QUADRATICS
6.2 The Full Power of Symmetry

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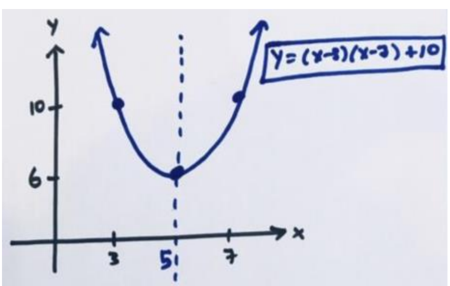
SETTING THE SCENE

Through the power of symmetry we've started graphing quadratic equations with ease. We noticed, for instance, that for

$$y = (x-3)(x-7)+10$$

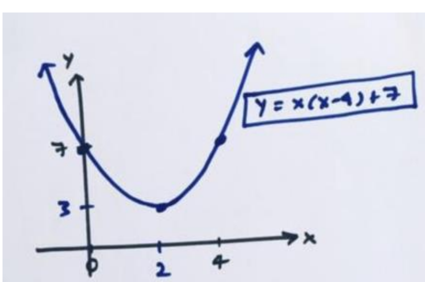
the x values 3 and 7 are interesting and thus led us to two symmetric points on the symmetric curve. Then common sense allowed us to swiftly sketch the equation's graph.



And for

$$y = x(x-4)+7$$

the x values 0 and 4 are interesting.



But most quadratic equations are presented in a different, namely, in the form

$$y = ax^2 + bx + c.$$

Is there a way to identify interesting x values in such equations?

THE FULL POWER OF GRAPHING

Consider this example.

PROBLEM: *Sketch a graph of* $y = x^2 - 4x + 7$.

There are no interesting x values immediately staring us in the face here. What can we do?

Since we are focused on x values, is there anything we can do with the "$x^2 - 4x$" part of the equation?

The only thing I can think to try is to factor out a common factor of x from those two terms. Write

$$y = x(x-4)+7.$$

Oh! And now we see this is the previous problem we just discussed with interesting x values 0 and 4.

So that's the key!

If presented with a quadratic expression for which you don't see any obvious interesting x values that lead to a pair of symmetric data points, try doing some basic algebra on the "x part" of the equation.

PROBLEM: *Sketch a graph of* $y = 3x^2 + 9x + 4$.

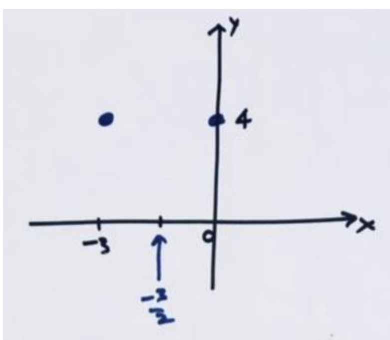
To find some interesting x values, let's play with the "$3x^2 + 9x$" part of the equation. Shall we factor out a common x or $3x$? Well, let's just try $3x$ first (and if it is not useful we can go back and then just try x).

$$y = 3x(x+3) + 4$$

Ah! This is good. We see that $x = 0$ and $x = -3$ are interesting.

$$x = 0 \quad \rightarrow \quad y = 4$$
$$x = -3 \quad \rightarrow \quad y = 4$$

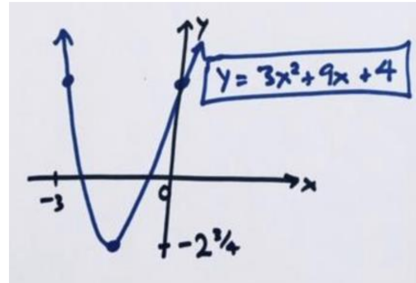
We now have two symmetrical points on a symmetrical graph. The line of symmetry is at $x = -\frac{3}{2}$.



The vertex of the parabola will be on this line of symmetry. Let's put $x = -\frac{3}{2}$ into the second form of the equation. (It looks a bit easier.)

$$x = -\frac{3}{2} \quad \rightarrow \quad y = 3\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right) + 4 = -\frac{11}{4}$$

The vertex is below the x axis.



PRACTICE 1: *Sketch a graph of* $y = 7x^2 + 7x - 100$.

Here's a curious challenge.

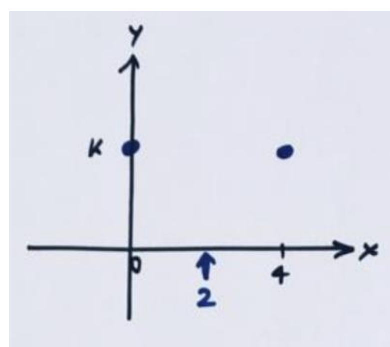
PROBLEM: *Find* k *so that* $y = -2x^2 + 8x + k$ *gives* 43 *as the largest possible value for* y.

Let's just follow our nose on this one and see how far we can get.

Let's factorise the first two terms, perhaps as

$$y = 2x(-x+4) + k.$$

We see that 0 and 4 are interesting x values: they both yield $y = k$.

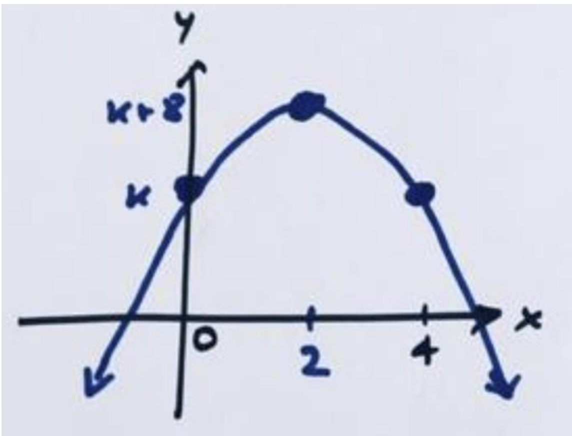


Here I've drawn a picture with k positive. This might not be right—the value could be negative—but let's just go with this for now.

The line of symmetry must be at $x = 2$. For this value we get

$$y = 2(2)(-2+4) + k = k + 8.$$

Hmm. This is a larger than k. Oh, we must have a picture as follows.



The graph is meant to have largest value 43 so we must have $k + 8 = 43$ giving $k = 35$. Done!

PRACTICE 2: *Which value of r forces the graph of $y = 3x^2 + 6x + r$ have 5 as the smallest possible y value?*

PRACTICE 3: *Find a negative value for a so that $y = x^2 + ax + a$ has smallest possible value -3.*

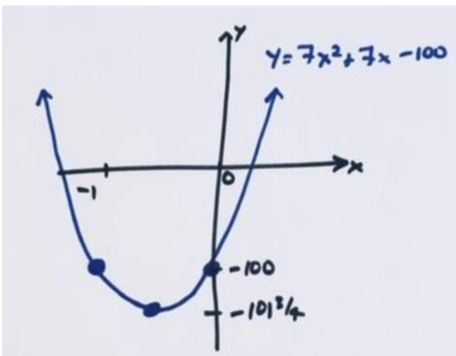
PRACTICE 4: *Find a formula for the location of the line of symmetry of a general quadratic equation $y = ax^2 + bx + c$.*

PRACTICE 5: *Sketch the graph of $y = -x^2 + 8x + 21$. What is the largest possible y value this equation can produce? What is the vertex of this graph? Make a guess as to what it means to rewrite $y = -x^2 + 8x + 21$ in "vertex form."*

SOLUTIONS

PRACTICE 1: *Sketch a graph of* $y = 7x^2 + 7x - 100$.

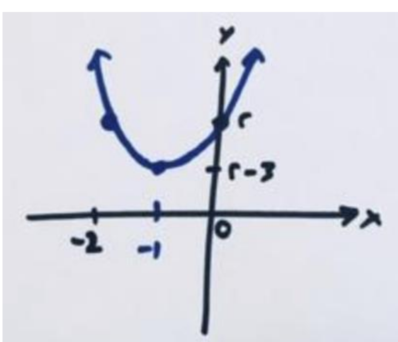
Answer: We have $y = 7x(x+1) - 100$ showing that both $x = 0$ and $x = -1$ are interesting values.



PRACTICE 2: *Which value of* r *forces the graph of* $y = 3x^2 + 6x + r$ *have* 5 *as the smallest possible* y *value?*

Answer: We have $y = 3x(x+2) + r$ showing that the line of symmetry is halfway between $x = 0$ and $x = -2$, namely, at $x = -1$. The vertex is on this line.

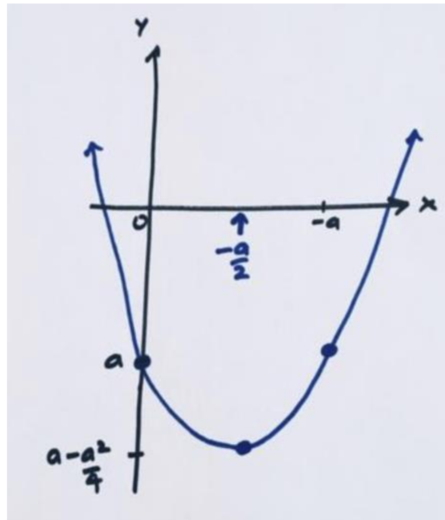
At $x = -1$ we have $y = 3 - 6 + r = r - 3$.



We must have $r - 3 = 5$ giving $r = 8$.

PRACTICE 3: *Find a negative value for* a *so that* $y = x^2 + ax + a$ *has smallest possible value* -3.

Answer: We have $y = x(x+a) + a$.



(The picture assumes a is negative.)

We see that the line of symmetry is at $-\dfrac{a}{2}$ and

for $x = -a/2$, $y = \left(-\dfrac{a}{2}\right)\left(\dfrac{a}{2}\right) + a = a - \dfrac{a^2}{4}$

We see we need

$$a - \frac{a^2}{4} = -3.$$

That is, we need $a^2 - 4a - 12 = 0$.

Solving

$$(a-2)^2 = 16$$
$$a - 2 = 4 \text{ or } -4$$
$$a = 6 \text{ or } -2.$$

Choose $a = -2$.

PRACTICE 4: *Find a formula for the location of the line of symmetry of a general quadratic equation* $y = ax^2 + bx + c$.

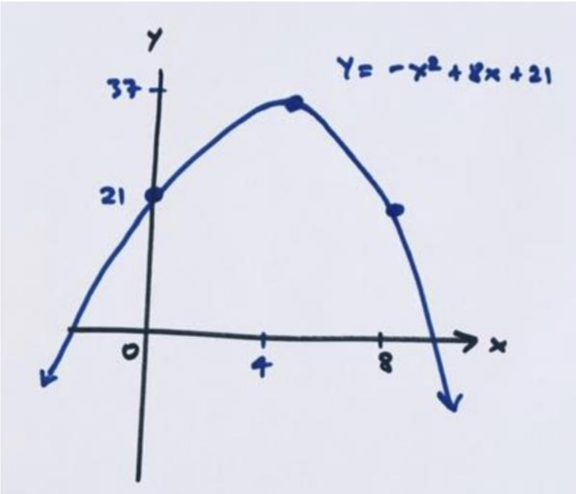
Answer: We have $y = x(ax+b)+c$ which shows

$$x = 0 \quad \to \quad y = c$$
$$x = -\frac{b}{a} \quad \to \quad y = c$$

The line of symmetry is halfway between $x = 0$ and $x = -\frac{b}{a}$, which is at $x = -\frac{b}{2a}$.

PRACTICE 5: *Sketch the graph of* $y = -x^2 + 8x + 21$. *What is the largest possible* y *value this equation can produce? What is the vertex of this graph? Make a guess as to what it means to rewrite* $y = -x^2 + 8x + 21$ *in "vertex form."*

Answer: We have $y = x(-x+8)+21$, with line of symmetry at $x = 4$.



The largest value y value occurs at $x = 4$ with $y = 4(4)+21 = 37$. The vertex is $(4,37)$.

"Vertex form" of the equation probably means a form of the equation that makes the vertex clear in the equation. Going back to last lecture, we see this graph as the $y = x^2$ shifted in the plane with negative steepness. We have

$$y = a(x-4)^2 + 37.$$

When $x = 0$ we should have $y = 21$. This gives

$$21 = 16a + 37$$

Showing that $a = -1$.

Thus

$$\begin{aligned} y &= -x^2 + 8x + 21 \\ &= -(x-4)^2 + 37 \end{aligned}$$

Note: We could have deduced the steepness was going to be $a = -1$ by looking at the coefficient of the x^2 term in the original equation. The x^2 terms here

$$-x^2 + 8x + 21 = a(x-4)^2 + 37$$

will match only for $a = -1$.