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QUADRATICS
6.1 An Opening Puzzle;
Putting a BIG RESULT to Use

James Tanton

SETTING THE SCENE

We ended the last essay on a big note:

The graph of <u>any</u> quadratic equation $y = ax^2 + bx + c$ is sure to be a symmetric U-shaped curve, situated somewhere in the plane.

And the important word here is *symmetry*. As we full well know, once we have symmetry in a scenario, we have a good friend at our aid!

In this lecture, we'll show how the simple power of symmetry makes the graphing of quadratic equations beautifully natural and stunningly straightforward!

But first, here is something curious.

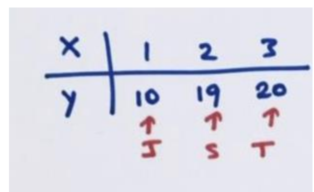
My full names is *James Stuart Tanton* and my initials thus are JST. So consider this quadratic equation:

$$y = -4x^2 + 21x - 7 .$$

Put in the values 1, 2, and 3 in turn and out come the y values:

$$-4 \cdot 1^2 + 21 \cdot 1 - 7 = 10$$
$$-4 \cdot 2^2 + 21 \cdot 2 - 7 = 19$$
$$-4 \cdot 3^2 + 21 \cdot 3 - 7 = 20$$

And notice: The 10th letter of the alphabet is J, the 19th letter of the alphabet is S; the 20th letter of the alphabet is T. This quadratic spells out my initials!



Many people like to call me JIM. So consider

$$y = \frac{5}{2}x^2 - \frac{17}{2}x + 16 .$$

Put in $x = 1$ and one gets

$$y = \frac{5}{2} - \frac{17}{2} + 16 = 10 .$$

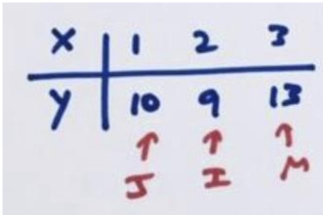
Put in $x = 2$ and one gets

$$y = \frac{5}{2} \cdot 4 - \frac{17}{2} \cdot 2 + 16 = 9 .$$

Put in $x = 3$ and one gets

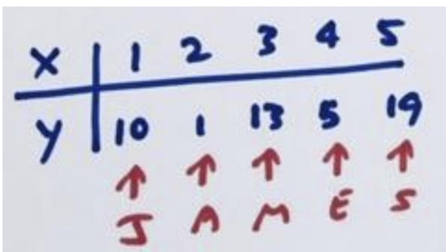
$$y = \frac{5}{2} \cdot 9 - \frac{17}{2} \cdot 3 + 16 = 13 .$$

This quadratic spells my nickname!



I actually prefer to be called JAMES. Going beyond quadratics I can write down

$$y = \frac{83}{24}x^4 - \frac{497}{12}x^3 + \frac{4141}{24}x^2 - \frac{3463}{12}x + 164$$



WHOA!

How am I coming up with these crazy equations that spell initials and names?

So my puzzle to you is:

Choose three letters (a three-letter word, three initials) and try to find a quadratic equation that "spells" those three letters. Can you do it?

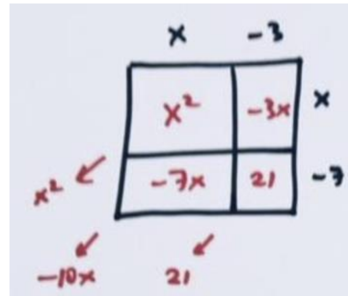
PUTTING SYMMETRY TO USE

We know from last time that every quadratic equation produces a symmetric U-shaped graph, somewhere in the plane, with some steepness. So let's see now how we can put the power of that symmetry to use.

PROBLEM: *Sketch a graph of* $y = (x-3)(x-7)+10$.

First issue: Is this equation a quadratic equation? It looks a little strange.

Well, if you were to expand out the product in the right side we would see it is a quadratic equation. (It's actually $y = x^2 - 10x + 31$.)



So we can be sure now that its graph will be a symmetric U-shaped graph.

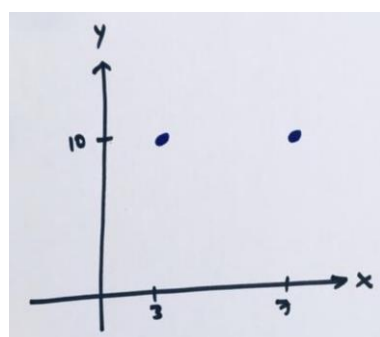
Now stare at the equation

$$y = (x-3)(x-7)+10.$$

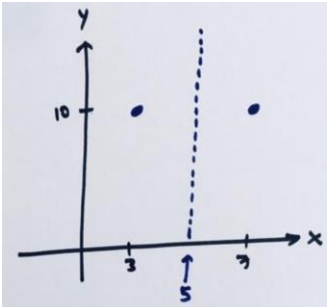
It seems irresistible to put in the x values $x = 3$ and $x = 7$. And they each give the same y value of 10.

$$x = 3 \quad \rightarrow \quad y = 0 + 10 = 10$$
$$x = 7 \quad \rightarrow \quad y = 0 + 10 = 10$$

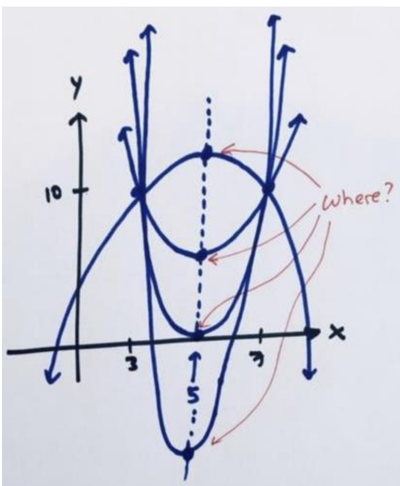
We can plot these two points at least!



Ooh! We have two symmetrical points on a graph that we know is going to be symmetrical. Common sense tells us that the line of symmetry of the graph is going to have to be right between $x = 3$ and $x = 7$, namely, at $x = 5$.



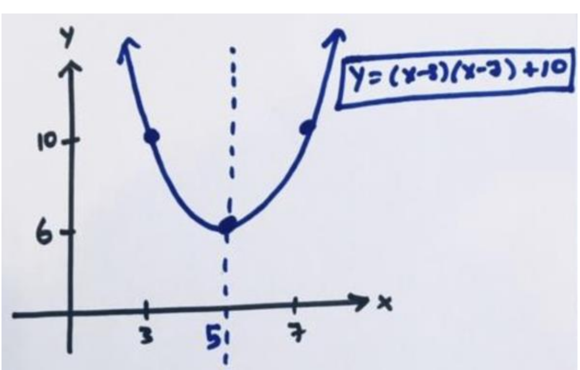
So we have a U-shaped graph with vertex somewhere along this line of symmetry. Where on that line? This will affect the shape of the curve.



Well, we can find out by putting $x = 5$ into the equation!

$$x = 5 \quad \rightarrow \quad y = (2)(-2) + 10 = 6$$

Now it is clear what the graph must be. Wow!



So here's the key idea:

To sketch the graph of a quadratic equation just find two symmetrical points to go on that symmetrical graph. Then common sense will tell you the line of symmetry, the location of the vertex, and thus the shape of the graph!

Just look for interesting x values!

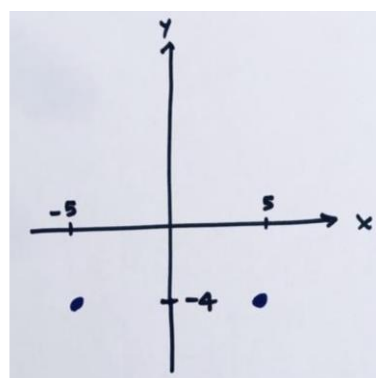
PROBLEM: *Sketch a graph of* $y = -2(x+5)(x-5) - 4$.

If we were to expand out the product on the right side we would see that this is indeed a quadratic equation. It thus will have a symmetrical U-shaped graph.

Are there any obvious and interesting x values? Yes!

$$x = -5 \quad \rightarrow \quad y = 0 - 4 = -4$$
$$x = 5 \quad \rightarrow \quad y = 0 - 4 = -4$$

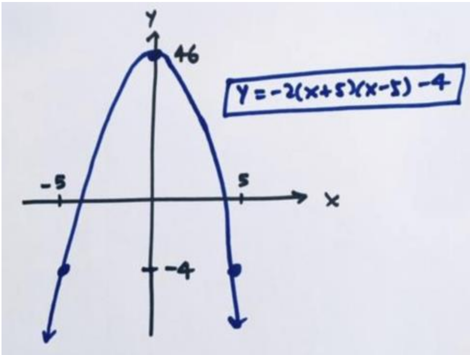
We have two symmetrical points on a symmetrical graph.



The line of symmetry must be the vertical axis and so the vertex must be in this line: $x = 0$.

$$x = 0 \quad \rightarrow \quad y = -2(5)(-5) - 4 = 46$$

We now have a good sketch of the graph.



So swift! So clear! So cool!

Let's do another example.

PROBLEM: *Sketch a graph of* $y = x(x-4)+7$.

Do we see two interesting x values staring us in the face?

This equation is a little trickier. But, if you like, you can think of it as

$$y = (x-0)(x-4)+7$$

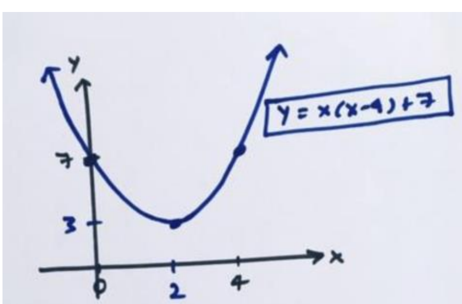
and see directly that $x=0$ and $x=4$ are interesting.

$$x=0 \quad \to \quad y=7$$
$$x=4 \quad \to \quad y=7$$

We have two symmetrical points on a symmetrical graph. The line of symmetry is at $x=2$ and the vertex must be on this line.

$$x=2 \quad \to \quad y=2(-2)+7=3$$

The graph of the equation is now clear.



PRACTICE 1: *The graph of a quadratic equation passes through the points* $(3,81)$, $(4,9)$, *and* $(-10,9)$. *What is the* x*-coordinate of its vertex?*

PRACTICE 2: *Sketch a graph of* $y = 2(x-3)(x-23)+200$.

PRACTICE 3: *Sketch a graph of* $y = \frac{1}{25}(x-1)(x-11)-1$.

PRACTICE 4: *Sketch a graph of* $y = -(x-3)(x+5)+6$.

PRACTICE 5: *Sketch a graph of* $y = -2x(x-80)+3$.

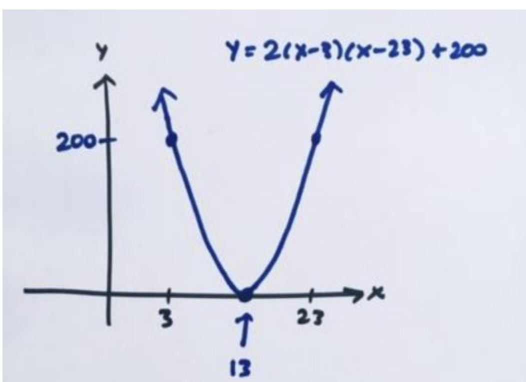
SOLUTIONS

PRACTICE 1: *The graph of a quadratic equation passes through the points* $(3,81)$, $(4,9)$, *and* $(-10,9)$. *What is the* x*-coordinate of its vertex?*

Answer: We see that $(4,9)$ and $(-10,9)$ are two symmetrical points on a symmetrical graph, and so its line of symmetry must be halfway between $x=4$ and $x=-10$, namely, at $x=-3$. And $x=-3$ must be the x-coordinate of the vertex.

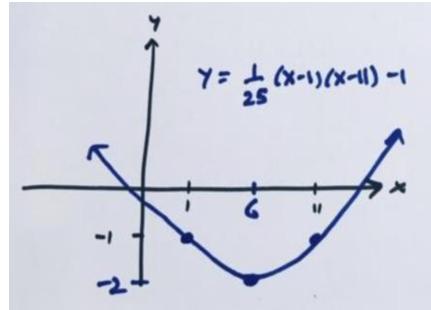
PRACTICE 2: *Sketch a graph of* $y=2(x-3)(x-23)+200$.

Answer:



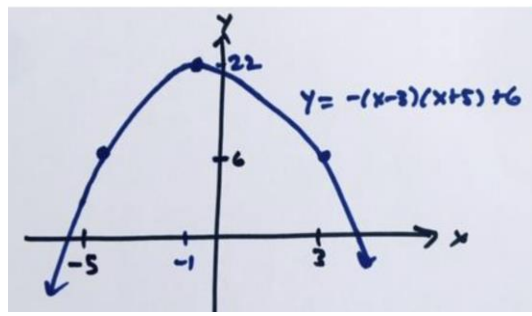
PRACTICE 3: *Sketch a graph of* $y=\frac{1}{25}(x-1)(x-11)-1$.

Answer:



PRACTICE 4: *Sketch a graph of* $y=-(x-3)(x+5)+6$.

Answer:



PRACTICE 5: *Sketch a graph of* $y=-2x(x-80)+3$.

Answer: