QUADRATICS
5.3 Steepness

James Tanton

SETTING THE SCENE

We've played with the equation $y = x^2$ and adjust it to create new equations with the same symmetrical U-shaped graph but shifted to different positions in the plane. But as we know from our algebra work, quadratic equations could also possess a coefficient attached to the x^2 term. How do the graphs of the equations $y = ax^2$ appear for different values of a?

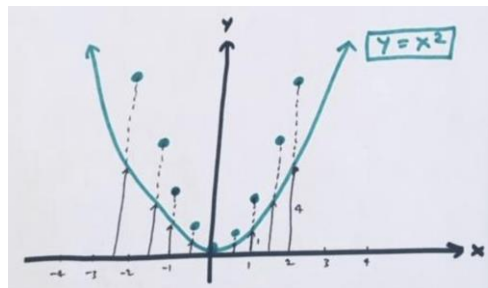
We'll explore that issue in this essay, and see that there are an infinitude of U-shaped curves I can choose from to have balance on my head!

STEEPNESS

Consider the equation $y = 2x^2$.

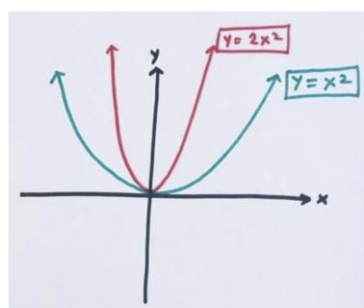
At $x = 1$, we have $y = 2(1)^2$, double the value we had for the $y = x^2$ equation.
At $x = 2$, we have $y = 2(2)^2$, double the value we had for the $y = x^2$ equation.
At $x = -33$, we have $y = 2(-33)^2$, double the value we had for the $y = x^2$ equation.

We see that the heights of all our data points for the equation $y = 2x^2$ are double the heights of the data points for $y = x^2$.

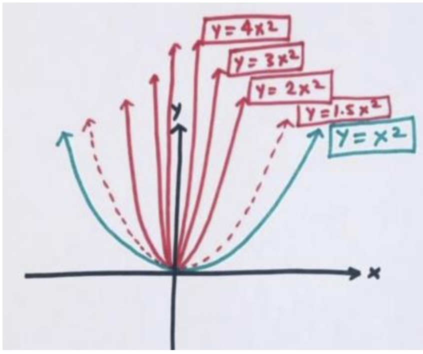


The point at $x = 0$ is still at height zero.

The graph of $y = 2x^2$ is again a symmetrical U-shaped graph, still centered with vertex at the origin, but it is a steeper curve.



We can make an even steeper graph by playing with $y = 3x^2$ or $y = 20x^2$ or $y = 10000000x^2$! (This third equation will have a graph that hugs the vertical axis very tightly if we tried to sketch it accurately.)



We can make a shallower U-shaped graph by using a coefficient small than 1. For instance, the equation

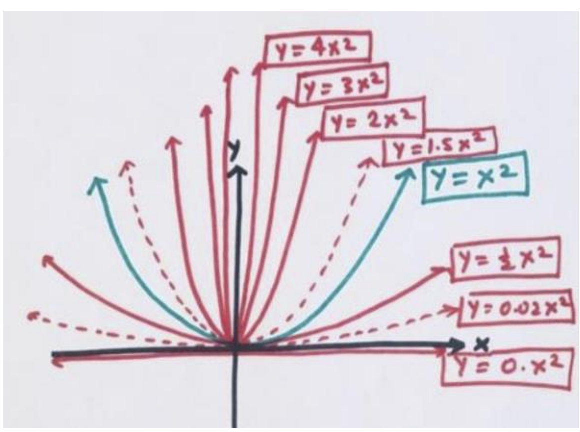
$$y=\frac{1}{2}x^2$$

has data points at half the heights of those of $y=x^2$, and

$$y=0.02x^2$$

has data points at one-fiftieth the heights.

All the data points for $y=0\cdot x^2$ have zero height and so this is U-shaped graph that is so shallow that it is flat!



What if we choose a coefficient that is even lower that zero? That is, what if we worked with a negative coefficient?

Consider, for instance, $y=-x^2$.

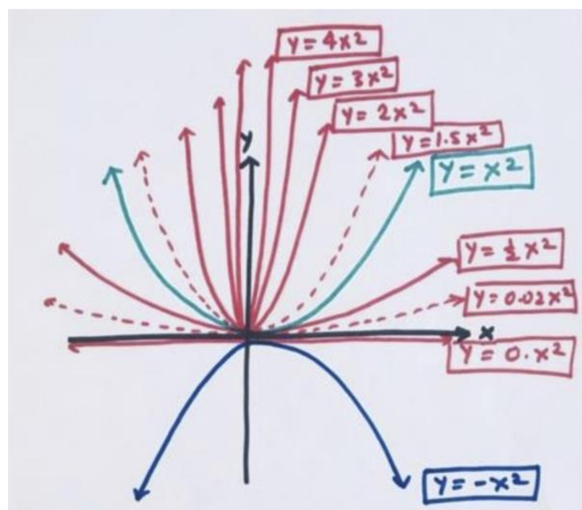
At $x=1$, we have $y=-(1)^2$, the opposite value we had for the $y=x^2$ equation.

At $x=2$, we have $y=-(2)^2$, the opposite value we had for the $y=x^2$ equation.

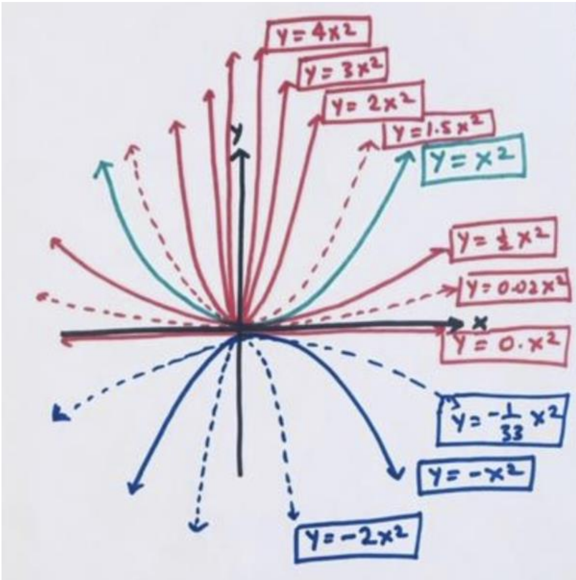
At $x=-33$, we have $y=-(-33)^2$, the opposite value we had for the $y=x^2$ equation.

And at $x=0$ we still have $y=-(0)^2=0$.

The graph is the same as the original $y=x^2$ but now pointing in the negatives.



And we can see now that $y=-2x^2$, for instance, would give a steeper U-shaped graph pointing downwards, and $y=-\frac{1}{33}x^2$ would give a shallow downward pointing graph.

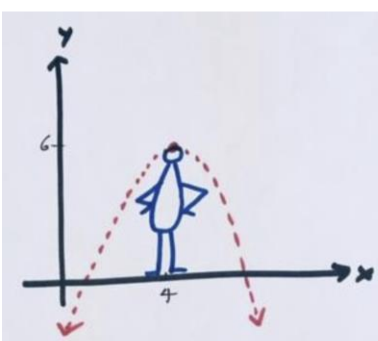


So we have

The graph of $y = ax^2$ is a symmetrical U-shaped graph based at the origin with the value a affecting the steepness of the graph.

If a is positive, the U-shape is upward pointing.
If a is negative, the U-shape is downward pointing.

PRACTICE 1: *Find three different equations that give U-shaped graphs that balance on my head this way.*



PRACTICE 2: *Draw, on the same sets of axes, rough sketches of each the following equations.*

$y = x^2$ $\quad y = 1.1x^2$ $\quad y = 0.9x^2$

$y = -x^2$ $\quad y = -1.1x^2$ $\quad y = -0.9x^2$

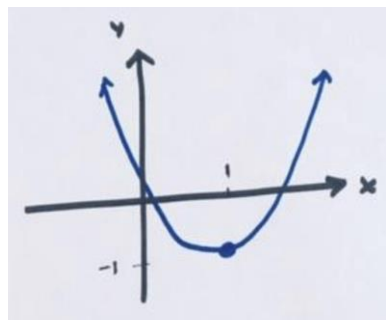
PRACTICE 3: *Sketch graphs of*

a) $y = 3(x-5)^2$

b) $y = 3(x-5)^2 + 4$

c) $y = -2(x+4)^2 + 40$.

PRACTICE 4: *Which of the following equations could have the graph shown?*



a) $y = \frac{4}{3}(x+1)^2 + 1$

b) $y = -\frac{4}{3}(x+1)^2 + 1$

c) $y = \frac{4}{3}(x-1)^2 + 1$

d) $y = -\frac{4}{3}(x-1)^2 + 1$

e) $y = \frac{4}{3}(x+1)^2 - 1$

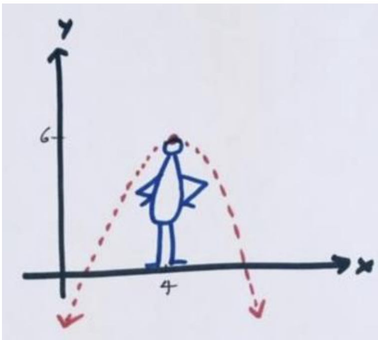
f) $y = -\frac{4}{3}(x+1)^2 - 1$

g) $y = \frac{4}{3}(x-1)^2 - 1$

h) $y = -\frac{4}{3}(x-1)^2 - 1$

SOLUTIONS

PRACTICE 1: *Find three different equations that give U-shaped graphs that balance on my head this way.*



Answer: We need to take the equation $y = -x^2$ and adjust it so that $x = 4$ behaves like zero and all data points are shifted 6 units higher. $y = -(x-4)^2 + 6$ works.

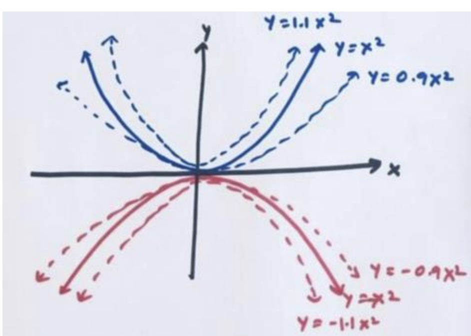
Actually, $y = -2(x-4)^2 + 6$,

$y = -\frac{1}{3}(x-4)^2 + 6$, and $y = -7(x-4)^2 + 6$ work too.

PRACTICE 2: *Draw, on the same sets of axes, rough sketches of each the following equations.*

$y = x^2$	$y = 1.1x^2$	$y = 0.9x^2$
$y = -x^2$	$y = -1.1x^2$	$y = -0.9x^2$

Answer: Roughly, we get:



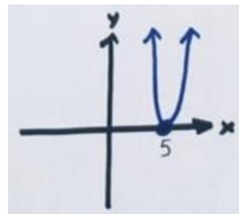
PRACTICE 3: *Sketch graphs of*

a) $y = 3(x-5)^2$

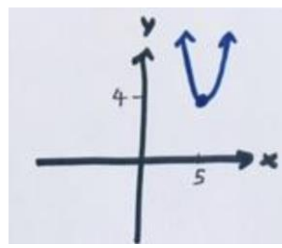
b) $y = 3(x-5)^2 + 4$

c) $y = -2(x+4)^2 + 40$.

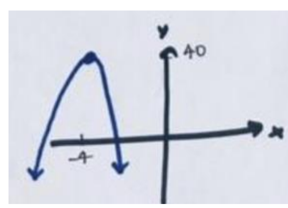
Answer: a) A steep U-shaped graph shifted so that $x = 5$ behaves like zero.



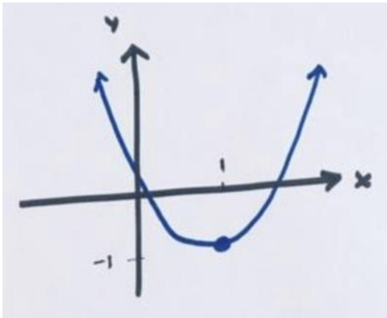
b) The same as the previous graph, except all data points are 4 units higher.



c) This is a steep upside-down U graph, with $x = -4$ behaving like zero, and all data points 40 units up.



PRACTICE 4: *Which of the following equations could have the graph shown?*



a) $y = \frac{4}{3}(x+1)^2 + 1$

b) $y = -\frac{4}{3}(x+1)^2 + 1$

c) $y = \frac{4}{3}(x-1)^2 + 1$

d) $y = -\frac{4}{3}(x-1)^2 + 1$

e) $y = \frac{4}{3}(x+1)^2 - 1$

f) $y = -\frac{4}{3}(x+1)^2 - 1$

g) $y = \frac{4}{3}(x-1)^2 - 1$

h) $y = -\frac{4}{3}(x-1)^2 - 1$

Answer: This is an upward-facing U-shaped graph with $x = 1$ behaving like zero and all data points shifted down 1. Only option g) can work.