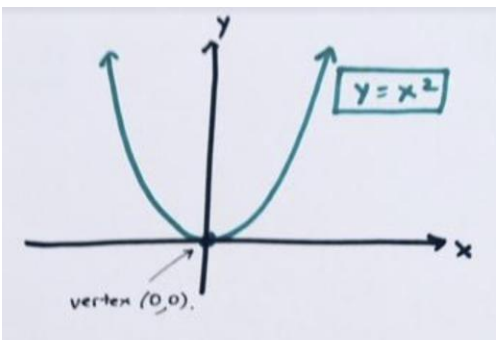
QUADRATICS
5.2 Another Graphing Challenge

James Tanton

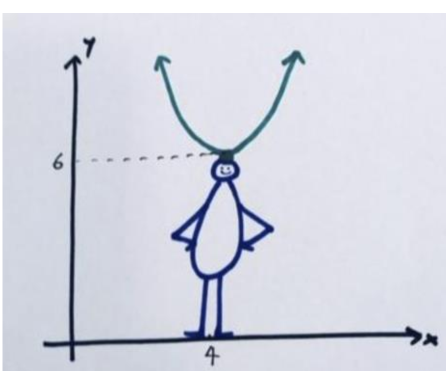
SETTING THE SCENE

Consider the lovely symmetrical U-shaped graph from last essay.



I like this graph so much that I would like to have one balancing on my head please!

Just to be clear, I am six feet tall and I will stand at position 4 on the x-axis. So this means I would like to have one of these symmetrical curves balanced at the point $(4,6)$ (that is, at the point with $x=4$, $y=6$) rather than at the point $(0,0)$.



Jargon: Given a U-shaped curve like the one we see, the point at which it curve dips down to its lowest value is called the *vertex* of the curve. The vertex of $y=x^2$ is the point $(0,0)$.

So we want a new U-shaped curve with vertex $(4,6)$. And that is the challenge of this essay.

Find an equation $y=$ _______ whose graph is a symmetrical U-shaped curve with vertex on the top of my head.

And this challenge is hard! It takes hours, days, even weeks of deep thinking and trying different things to eventually find success with it. (Such is the nature of doing mathematics.)

Please feel free to stop reading this essay and sit with this challenge for a good long while. Try graphing different variations of the $y=x^2$ equation, say, $y=x^2+2$, $y=4x^2$, $4y=x^2$, $6y=4x^2+2$ and the like, observe the results you get, and see if any helpful ideas come your way. Unfortunately, I am going to give all the answers away on the next few page and so take away your joy in playing and figuring things out for yourself. (Sorry!)

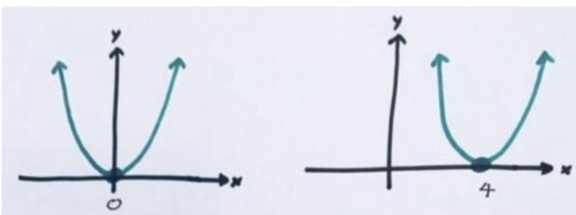
SHIFTING CURVES

We need to apply two motions to the picture of the $y = x^2$ graph: shift it rightwards four units and shift it upwards six units.

Let's think of each of these motions in turn.

1. *A Horizontal Shift*

What equation would give the same graph as $y = x^2$ but shifted four units over?



The key is to realise that in the $y = x^2$ graph all the "interesting action" is happening at $x = 0$. We want all that action to happen at $x = 4$ instead. So let's adjust the equation $y = x^2$ so that the number 4 now "behaves like zero."

How?

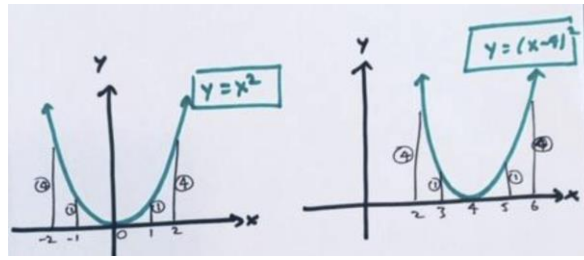
It takes a while for a flash of insight to come, but eventually one thinks of writing this equation.

$$y = (x-4)^2$$

Put in $x = 4$ and we get $y = 0^2$ and so, in some sense, 4 is behaving like 0. Wow!

And we see that putting in $x = 5$ and $x = 3$ we get $y = (1)^2$ and $y = (-1)^2$, and putting in $x = 6$ and $x = 2$, we get $y = (2)^2$ and $y = (-2)^2$, and so on. The values to the right and left of 4 in $y = (x-4)^2$ are matching the values to the left and right of 0 in $y = x^2$.

The graph of $y = (x-4)^2$ really is the graph of $y = x^2$ shifted horizontally so that 4 behaves like zero.



To shift the graph of an equation horizontally just ask "Which number would I like to behave like zero?" and then adjust the equation to make that happen!

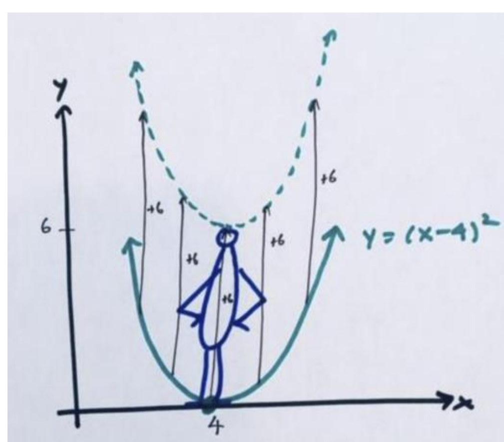
PRACTICE 1:

a) *Sketch the graph of* $y = (x-10)^2$.

b) *Sketch the graph of* $y = (x+5)^2$.

2. *A Vertical Shift*

We now want to shift the graph of $y = (x-4)^2$ vertically upwards 6 values.



Instead of being $0^2 = 0$ units high at $x = 4$, we want it to be $0^2 + 6 = 6$ units high.

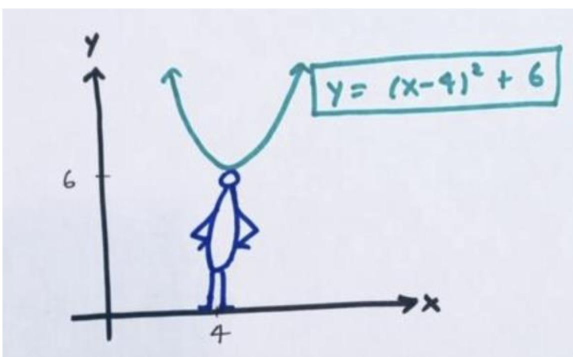
Instead of being $1^2 = 1$ unit high at $x = 5$, we want it to be $1^2 + 6 = 7$ units high.

Instead of being $(-2)^2 = 4$ units high at $x = 2$, we want it to be $(-2)^2 + 6 = 10$ units high.

And so on.

This suggests we work with the equation

$$y = (x-4)^2 + 6.$$



And one can check with a table of data values that this equation does indeed do the trick.

And when we look at the equation $y = (x-4)^2 + 6$ we can see that is it is basically the $y = x^2$ equation adjusted two ways.

1. The value $x = 4$ has been made to "behave like 0" (and so there is a horizontal shift of four units),
2. 6 has been added to the formula for the heights (and so there is a vertical shift six units upwards).

Magical!

PRACTICE 2:

a) Sketch the graph of $y = (x-5)^2 + 2$.

b) Sketch the graph of $y = (x+5)^2 - 2$.

PRACTICE 3: *When we say that the graph of* $y = x^2$ *is a U-shaped graph, is that a correct analogy? The two sides of the letter "U" are vertical. Does the graph of* $y = x^2$ *possess vertical lines?*

SOLUTIONS

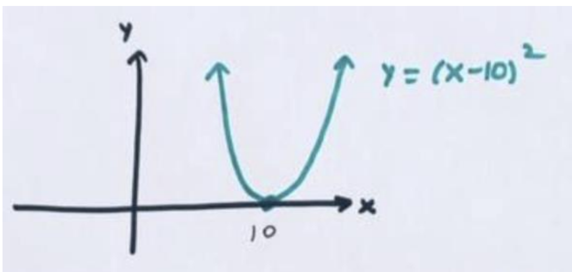
PRACTICE 1:

a) Sketch the graph of $y=(x-10)^2$.

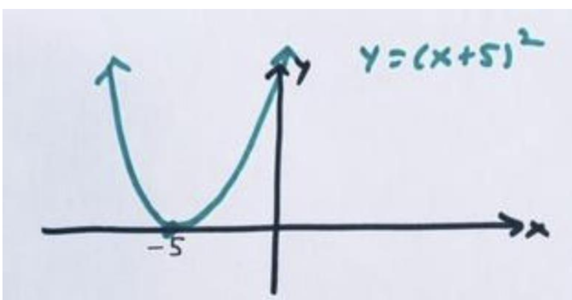
b) Sketch the graph of $y=(x+5)^2$.

Answer:

a) We see $x=10$ is behaving like zero.



b) We see $x=-5$ is behaving like zero.



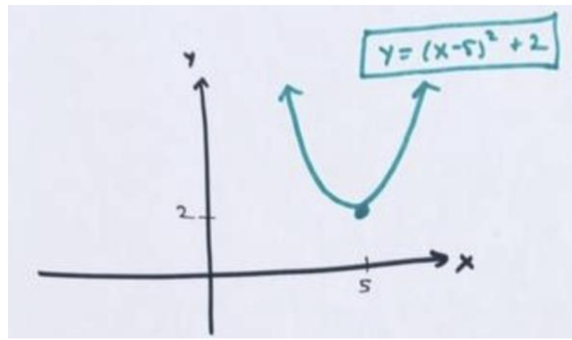
PRACTICE 2:

a) Sketch the graph of $y=(x-5)^2+2$.

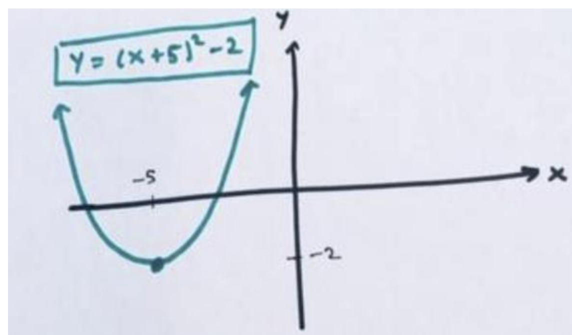
b) Sketch the graph of $y=(x+5)^2-2$.

Answer:

a) We see $x=5$ is behaving like zero and everything is shifted upwards 2 units.



b) We see $x=-5$ is behaving like zero and everything is shifted upwards -2 units, that is, down two units.



PRACTICE 3: *When we say that the graph of* $y=x^2$ *is a U-shaped graph, is that a correct analogy? The two sides of the letter "U" are vertical. Does the graph of* $y=x^2$ *possess vertical lines?*

Answer: The graph does not possess vertical lines. For instance, if the graph became vertical at say $x=100$, then the graph never extends to x-values beyond 100, which is absurd, because we know for example that $x=101$, $y=10201$ is a data point to be plotted.