QUADRATICS
5.4 Practicing Graphing

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SETTING THE SCENE

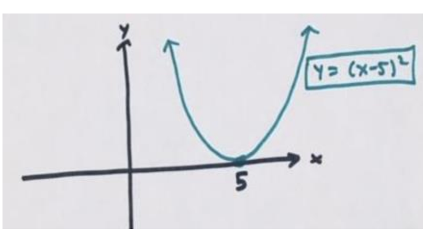
We've been adjusting the equation $y = x^2$ to give new equations whose graphs are the same U-shaped graph we see from $y = x^2$ but shifted horizontally, shifted vertically, and made steeper or broader and upward or downward facing.

For example, if asked to

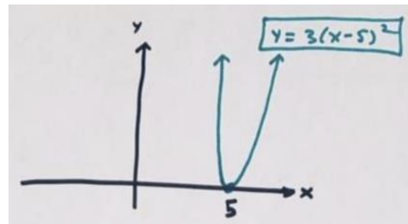
Sketch $y = 3(x-5)^2 + 10$,

we can first recognize this as the basic $y = x^2$ equation

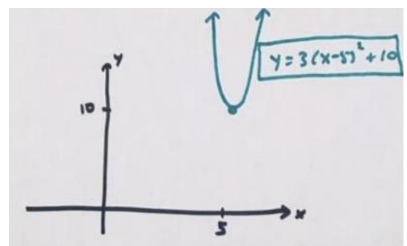
but with $x = 5$ behaving like zero



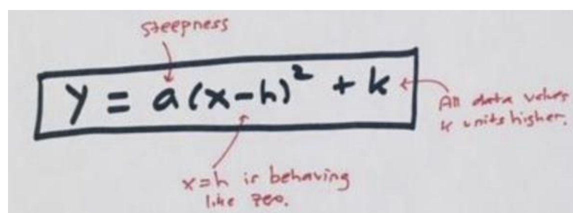
and made steeper with a steepness factor of 3



and with all data points shifted 10 units higher.



The following picture summarises all that we just saw!



We have a steepness factor a, we have $x = h$ behaving like zero, and we have all data values shifted k units higher.

PRACTICE 1: *Sketch the graph of*

$$y = -2(x+10)^2 - 7.$$

PRACTICE 2: *The graph of a quadratic equation has a vertical line of symmetry at* $x=3$*, and has highest value* $y=17$*. Which of the following could be an equation for that quadratic?*

a) $y=200(x-3)^2+17$

b) $y=-200(x-3)^2+17$

c) $y=200(x-3)^2-17$

d) $y=-200(x-3)^2-17$

PRACTICE 3: *Sketch a graph for each of the following equations.*

a) $y=2-x^2$

b) $y=\frac{1}{3}\left(x-\frac{1}{2}\right)^2-4$

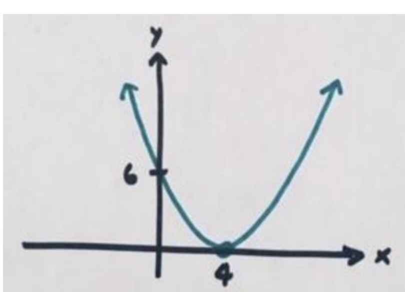
c) $y=0.0034(x+0.276)^2+0.778$

d) $y=200000(x-200000)^2-200000$

PRACTICE 4: *If* $y=a(x+b)^2+c$ *has a graph passing through the origin and with* $(2,3)$ *as the vertex, then what is the value of* $a+b+c$*?*

a) $\frac{1}{4}$ b) $1\frac{3}{4}$ c) $4\frac{1}{4}$ d) $5\frac{1}{4}$

PRACTICE 5: *Write a quadratic equation that fits this graph.*



PRACTICE 6: *Write down a quadratic equation whose graph passes through the points* $(3,18)$ *and* $(17,18)$ *and has lowest value* 5*.*

PRACTICE 7: *Write down a quadratic equation whose graph passes through the* x *axis at* $x=-2$ *and at* $x=10$ *and passes through the* y *axis at* $y=-6$*.*

PRACTICE 8: *Write down quadratic equations with symmetrical U-shaped graphs possessing the following properties:*

a) *Crosses the* x*-axis at* 3 *and* 5 *and the* y*-axis at* 1000*.*

b) *Passes through* $(4,10)$*,* $(6,10)$ *and* $(8,13)$*.*

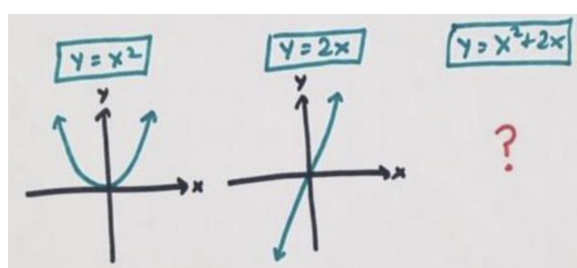
c) *Has vertex* $(5,5)$ *and passes through* $(4,4)$*.*

d) *Has vertex the origin and passes through the point* $(\sqrt{2},\pi)$*.*

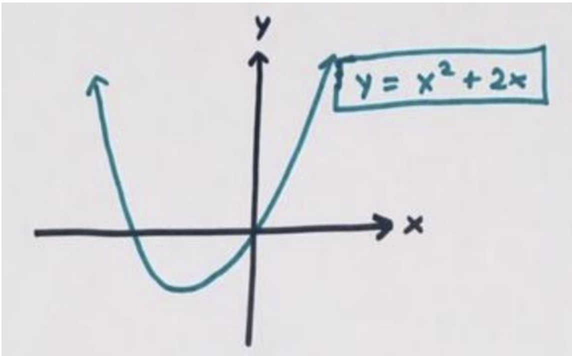
DIFFERENT-LOOKING EQUATIONS

Let's go back to the opening graphing challenge.

What graph results if we add together the matching data heights of the $y=x^2$ *and* $y=2x$ *graphs?*



If you actually made a table of data values that make $y=x^2+2x$ a true number sentence and plotted those points, then you may have been surprised to see another symmetrical U-shaped curve appear.



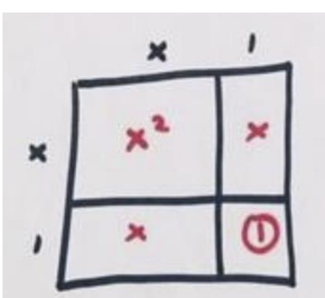
And this seems shocking! The graph of $y = x^2$ has perfect vertical symmetry, the graph of $y = 2x$ has perfect vertical anti-symmetry, and so there is no reason whatsoever to believe that a graph with perfect vertical symmetry would appear.

Let's prove that the curve we see really is a perfectly symmetric U-shaped curve. We can use our algebra of quadratics for this.

Consider $y = x^2 + 2x$.

Let's analyse the "$x^2 + 2x$" part of this equation using our quadrus method, back in lectures 1 and 2.

Let's draw a square with one piece of area x^2.



We're keeping things symmetrical, so we can say that this comes from $x \times x$. We also split the $2x$ piece into two equal areas. And then we see we need an additional area of 1.

Adding 1 through our original equation gives

$$y + 1 = x^2 + 2x + 1$$

which we now recognize as

$$y + 1 = (x+1)^2.$$

Subtracting 1 shows that the equation

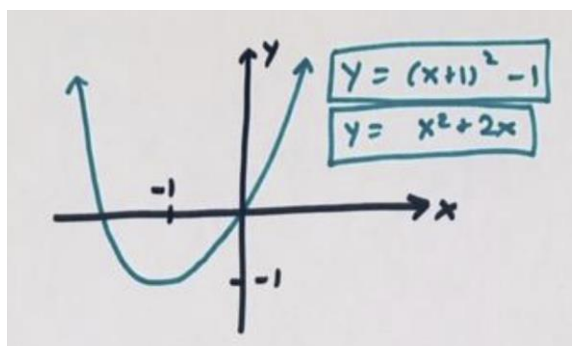
$$y = x^2 + 2x$$

is really the equation

$$y = (x+1)^2 - 1$$

in disguise.

And this is fabulous! We can now see that the graph of our equation is indeed the perfectly symmetrical U-shaped graph of $y = x^2$ with $x = -1$ behaving like zero for the x-values and with all heights shifted down by 1.



PRACTICE 9: *Use algebra to prove that the graph of* $y = x^2 - 6x + 10$ *is sure to be a symmetrical U-shaped graph.*

PRACTICE 10: *Use algebra to prove that the graph of* $y = x^2 + 8x - 7$ *is sure to be a symmetrical U-shaped graph.*

Even if there is a coefficient in front of the x^2 term in a quadratic equation, we can use the quadrus method to rewrite quadratic equations.

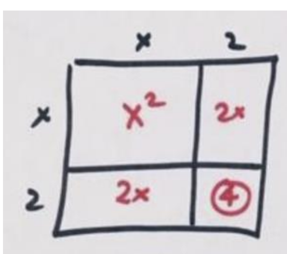
PROBLEM: *Use algebra to prove that the graph of* $y = 2x^2 + 8x + 6$ *is sure to be a symmetrical U-shaped graph.*

Answer: Notice here that all the terms on the right have even coefficients so it feels natural to divide everything throughout by 2. (We could follow the quadrus method right away and multiply through by 2 take make a perfect square up front.)

$$\frac{1}{2}y = x^2 + 4x + 3$$

The quadrus method suggests we rewrite this as

$$\frac{1}{2}y + 1 = x^2 + 4x + 4$$



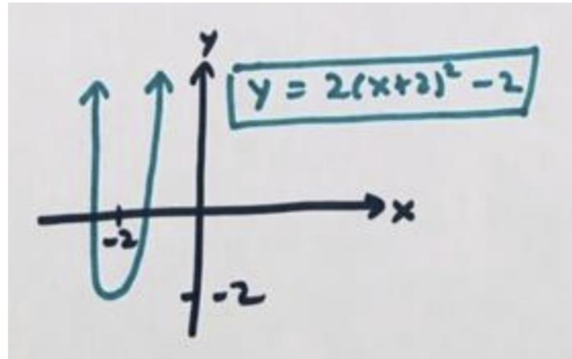
So we have

$$\frac{1}{2}y + 1 = (x+2)^2$$

$$\frac{1}{2}y = (x+2)^2 - 1$$

$$y = 2(x+2)^2 - 2$$

And the graph of this is indeed a symmetrical U-shaped curve. It has $x = -2$ behaving as zero, all heights are shifted down 2 places, and there is a steepness of 2.

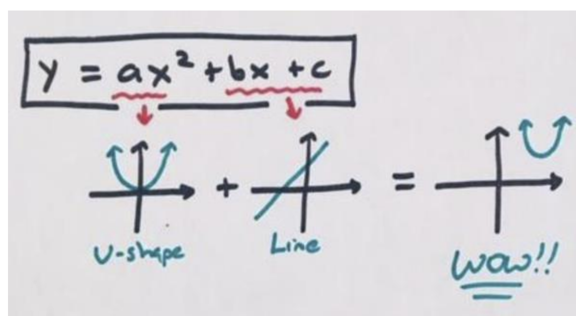


Even if the algebra is a little more complicated, the quadrus method will show each and every time that the graph of a quadratic equation is sure to be a symmetrical U-shaped graph. It will likely be shifted in the plane. There will likely be a steepness factor. But it is guaranteed to be symmetrical!

THE BIG KEY POINT OF THIS ENTIRE LECTURE

Every quadratic equation $y = ax^2 + bx + c$ has a symmetrical U-shaped graph.

This is just astounding!



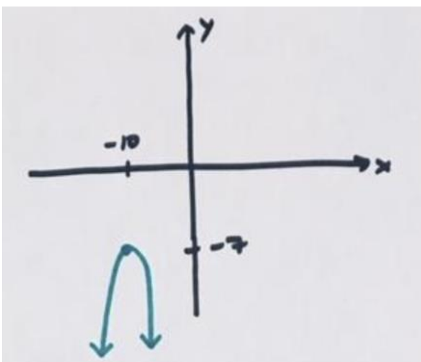
PRACTICE 11 (OPTIONAL): *Show that* $y = 3x^2 + 5x + 1$ *can be rewritten as* $y = 3\left(x + \frac{5}{6}\right)^2 - \frac{13}{12}$*, and so is sure to have a symmetrical U-shaped graph.*

SOLUTIONS

PRACTICE 1: *Sketch the graph of*

$$y = -2(x+10)^2 - 7.$$

Answer: We see $x = -10$ is behaving like zero, with a steepness factor of -2 at play (so the graph will be a steep downward-facing U-shape), with all data values shifted down 7 units.



PRACTICE 2: *The graph of a quadratic equation has a vertical line of symmetry at $x = 3$, and has highest value $y = 17$. Which of the following could be an equation for that quadratic?*

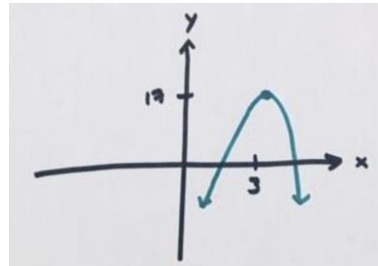
a) $y = 200(x-3)^2 + 17$

b) $y = -200(x-3)^2 + 17$

c) $y = 200(x-3)^2 - 17$

d) $y = -200(x-3)^2 - 17$

Answer: We must have a graph like this:



Only option b) can work.

PRACTICE 3: *Sketch a graph for each of the following equations.*

a) $y = 2 - x^2$

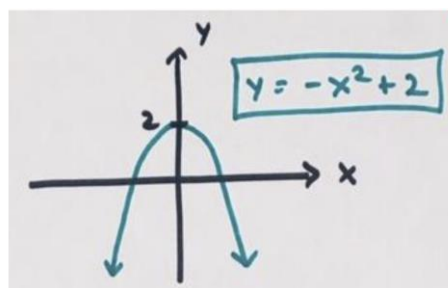
b) $y = \frac{1}{3}\left(x - \frac{1}{2}\right)^2 - 4$

c) $y = 0.0034(x+0.276)^2 + 0.778$

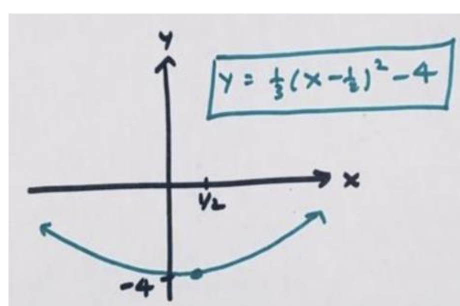
d) $y = 200000(x-200000)^2 - 200000$

Answer:

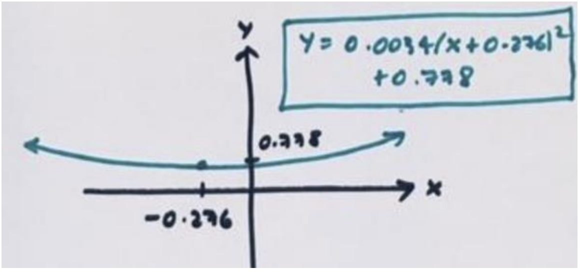
a)



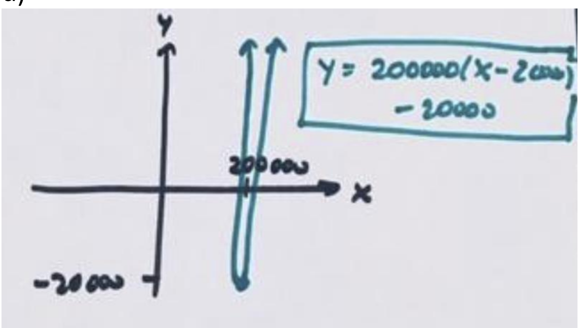
b)



c)



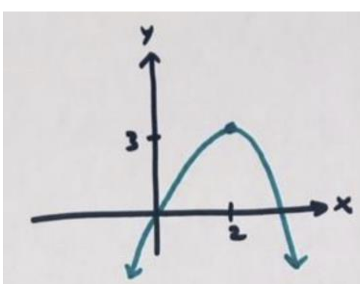
d)



PRACTICE 4: *If* $y = a(x+b)^2 + c$ *has a graph passing through the origin and with* $(2,3)$ *as the vertex, then what is the value of* $a+b+c$ *?*

a) $\frac{1}{4}$ b) $1\frac{3}{4}$ c) $4\frac{1}{4}$ d) $5\frac{1}{4}$

Answer: The graph must look like this:



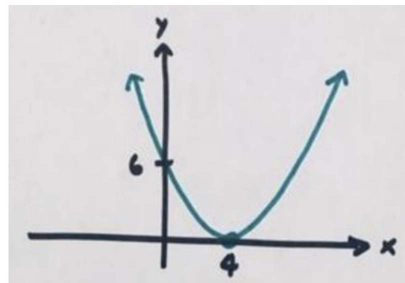
So we see the equation must be of the form $y = a(x-2)^2 + 3$, and so $b = -2$ and $c = 3$.

The graph passes through $x = 0$, $y = 0$ and so

$$0 = a(-2)^2 + 3$$

must be a true sentence about numbers. This forces $a = -\frac{3}{4}$ and so $a+b+c = \frac{1}{4}$, option a).

PRACTICE 5: *Write a quadratic equation that fits this graph.*



Answer: We see that it is a graph basically coming from $y = x^2$ but with $x = 4$ behaving like zero. So we can try

$$y = (x-4)^2.$$

But we see this is not right: when $x = 0$ we get $y = 16$, not 6. We are missing a steepness factor!

Try $y = a(x-4)^2$.

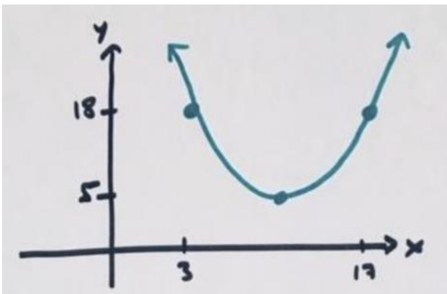
Now $x = 0$, $y = 6$ should yield a true number sentence, so $6 = 16a$ forcing $a = \frac{3}{8}$.

We have

$$y = \frac{3}{8}(x-4)^2.$$

PRACTICE 6: *Write down a quadratic equation whose graph passes through the points* $(3,18)$ *and* $(17,18)$ *and has lowest value* 5.

Answer: We must have a graph that looks like this:



So let's just use common sense to figure things out.

We want a symmetrical graph and so the line of symmetry must be at $x=10$, midway between 3 and 17. So the quadratic equation producing this graph must have the form

$$y=a(x-10)^2+5$$

for some steepness a.

When $x=3$ we should have $y=18$, showing that

$$18=a(-7)^2+5$$

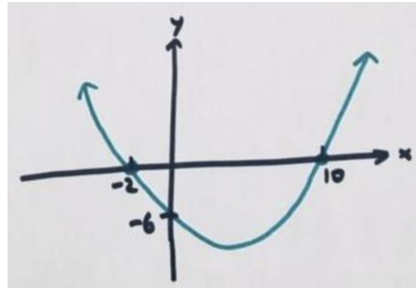
should be a true number sentence. This forces

$$a=\frac{13}{49}.$$

So $y=\frac{13}{49}(x-10)^2+5$ works.

PRACTICE 7: *Write down a quadratic equation whose graph passes through the* x *axis at* $x=-2$ *and at* $x=10$ *and passes through the* y *axis at* $y=-6$.

Answer: We have this picture.



The equation must be of the form

$$y=a(x-4)^2+b$$

When $x=10$, $y=0$ and so

$$0=36a+b$$

must be a true number sentence. We have $b=-36a$.

When $x=0$, $y=-6$ and so

$$-6=16a+b$$

must also be a true number sentence. We see that

$$-6=16a-36a$$

showing that $a=\frac{3}{10}$. Consequently $b=-\frac{54}{5}$.

The equation we need is

$$y=\frac{3}{10}(x-4)^2-\frac{54}{5}.$$

PRACTICE 8: *Write down quadratic equations with symmetrical U-shaped graphs possessing the following properties:*

a) *Crosses the* x*-axis at* 3 *and* 5 *and the* y*-axis at* 1000.

b) *Passes through* $(4,10)$, $(6,10)$ *and* $(8,13)$.

c) *Has vertex* $(5,5)$ *and passes through* $(4,4)$.

d) *Has vertex the origin and passes through the point* $(\sqrt{2},\pi)$.

Brief Answers: Sketch a picture of each scenario and use logic to determine which x-value is behaving like zero in each case. Then go from there!

a) $y=\frac{200}{3}(x-4)^2-\frac{200}{3}$

b) $y=\frac{3}{8}(x-5)^2+\frac{77}{8}$

c) $y=-(x-5)^2+5$

d) $y=\frac{\pi}{2}x^2$

PRACTICE 9: *Use algebra to prove that the graph of* $y=x^2-6x+10$ *is sure to be a symmetrical U-shaped graph.*

Answer: Following the quadrus method we have

$$y-1=x^2-6x+9$$

$$y-1=(x-3)^2$$

and so

$$y=(x-3)^2+1.$$

The graph of $y=x^2-6x+10$ is thus the graph of $y=x^2$ with $x=3$ behaving like zero and with all heights shifted 1 higher. It is thus the same symmetrical U-shape!

PRACTICE 10: *Use algebra to prove that the graph of* $y=x^2+8x-7$ *is sure to be a symmetrical U-shaped graph.*

Brief Answer: We get $y+23=(x+4)^2$ and so

$$y=(x+4)^2-23.$$

We get the same symmetrical U-shaped graph just shifted in the plane.

PRACTICE 11 (OPTIONAL): *Show that* $y=3x^2+5x+1$ *can be rewritten as* $y=3\left(x+\frac{5}{6}\right)^2-\frac{13}{12}$, *and so is sure to have a symmetrical U-shaped graph.*

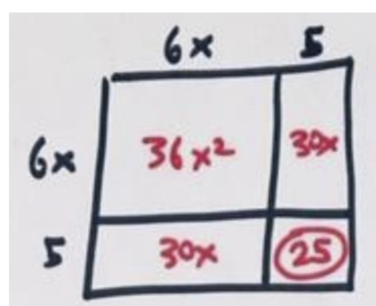
Answer: Let's follow the quadrus method. First multiply three by 3 to create a perfect square up front.

$$3y=9x^2+15x+3$$

Now multiply though by 4 to make an even middle term.

$$12y=36x^2+60x+12$$

Draw the square.



$$12y+13=(6x+5)^2$$

Pause here as we have $6x$ in the parentheses, and not just x. Since $(6x+5)=6\left(x+\frac{5}{6}\right)$ we can rewrite matters as

$$12y+13=36\left(x+\frac{5}{6}\right)^2 .$$

So

$$12y=36\left(x+\frac{5}{6}\right)^2-13$$

giving

$$y=3\left(x+\frac{5}{6}\right)^2-\frac{13}{12}.$$