

c) *Sketch a graph of* $x = 3$ *thinking of it as a three-variable equation. (Imagine it as* $x + 0 \cdot y + 0 \cdot z = 3$ *if you like.) How will you draw your three number lines?*

Answer:
a) There is only one value for x that makes "$x = 3$" a true sentence, namely, x being 3.



b)



c) Draw three mutually perpendicular number lines. The graph is an entire plane of points.