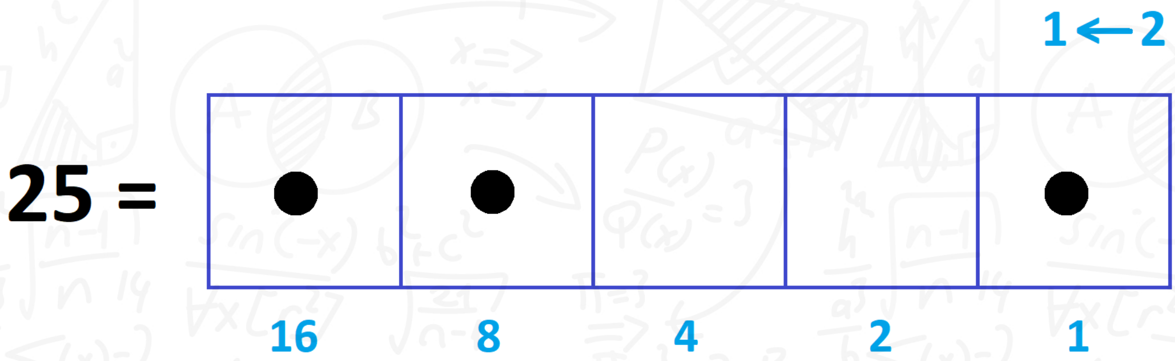
THE FIVE CARD MINDREADING TRICK

SOLUTION

The key is to look at the 1←2 machine codes (that is, the binary codes) of the numbers from 1 up to 31.



1: 1	9: 1001	17: 10001	25: 11001
2: 10	10: 1010	18: 10010	26: 11010
3: 11	11: 1011	19: 10011	27: 11011
4: 100	12: 1100	20: 10100	28: 11100
5: 101	13: 1101	21: 10101	29: 11101
6: 110	14: 1110	22: 10110	30: 11110
7: 111	15: 1111	23: 10111	31: 11111
8: 1000	16: 10000	24: 11000	

These are the codes with at most five digits. (The codes for the numbers 32 and above are six or more digits long.)

The numbers listed in group E are precisely the numbers with a 1 in the rightmost box for their 1←2 machine codes.

The numbers listed in group D are precisely the numbers with a 1 in the second-to-right box for their 1←2 machine codes.

The numbers listed in group C are precisely the numbers with a 1 in the third-to-right box for their 1←2 machine codes.

The numbers listed in group B are precisely the numbers with a 1 in the fourth-to-right box for their 1←2 machine codes.

The numbers listed in group A are precisely the numbers with a 1 in the fifth-to-right box for their 1←2 machine codes.

So, when a student answers “YES” to a particular group, she is informing you that there is a 1 in that matching box of her secret number’s code. When she answers “NO,” there is no 1.

For example, Suzzy informed us that the 1←2 code of her number is 11001. That the top left corner number in each group corresponds to the value of a dot in that group’s matching box makes it quick to figure out which number has that code!

GROUP A

16	20	24	28
17	21	25	29
18	22	26	30
19	23	27	31

YES

GROUP B

8	12	24	28
9	13	25	29
10	14	26	30
11	15	27	31

YES

GROUP C

4	12	20	28
5	13	21	29
6	14	22	30
7	15	23	31

NO

GROUP D

2	10	18	26
3	11	19	27
6	14	22	30
7	15	23	31

NO

GROUP E

1	9	17	25
3	11	19	27
5	13	21	29
7	15	23	31

YES

1←2

25 =	●	●			●
	16	8	4	2	1

Question 1: Does it make sense that only the odd numbers have a 1 in the rightmost position for their 1←2 codes?

Question 2: Does it make sense that group A matches the numbers 16 through 31?

Question 3: Take the eight numbers in one group that are less than 16. Double them to obtain a new set of eight numbers. Now add one to each of those answers to obtain a second set of eight numbers. What do you notice about this collection of 16 numbers? Can you explain what you observe?

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: What if we were willing to play with six groups of numbers? What would be a six-group version of this mind-reading trick? "Think of a number between 1 and ___"?

Wild Exploration 2: Here's a second mindreading trick. It works exactly the same way as the previous trick (all the mind-reader has to do is again look at the top left corner numbers to each group with a YES answer and sum those numbers to determine the secret number) but this time a participant is asked to think of a number between -10 and 21 .

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 8 12	-8 -4 9 12	4 -4 20 12	-2 -10 14 6	1 5 9 15
17 21 9 13	-7 -3 8 13	5 -3 21 13	-1 -9 15 7	-1 -5 -9 17
14 18 6 10	-10 -6 6 10	2 -6 18 10	2 -6 18 10	3 7 11 19
15 19 7 11	-9 -5 7 11	3 -5 19 11	3 -5 19 11	-3 -7 13 21

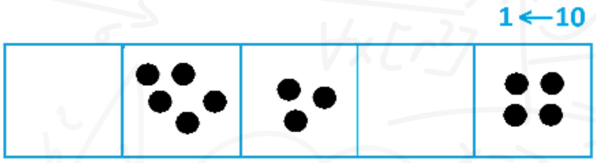
Can you figure out why this trick works?

A TWO-BUTTON CALCULATOR

SOLUTION

Let’s focus on puzzle 1 for now.

Consider representing numbers in a 1←10 machine.



Now each press of the + adds a dot to the rightmost box. Each press of the x button has the effect of shifting all the dots in the machine one place to the left.

1←10

				0

Press + five times.

				5

Press ×.

				50

			5	0

Press + three times.

			5	3

Press ×.

			50	30

		5	3	0

Press ×.

		50	30	0

	5	3	0	0

Press + four times.

	5	3	0	4

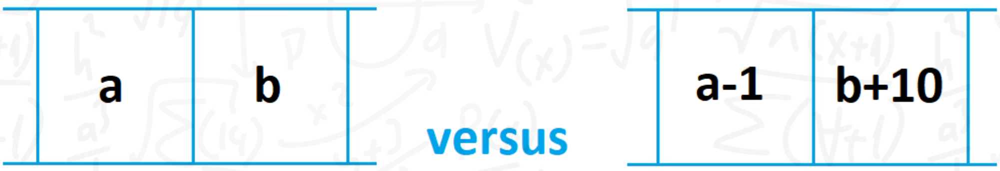
We see we can reach the number 5304 in fifteen button presses.

We can reach any four-digit number a | b | c | d by

Pressing +, a times
Pressing x
Pressing +, b times
Pressing x
Pressing +, c times
Pressing x
Pressing +, d times

for a total of a + b + c + d + 3 button presses, even if the “digits” a , b , c , d are larger than nine. For example, in Exploding Dots notation we can create 4 | 13 | 0 | 4 in 24 button presses, which will display as 5304 on the calculator.

We could also create 4 | 12 | 10 | 4 or 3 | 21 | 19 | 14 or any other representation of 5304 . The question is: Would it ever behoove us to create more than ten dots in box by pressing the + button more than ten times in a row?



Creating a | b within part of a display requires pressing + , a times, pressing x, pressing +, b times, for a total of a+b+1 button presses.

Creating a - 1 | b+10 within part of a display requires pressing +, a - 1times, pressing x, pressing +, b+10 times, for a total of a+b+10 button presses.

It is thus always more efficient to create the version of 5304 for which all explosions have occurred, that is, the version of 5 | 3 | 0 | 4 the number for a total of 15 button presses. This number cannot be created in fewer button presses.

Puzzles 2 and 3:

Puzzle 2 is the same puzzle but based on base two: the + adds a dot in the rightmost box of a 1←2 machine code and the x button shifts all the dots in the machine one place to the left.

The number 244 in base two is 11110100 and this can be obtained by pressing the buttons +x+x+x+xx+xx in order for a total of twelve button presses. This is optimal by essentially the same argument as before.

Puzzle 3 is the same puzzle but based on base three: the + adds a dot in the rightmost box of a 1←3 machine code and the x button shifts all the dots in the machine one place to the left.

The number 244 in base three is 100001 and this can be obtained by pressing the buttons +xxxxx+ in order for a total of seven button presses. This is optimal by essentially the same argument as before.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: A calculator, currently displaying 0, has just two buttons.

[+] Pressing this button adds 1 to the number currently on display.

[×] Pressing this button multiplies the number currently on display by -2.

a) Is it possible for each and every integer (positive or negative) to be on display?

b) Analyze the minimum number of button presses needed to reach given integers.

Wild Exploration 2: A calculator, currently displaying 0, has just two buttons.

[+] Pressing this button adds 1 to the number currently on display.

[×] Pressing this button multiplies the number currently on display by $\frac{3}{2}$ if the number is even, does nothing if the number is odd.

Which numbers are possible to display on this calculator?

PEASANT MULTIPLICATION

SOLUTION

Scholars of the region on ancient Egypt some 3500 years ago were aware that it is not too taxing mentally to repeatedly double a number, and from this, one can perform multiplication. For example, to compute 37 times a number ***N*** start by listing the numbers that arise from repeatedly doubling ***N***.

$$\begin{array}{r} N \\ 2N \\ 4N \\ 8N \\ 16N \\ 32N \\ 64N \\ \vdots \end{array}$$

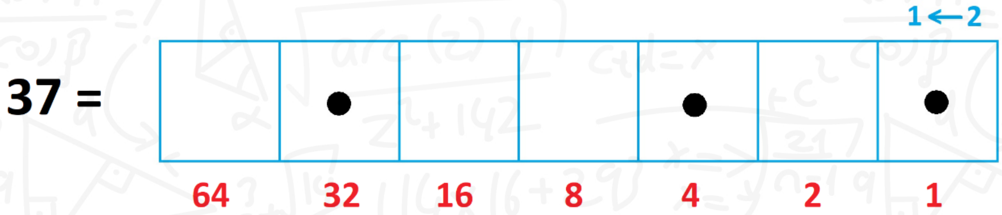
and since 37= 32+4+1 we can compute 37x N simply by adding 32N + 4N + N .

For example,

$$37 \times 10 = (32+4+1) \times 10$$
$$= 32 \times 10 + 4 \times 10 + 1 \times 10 = 370$$

This ancient technique of adding repeated doubles of a given number to compute a product is sometimes called today “Egyptian Multiplication.” It feels closely related to the multiplication technique described in today’s puzzle.

To work out a product M x N, Egyptian multiplication relies on writing M as a sum of powers of two. That is—in Exploding Dots thinking—it relies on knowing the 1←2 machine code for M .

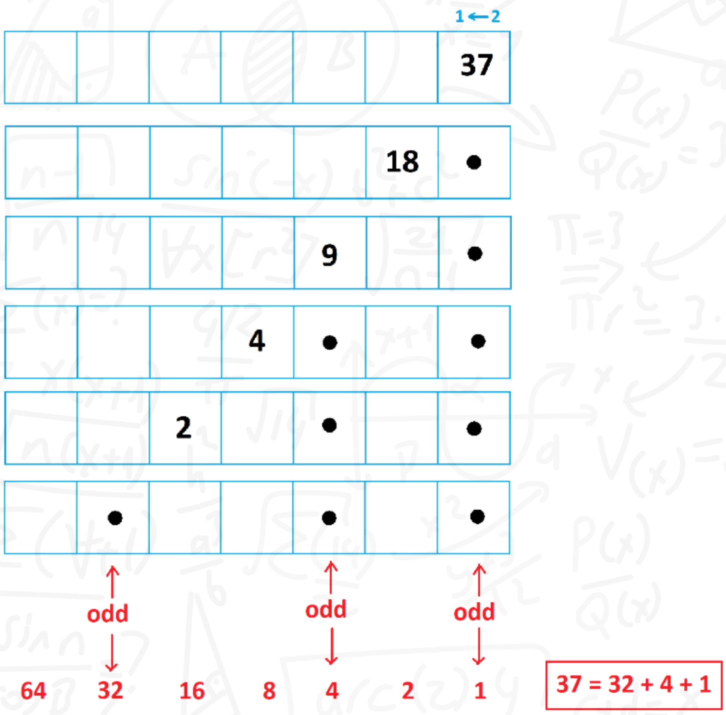


And how do we find the 1←2 code for a number M ? Well, we just put in M dots in the right box of a 1←2 machine!

Here's the thing to note: In a 1←2 machine, if the count of dots in a box is …

EVEN, then all the dots explode leaving 0 behind and half the number of dots appear in the box one place to the left,

ODD, then all but one of the dots explode leaving 1 behind and just under half the number of dots appear in the box one place to the left.



That is, in working out the **1**←**2** machine code for 37, say, we place 37 dots in the machine, and repeatedly halve the count on dots as we move to the left (ignoring remainders) and note that a dot remains (the remainder!) at each position that the count was odd.

The positions of the odd counts match the powers of two that appear in the binary code of the number.

	Power of Two
37	1
~~18~~	~~2~~
9	4
~~4~~	~~8~~
~~2~~	~~16~~
1	32

37 = 32 + 4 + 1

And we thus we see that computing 37 × ***N*** matches the Peasant Multiplication method presented.

	Power of Two	
37	1	N
~~18~~	~~2~~	~~2N~~
9	4	4N
~~4~~	~~8~~	~~8N~~
~~2~~	~~16~~	~~16N~~
1	32	32N

$\boxed{37 \text{ x } N = 32\text{x}N + 4\text{x}N + 1\text{x}N}$

Of course, there is nothing special about the number 37 here.

> To compute ***M* x *N***, repeatedly halve ***M*** (ignoring remainders). Focusing on the odd entries gives that powers of two that appear when writing ***M*** as a sum of powers of two. Thus ***M* x *N*** is the sum of these powers of two each multiplied by ***N***. We see these multiples of two multiplied by ***N*** when we repeatedly double ***N***.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: We have a quick method for finding the binary code of number: Write the number to the right of the page and, working left, repeatedly halve the number ignoring remainders. Writing a 1 under every odd entry and 0 under every even entry gives the binary code of the number.

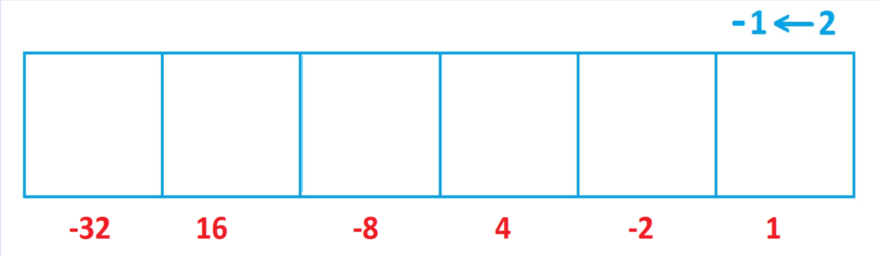
$$1 \leftarrow 3 \leftarrow 6 \leftarrow 13 \leftarrow 26$$

1	1	0	1	0

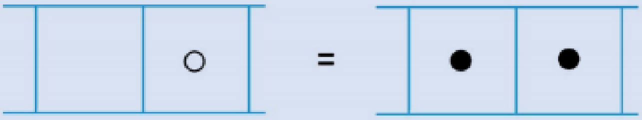
The binary code of 26 is 11010.

Devise a similar method for swiftly finding the base-three code of a number. Can you create new multiplication technique based on these base-three codes?

Wild Exploration 2: Consider a $-1 \leftarrow 2$ machine in which two dots in a box explode to be replaced by an antidot one place to their left, and two antidots in a box explode to be replaced by a dot one place to their left. This is a base negative-two machine.



This machine has the added feature that any antidot can be replaced by a dot along with a second dot one place to its left.



a) Put ten dots in the machine and show you obtain the code 1|1|1|1|0 for it.

b) What is the code for negative ten in this machine using only 0 s and 1s?

One can prove that every positive integer has a unique code in this machine using only the digits 0 and 1. (Care to prove this?) These codes are called negabinary codes.

c) Can you find a swift way to compute the negabinary code of a number along the lines of this puzzle? If so, can you create an interesting multiplication method from it?

DIVISIBILITY BY 9

SOLUTION

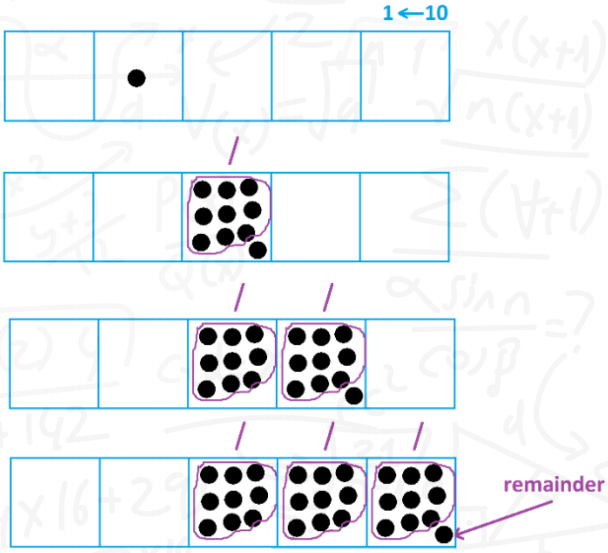
Let's look at division by 9 in a 1←10 machine.

To get a feel for what is going on, perhaps have students draw dots and boxes to compute 210÷9. It will be tedious, as one must "unexplode" multiple times to find groups of 9, but the key is to notice that there will be 2 +1 dots left over in the rightmost box as remainder. (Try it!)

This prepares us to see

> Each dot in a 1←10 machine leaves a remainder of 1 upon division by 9.

This picture illustrates why.



Thus we have:

If a number is represented by a total of ***N*** *dots in a 1←10 machine, dividing by 9 leaves us with* ***N*** *dots in the rightmost box. (And there might be some more groups of 9 we can circle there.)*

But let's think about what N is in this statement: it's the sum of the digits of the original number. And looking for groups of 9 in the rightmost box with N dots in it is the precisely the act of dividing N itself by 9.

> *The act of dividing a number by 9 in a 1←10 machine reduces to the equivalent act of dividing its sum of digits by 9.*

Thus the original number and the sum of its digits leave the same remainder upon division by 9.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Many people know the rule: A number is divisible by 3 only if its sum of digits is. Is there a stronger version of this rule to consider? If so, can you prove it? (If not, can you at least prove the first version of the rule?)

Wild Exploration 2: Martians have three fingers on each of two hands and so naturally write all their numbers in base 6 using a 1←6 machine. Is there a divisibility rule for some special Martian number like the one we Earthlings have for 9 in our base-10 system?

Wild Exploration 3: What remainder(s) does a single dot in a 1←10 machine leave upon division by 11? Can you devise, and explain, a divisibility rule for 11? (Or look up a rule on the internet and see if you can explain it using a 1←10 machine.)

Wild Exploration 4: Which numbers k have the property that if N is divisible by k, then so are all the numbers obtained by rearranging the digits of N? (For example, k = 9 is one such number. So is k = 1.)

DIVISIBILITY BY 9 – AGAIN!

SOLUTION

Let's look at division by 9 in a 1←10 machine.

To get students going have them look at

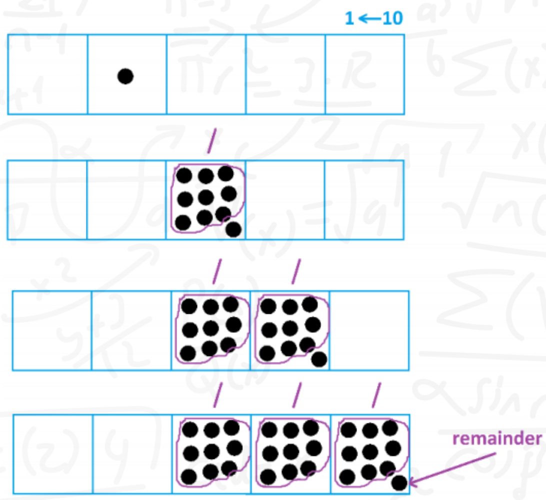
10÷9

100÷9

1000÷9

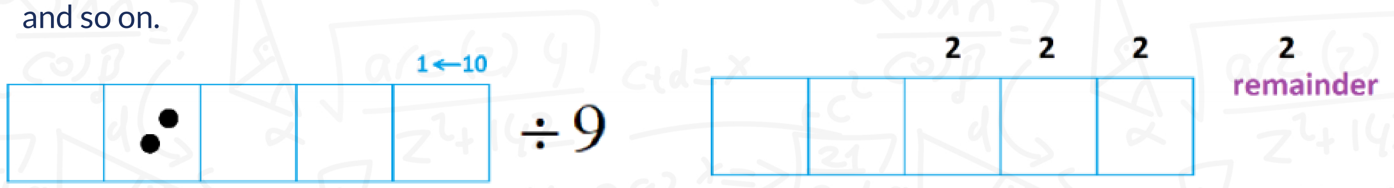
10000÷9

and so on in a 1←10 machine and see that they each given an answer of the form 1111...1 R 1 .

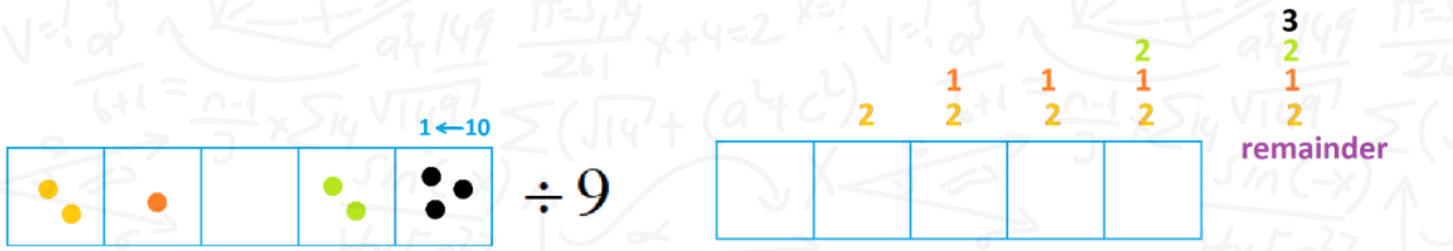


So each dot in a 1←10 machine, upon the act of dividing by 9, gives one tally mark in each column to its right and an extra tally mark in the "remainder section" at the end.

Two dots in a box will this give 2 tally marks in each of these positions; three dots in a box, three, and so on.



So dots representing the number 21023 in a 1←10 machine give tally marks as shown when divided by 9



and the answer 2| 2+1| 2+1| 2+1+2 R 2+1+2+3 emerges. This matches the opening example (except that we're not recording zeros here). We're also observing that 3 dots in the rightmost box match a remainder of 3 upon division by 9.

We could also take a purely arithmetical approach. Noticing, for example, that 10000=9999+1, we readily see

1÷9 = 0 R 1
10÷9 = 1 R 1
100÷9 = 11 R 1
1000÷9 = 111 R 1

etc.

We deduce then, from 20000 = 10000 + 10000 and 300=100+100+100 , for example, that

2÷9 = 0 R 2	3÷9 = 0 R 3	
20÷9 = 2 R 2	30÷9 = 3 R 3	etc.
200÷9 = 22 R 2	300÷9 = 33 R 3	
2000÷9 = 222 R 2	3000÷9 = 333 R 3	

Thus

2312÷9=(2000+300+10+2)÷9=222 R 2 + 33 R 3 + 1 R 1 + 0 R 2

and we see that this gives the answer 2 | 2+3 | 2+3+1 R 2+3+1+2=256 R 8

	2	**2**	**2**	**R 2**
		3	**3**	**R 3**
			1	**R 1**
+				**R 2**
=	**2**	**|2+3**	**|2+3+1**	**R 2+3+1+2**

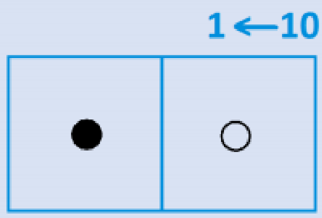
The partial sums of the digits naturally appear!

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Okay. What if we do start with a multiple of 9. How do we deduce from this algorithm that there is zero remainder? [Can you explain this true statement: A number is divisible 9 only if the sum of its digits is a multiple of 9?]

Wild Exploration 2: The number 9 can be also represented in a 1←10 machine by a dot and an antidot as shown.



Look at the number 21023 in a 1←10 machine and, starting at the left, make dot/anti-dot copies of 9 appear in your machine. (Add some antidots in your picture, and some dots to counteract them.) Do you see the computation 21023÷9 = 2 | 2+1 | 2+1+0 | 2+1+0+2 ***R*** 2+1+0+2+3 naturally emerging?

Wild Exploration 3: Here's a strange way to divide by 8. Write down the first digit of the number. Double it and add it to the next digit. Double this answer and add it to the third digit. And so on. Keep doubling the previous obtained answer and add it to the next digit of the number. You are now ready to read off the answer to your division problem!

$\boxed{21023 \div 8}$

2
1 + 2x2 = 5
0 + 2x5 = 10
2 + 2x10 = 22
3 + 2x22 = 47

= 2|5|10|22 R 47
= 2627 R 7

This is by no means an efficient way to divide by 8, but the interesting question is: Why does this method work?

Care to make an inefficient approach to dividing by 7?

Wild Exploration 4: Might you wish to explore decimals? Can this dividing-by-nine algorithm be used to compute 214.32÷9, for instance? How? Do you get an infinitely long decimal result?

DIVISIBILITY BY 7

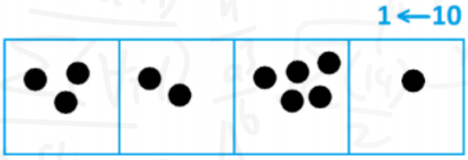
SOLUTION

Let's look numbers represented in 1←10 machines.

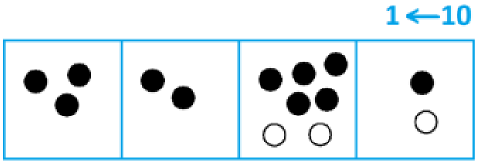
We're interested in finding groups of 7 and seeing if there will be any remainders or not. The procedure doesn't claim to tell us what the answer to the division problem shall be nor what remainder we'll get, only whether or not well be able to find complete groups of seven.

Since we don't care about how many groups of 7 we can find, we are thus free to add extra groups of seven in any way we like to our pictures to help us answer our basic YES/NO question. So let's be clever!

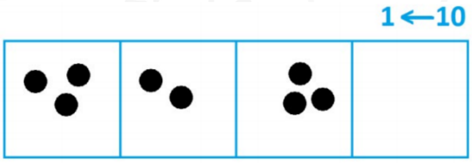
Suppose we have a single dot in the rightmost box of a number in a1←10 machine. This picture shows the number 3251.



Let's add three groups of 7 to this picture, actually, three anti-groups of 7!
That is, let's add 21 antidots.



This has us now looking for whole groups of 7 in the left three boxes: we made the rightmost box empty. That is, we're now looking for groups of 7 in a picture of the number 325−2=323.



Are there a whole number group of 7s here? If the answer is YES, then there were a whole number of groups of 7 in the original picture too. If the answer is NO, then we have 1, 2, 3, 4, 5, or 6 dots left over in that second-to-last box, which correspond to 10, 20, 30, 40, 50, or 60 dots in the rightmost box. None of these counts yield whole groups of seven, so our original number also leaves a remainder upon division by 7. This shows that answering our YES/NO question for these three left boxes precisely matches answering the YES/NO question for the original number.

This reasoning extends.

If we have a number with d dots in the rightmost box, add d copies of 21antidots. That leaves us answering the same YES/NO question for a number with the final digit d deleted and 2d subtracted from what remains.

a	**b**	**c** **-2d**	**d** **-d**

a	**b**	**c-2d**	

There are a whole number of groups of 7 in this new smaller number only if the original number had a whole number of groups of 7.

And this is it! This is the divisibility rule for 7 described!

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Here's a divisibility rule for the number 11. To determine if a number is divisible by 11, delete the final digit from the number and subtract that digit from what remains. Then the original number is divisible by 11 only if this new number is.

a) Can you make sense of the rule? Practice with some examples.
b) Can you explain the rule?

Extra: Use this rule to show that a number of the form a I b I c I d is divisible by 11 only if the alternating sum of its digits, a - b + c - d , is. (There is nothing special about four-digit numbers here.)

Wild Exploration 2: Here's a divisibility rule for the number 17. To determine if a number is divisible by 17, delete the final digit from the number and subtract five times that digit from what remains. Then the original number is divisible by 17 only if this new number is.

a) Can you make sense of the rule? Practice with some examples.
b) Can you explain the rule?

Wild Exploration 3: Here's a divisibility rule for the number 13. To determine if a number is divisible by 13, delete the final digit from the number and add four times that digit from what remains. Then the original number is divisible by 13 only if this new number is.

a) Can you make sense of the rule? Practice with some examples.
b) Can you explain the rule?

Wild Exploration 4: Here's yet another divisibility rule for the number 9. To determine if a number is divisible by 9, delete the final digit from the number and add that digit from what remains. Then the original number is divisible by 9 only if this new number is.

a) Can you make sense of the rule? Practice with some examples.
b) Can you explain the rule?

Wild Exploration 5: Make up a divisibility rule for the number 31. Make up ones for each of 19, 23, and 101 as well!

DIVISIBILITY BY 37

SOLUTION

The algorithm relies on the fact that 999 is a multiple of 37. (It's 37×27 .)
[This might be enough of a hint for your students to figure out what is going on on their own.]

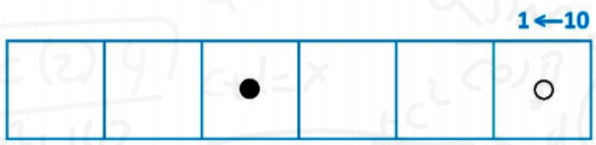
Let's look at numbers represented in 1←10 machines.

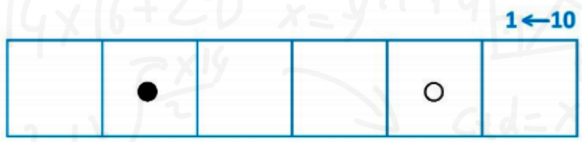
We want to assess whether or not, in identifying groups of 37, we'll see any remainders. No remainders present yields a YES answer to our question (the number is divisibly by 37), some remainders present yields a NO answer.

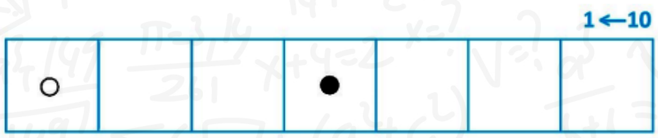
Since we don't care about how many groups of 37 we can find, we are free to add in extra groups of 37, or anti-groups of 37, in any way we like help answer our basic YES/NO question.

So, let's be clever! Let's work with groups of 999. (Each is a set of twenty-seven 37s.)

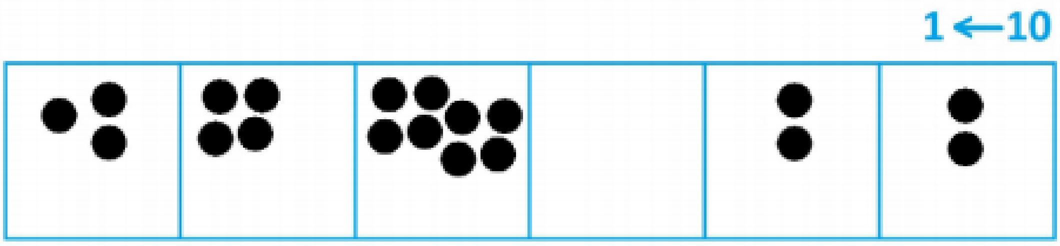
Think of 999 as 1000 - 1 . Here is one group of 999, ten groups of 999, and one-thousand groups of -999.



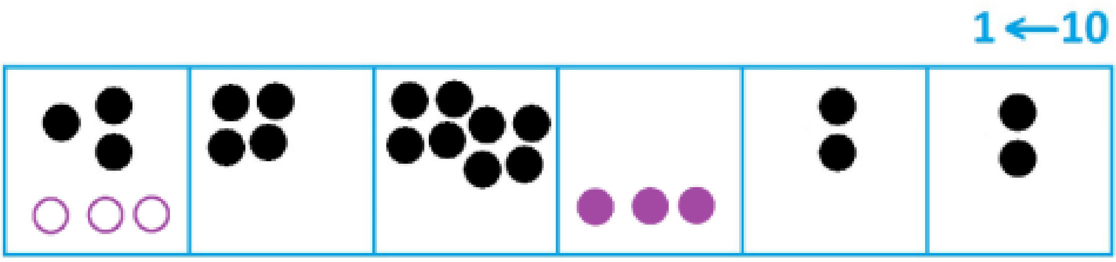




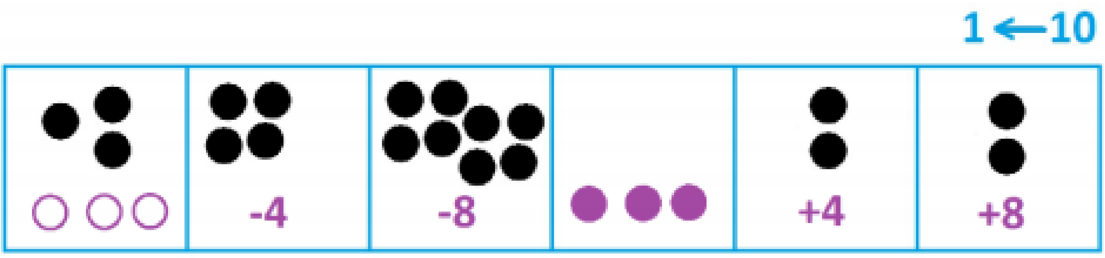
to a number such as 348022 in a 1←10 machine



add some groups of anti 999. This won't affect our analysis of whether or not finding groups of 37 leaves remainders. Let's add them in such a way that deletes the first digit of the number.



Thus analyzing the divisibility of 348022 has been reduced to analyzing the divisibility of 48322 . And in the same way this can reduced to analyzing 8362 and then to 370.



It is clear now how the divisibility rule works.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: The number 99999 is divisible by 271. What impractical divisibility rule can you construct from this?

Wild Exploration 2: The number 99 is divisible by 11. What divisibility rule can you create for the number 11 that first involves deleting the first digit of the given number. (How different is this rule from any rules for 11 you already know?)

Wild Exploration 3: You meet a Martian. It tells you, because they have three fingers on two hands, that they write all their numbers in base 6. But the Martian then claims that the very same divisibility rule of this essay works for the number forty-three (which they write as 111) in Martian: *Delete the first digit of the given number and add to the digit three places further in. The original number is divisible by "43" only if this new number is.* Is this true? Does this divisibility rule for work in Martian?

Wild Exploration 4: The number 1001 is divisible by 13. Use this to create a divisibility rule for the number 13 that starts by deleting the first digit of the given number.

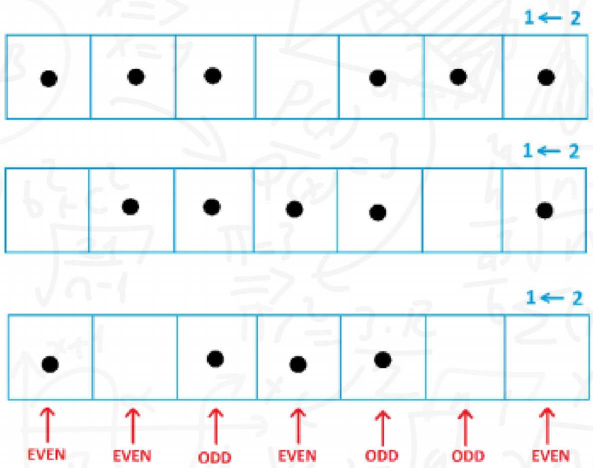
Wild Exploration 4: Look at all the factors of 9, 99, 999, 9999, 99999, … and of 11, 101, 1001, 10001, 10001, … to create new divisibility rules that start by deleting the first digit of the given number.

A STRATEGY FOR NIM

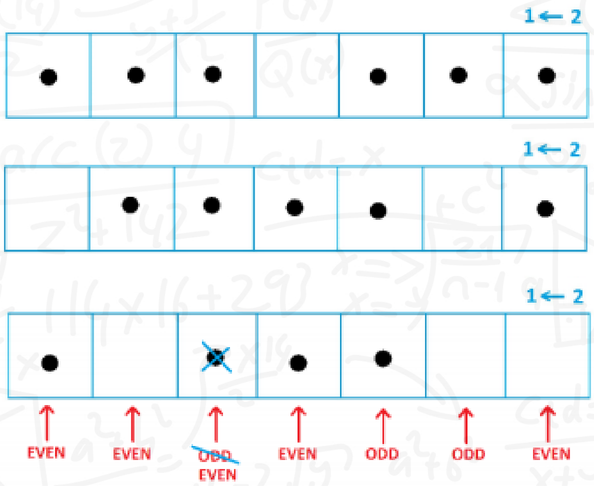
SOLUTION

No matter the number of piles or the size of the piles it is always possible to find a NIM move that turns a diagram with one or more columns containing an odd count of dots into a diagram with all columns containing an even count of dots.

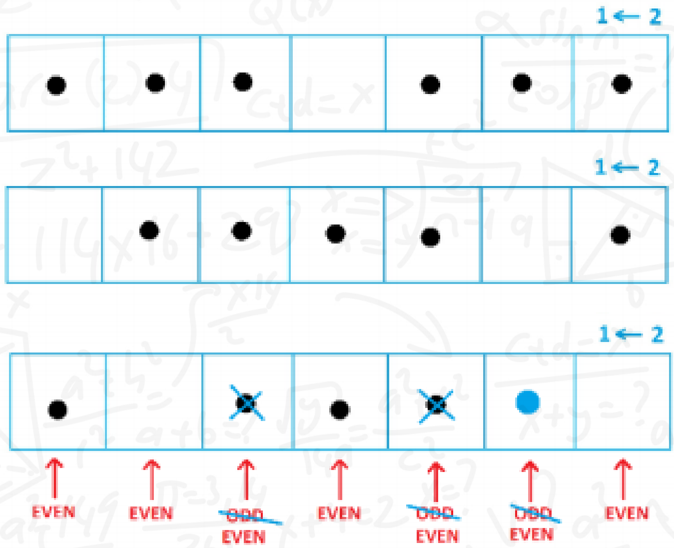
Consider this diagram with three piles each containing a large number of pebbles.



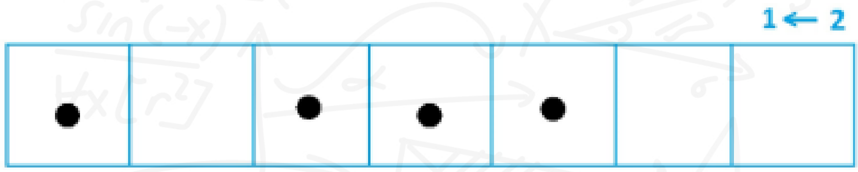
Choose any dot in the leftmost "odd column" and delete it.



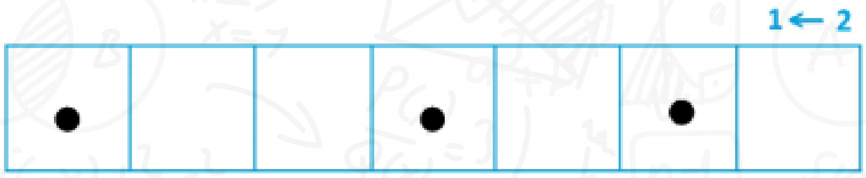
Next, either add or delete dots to the right of the deleted dot to give each column an even count of dots.



In this picture we have change the third pile from



to

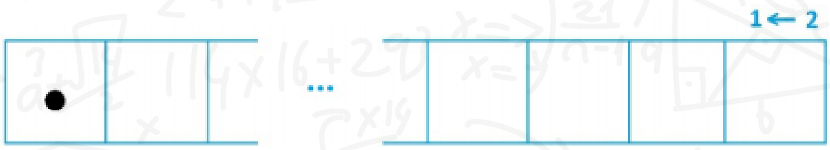


That is, we changed the third pile from 92 pebbles to 74 pebbles, a smaller number. This corresponds to taking pebbles out from a pile and so is a valid NIM move. But this begs the question:

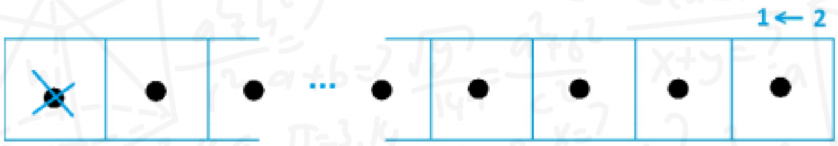
> In a 1←2 machine code, if we delete a dot and change some or all of the boxes to its right, is the result sure to be the code of a smaller number?

What's the worst possible case scenario? It would deleting just the one dot and adding the maximal number of dots possible to its right.

Does changing this picture



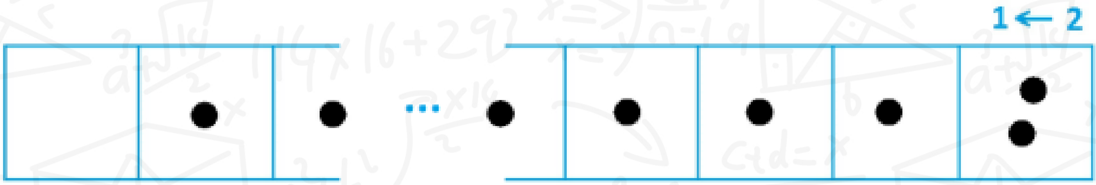
to this picture



give us a smaller number and so still represent the act of removing pebbles from a pile?

Well, yes!

Can you see that with unexplosions that the first of these pictures is equivalent to this picture?



And so it represents a number one larger than the second picture!

We have

> *Given a NIM game with a matching diagram with some columns containing an odd count of pebbles, there is always a valid NIM move that will yield a matching diagram with all columns containing an even count of pebbles.*

Now, on the other hand, suppose you are handed a NIM scenario with matching diagram all of whose columns possess an even count of dots. (Call this an EVEN SCENARIO.) Whatever move you make changes the dots in one row of the diagram. In fact, we can be sure that the state of at least one box will change (if not, you haven't made a move). If a box loses a dot or if a box gains a dot, the column containing that box has turned to an "odd column."

> *Given a NIM game with a matching diagram with all columns containing an even count of pebbles, any move is sure to produce a diagram with at least one column containing an odd count of pebbles.*

Let's call a NIM scenario with at least one column containing an odd count of pebbles an ODD SCENARIO. You, as a savvy player, now have a winning strategy if presented with an ODD SCENARIO: always play to give your opponent an EVEN SCENARIO. Your opponent will be forced to hand you back an ODD SCENARIO, which means there is at least one pebble remaining on the table. That is, your opponent simply cannot present you an empty set of piles. You have to be the one who does creates that, which means your win is certain with this strategy.

NEXT CHALLENGE: Can you create a mental schema so that you can do all the binary manipulations swiftly in your head while you play?

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Analyze misère NIM where the object of the game is NOT to win. Might there ever be a strategy for a player to ensure she never picks up the last pebble?

Wild Exploration 2: Every integer, positive or negative, can be uniquely represented in a $-1 \leftarrow 2$ with the digits 0 and 1. For example, ten is represented as 11110 and negative ten as 1010. These are called the negabinary codes of numbers.

Can one invent a NIM-like game with a winning strategy based on negabinary representations rather than binary representations?

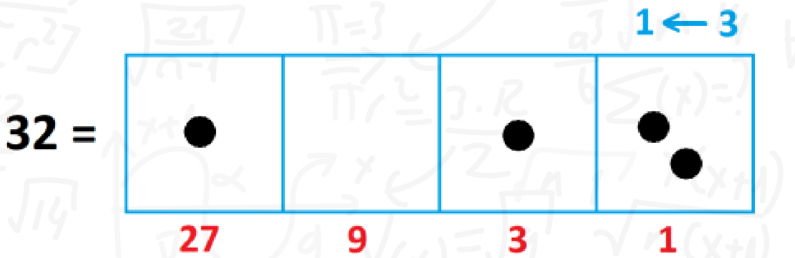
TWO-PAN BALANCE PUZZLES

SOLUTION

After some experimentation one comes to see that rocks of weights of 1 kg, 2 kg, 4 kg, and 8 kg—as for the binary codes of a 1←2 machine—don't give us Poindexter's results.

The key is to realize that placing rocks in a pan alongside the customer's rock has the effect of "subtracting weight" from the customer's rock: these rocks are behaving like "anti-rocks." And so perhaps if we go to a bigger machine and allow anti-dots in our codes, we can encode a larger range of values.

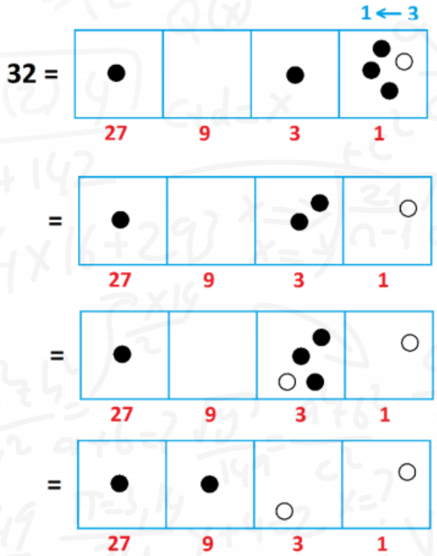
Let's not go too wild and just try the next sized machine, a 1←3 machine.



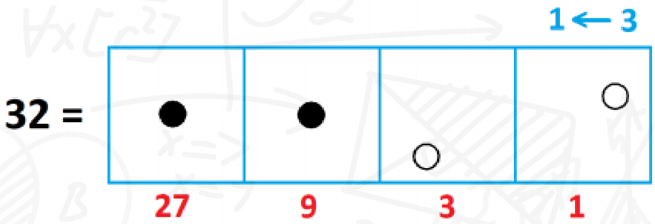
1: 0001	10: 0101	19: 0201	28: 1001	37: 1101
2: 0002	11: 0102	20: 0202	29: 1002	38: 1102
3: 0010	12: 0110	21: 0210	30: 1010	39: 1110
4: 0011	13: 0111	22: 0211	31: 1011	40: 1111
5: 0012	14: 0112	23: 0212	32: 1012	
6: 0020	15: 0120	24: 0220	33: 1020	
7: 0021	16: 0121	25: 0221	34: 1021	
8: 0022	17: 0122	26: 0222	35: 1022	
9: 0100	18: 0200	27: 1000	36: 1100	

The code for 32, say, is 1012. So, with four rocks of weights 1 kg, 1kg, 3 kg, and 27 kg in the left pan, we can balance of rock of weight 32 kg in the right pan. The code for 35 is 1022 and requires five rocks to balance … as it stands.

But if we do indeed allow for anti-rocks, that is, for antidots in our codes, then each 2 in a code can be "exploded away" by adding a dot and antidot to it.



This leaves us codes for numbers in a 1←3 with the digits 0 , 1, and −1. In particular, we see that each of the numbers 1 through 40 have codes with at most four digits. (What's the code for 41 with these digits?)

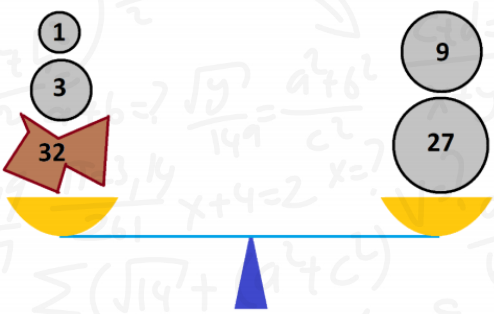


1: 0|0|0|1	10: 0|1|0|1	19: 1|-1|0|1	28: 1|0|0|1	37: 1|1|0|1
2: 0|0|1|-1	11: 0|1|1|-1	20: 1|-1|1|-1	29: 1|0|1|-1	38: 1|1|1|-1
3: 0|0|1|0	12: 0|1|1|0	21: 1|-1|1|0	30: 1|0|1|0	39: 1|1|1|0
4: 0|0|1|1	13: 0|1|1|1	22: 1|-1|1|1	31: 1|0|1|1	40: 1|1|1|1
5: 0|1|-1|-1	14: 1|-1|-1|-1	23: 1|0|-1|-1	32: 1|1|-1|-1	
6: 0|1|-1|0	15: 1|-1|-1|0	24: 1|0|-1|0	33: 1|1|-1|0	
7: 0|1|-1|1	16: 1|-1|-1|1	25: 1|0|-1|1	34: 1|1|-1|1	
8: 0|1|0|-1	17: 1|-1|0|-1	26: 1|0|0|-1	35: 1|1|0|-1	
9: 0|1|0|0	18: 1|-1|0|0	27: 1|0|0|0	36: 1|1|0|0	

Jargon: Base-three codes using the digits 0 , 1, and −1 are called "balanced ternary" codes.

These codes explain the system Poindexter was using. He had four special rocks of weights 1 kg, 3 kg, 9kg, and 27 kg, and, via trial-and-error, balanced a customer's rock with these four rocks. Each customer's rock will balance, and the balance is given by the balanced ternary code of its weight.

For example, 32 has balanced ternary code 1|1|−1|−1. Each non-zero digit means "use the rock whose weight matches the place of that digit", with +1 specifically meaning "place that rock opposite the customer's rock" and −1 meaning "place that rock with the customer's rock."



EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Grizelda Bumblesnort later put Poindexter out of business. She too used a simple two-pan balance and four special rocks, but made the following claim:

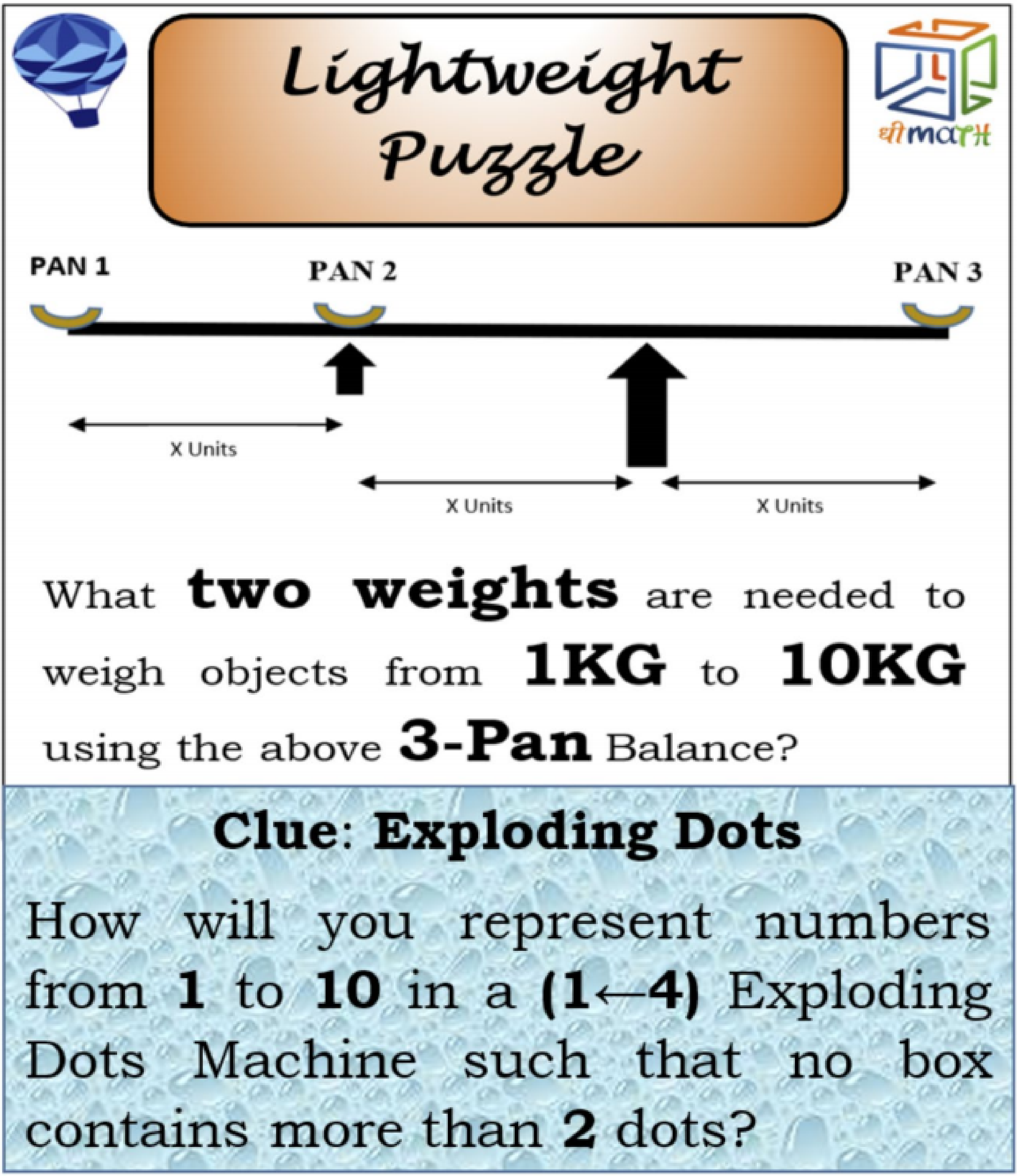
> *Do you have a rock? And do you want to know its weight? If it weighs a whole number of kilograms between 1 kg and 81 kg, come to me! I'll figure out its weight for you.*

What are the weights of her four special rocks?

[**Hint:** Grizelda never claims that she'll make her rocks balance with a customer's rock, only that she can deduce what the customer's rock-weight must be.

Wild Exploration 2: Global Math Project ambassador Kiran Bacche recently shared a puzzle on social media (@KiranABacche) he made up about weighing rocks. Can you solve his puzzle?

Can you make up your own special puzzles?



ANOTHER TWO-PAN BALANCE PUZZLES

SOLUTION

Digging into the number 50's "relationship" with the powers of two can be revealing.
We certainly have

$$32+16+2=50$$

but the weights 2, 16 ,and 32 kilograms are not permitted to sit on the same one side of the balance. The equation, visually, has 50 on the right, and it also has 16 on the left, which is permitted in terms of placing weight on the balance pan, but 2 and 32 are also on the left and that is not permitted.

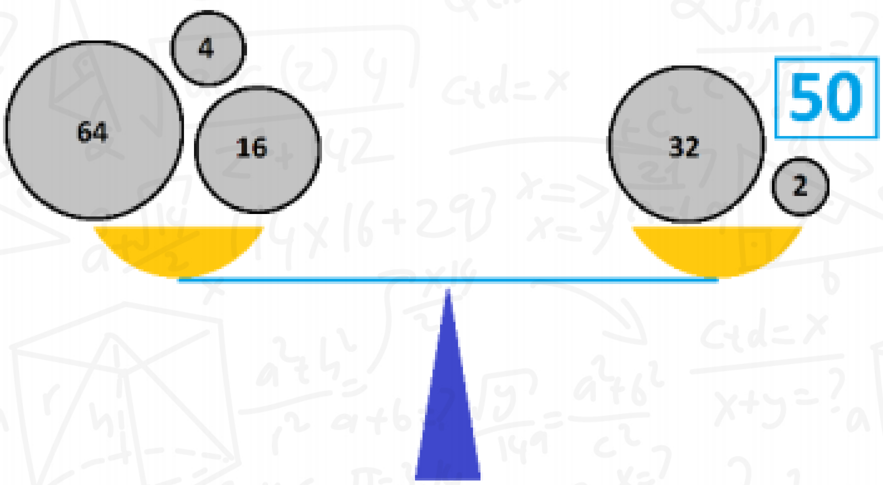
But adding 2 and 32 to each side of the equation gives

$$(32+32)+16+(2+2)=50+32+2$$

or

$$64+32+4=50+32+2,$$

which represents a balanced, permissible, equation.



Can we obtain a balanced solution with the weight 50 on the left?

Consider writing 50=32+16+2. We need 16 on the left, and we can obtain this by adding 16 to each side of the equation.

$$50+16=32+(16+16)+2$$

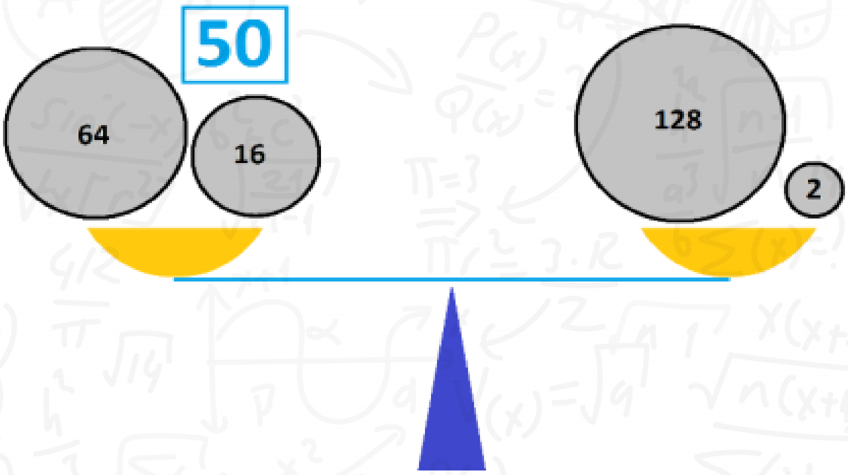
This reads

$$50+16=32+32+2=64+2.$$

But 64 cannot be on the right. So, let's try adding 64 to each side. This yields

$$50+64+16=128+2,$$

and we see we have a permissible balanced solution.



It now feels that it might be always be possible to balance a weight of any weight a whole number **N** of kilograms placed on either side of the balance scale with this sort of jiggling. Can we detail a procedure?

Given a weight of **N** kilograms

1. Write **N** as a sum of powers of two.

2. If we want the weight **N** to be in the left pan write the equation as " **N** =sum." If we want the weight **N** in the right pan, write the equation as "sum = **N** ."

For example, with ***N*** = 55 , we have ***N*** = 1+2+4+16+32.

The weight on the left.

$$\boxed{55} = 1 + 2 + 4 + 16 + 32$$

The weight on the right.

$$1 + 2 + 4 + 16 + 32 = \boxed{55}$$

3. Look at the smallest power of two in the sum. If is it on the “allowed” side of the equation, do nothing. If it is on the incorrect side of the equation, add that power of two to each side of the equation and perform any chain of computations that occur among the powers of two.

The weight on the left.

$$\boxed{55} = 1 + 2 + 4 + 16 + 32$$

$$\begin{aligned}\boxed{55} + 1 &= (1+1) + 2 + 4 + 16 + 32\\ &= (2+2) + 4 + 16 + 32\\ &= (4+4) + 16 + 32\\ &= 8 + 16 + 32\end{aligned}$$

The weight on the right.

$$1 + 2 + 4 + 16 + 32 = \boxed{55}$$

4. Repeat step 3 with the next smallest power of two that is on an incorrect side, and then the next smallest, and the next. Repeat until we have an expression with all powers of two in permissible positions.

The weight on the left.

$$\boxed{55} = 1 + 2 + 4 + 16 + 32$$

$$\begin{aligned}\boxed{55} + 1 &= (1+1) + 2 + 4 + 16 + 32\\ &= (2+2) + 4 + 16 + 32\\ &= (4+4) + 16 + 32\\ &= 8 + 16 + 32\end{aligned}$$

$$\boxed{55} + 1 + 16 = 8 + 64$$

$$\boxed{55} + 1 + 16 + 64 = 8 + 128$$

The weight on the right.

$$1 + 2 + 4 + 16 + 32 = \boxed{55}$$

$$1 + 8 + 16 + 32 = 2 + \boxed{55}$$

$$1 + 64 = 2 + 8 + \boxed{55}$$

The only trouble with this reasoning is that it is not clear that algorithm we’ve designed will stop. How do we know that each step of fixing doesn’t produce a larger power of two that needs fixing?

Hmm. This has me worried!

Let’s look at the two equations we have for ***N*** = 55 in the example. One equation says that

$$55 = -1 + 8 - 16 - 64 + 128$$

and the other says

$$55 = 1 - 2 - 8 + 64 .$$

Those minus signs have me thinking that maybe we should be working with the powers of negative two, not the powers of two. For example, the first of these equations can be rewritten as

$$-55 = 1 + (-2)^3 + (-2)^4 + (-2)^6 + (-2)^7,$$

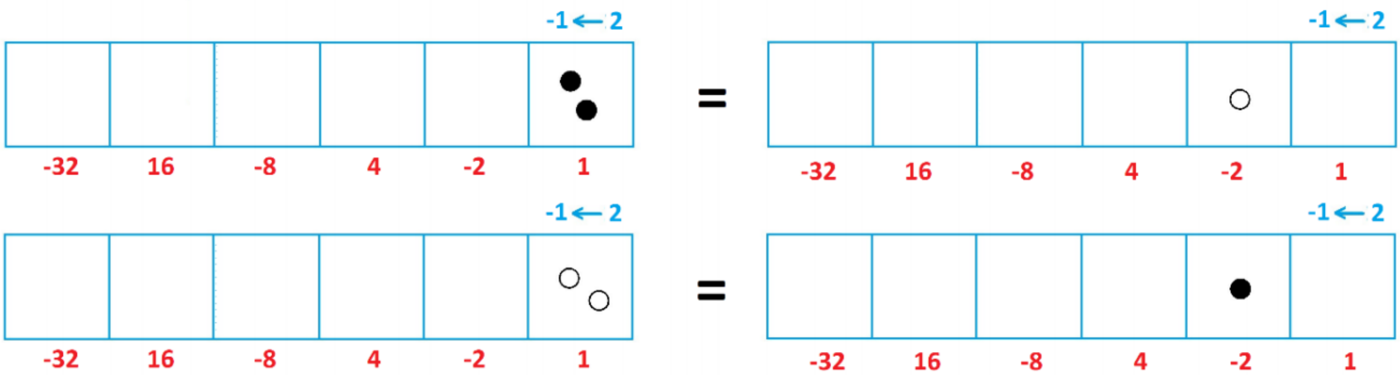
(I introduced a minus sign to look at −55 instead of positive 55 to align with these powers). And the second can be rewritten as

$$55 = 1 + (-2)^1 + (-2)^3 + (-2)^6.$$

So should we be looking at a −1←2 machine rather than a 1←2 machine that works with the powers of two?

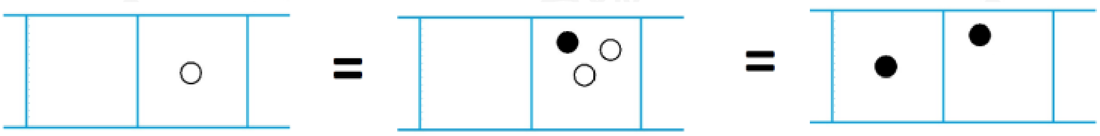
How does a −1←2 machine work?

Well, in such a machine, two dots in one box explode away to be replaced by one antidot, one place to their left, and two antidots in one box explode away to be replaced by a dot, one place to their left.

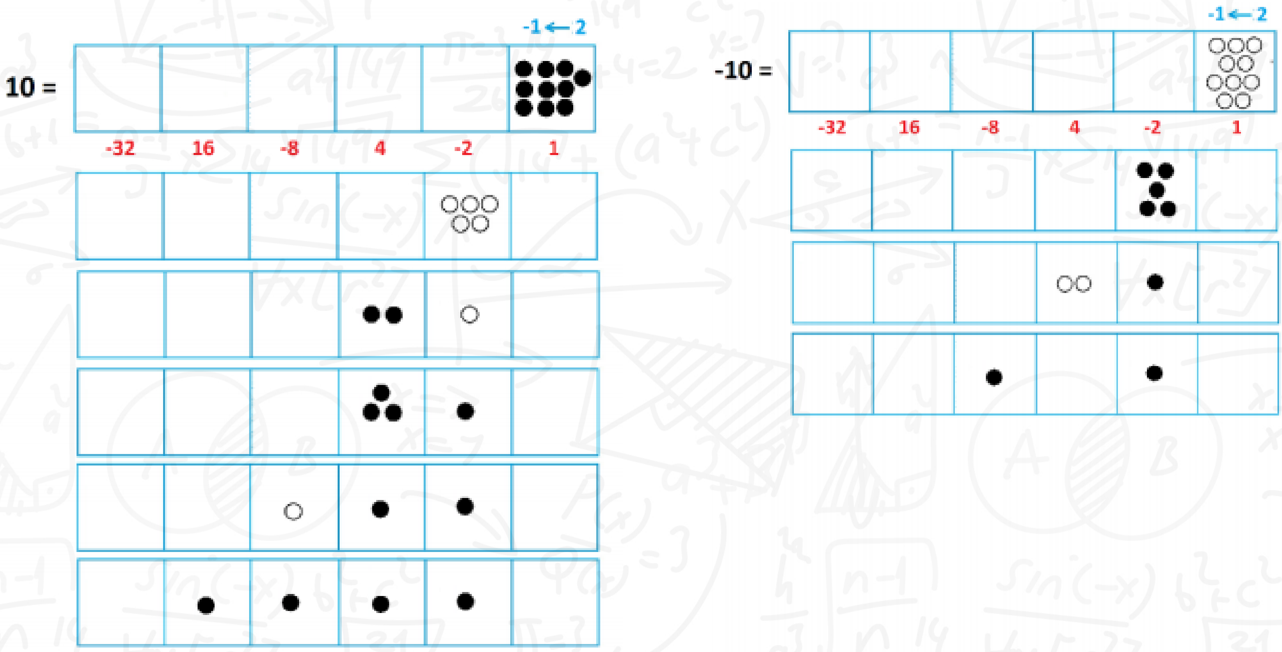


One sees that this is a machine giving codes of numbers in base negative two, with each box containing at most one dot or one antidot.

But we can go little bit further. Any box that contains an antidot, can be replaced by two single dots. We see this by adding a dot/antidot pair and performing one explosion.



This means that any number, positive or negative, can be represented as a code in a $-1 \leftarrow 2$ machine with nothing but dots with at most one dot per box. Either place **N** dots or **N** antidots in the rightmost box (for the positive integer **N** or the negative integer **−N**), explode away pairs of dots and antidots from the rightmost box. If an antidot is left behind, replace it with two dots as shown above, and then repeat this procedure for the second box from the right, then the third box from the right, and so on. What will be left behind is a representation of **N** or **−N** with single dots in boxes, that is, a representation of **N** or **−N** as a sum of single powers of -2 .

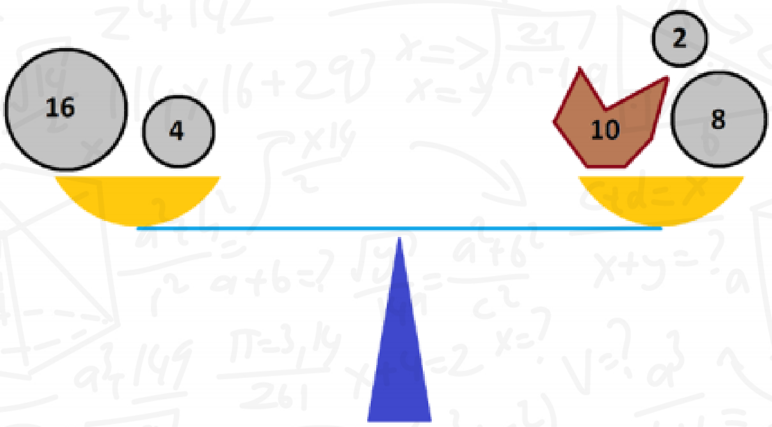


Positive ten is 11110 *in base negative two.*
Negative ten is 1010 *in base negative-two*

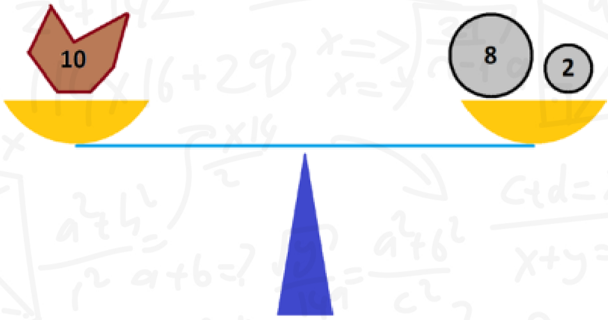
So this gives us a means to create balance. For example, from “positive ten = 11110 ,” that is, from

$$16+(-8)+4+(-2)=10$$

move the negative numbers to the other side of the equation to rewrite this as 4+16=10+2+8, to be read off as balanced scenario!



From “negatative ten = 1010,” that is, from (−2) + (-8) = -10 move the negative quantities to the other side of the equation to rewrite as 10=2+8. Balance!



Base negative-two representations give balanced solutions for any given weight placed on either side of the scale!

Jargon: A representation of a number in base negative-two using the coefficients 0 and 1 is called is negabinary represention.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Are balanced solutions unique? Given a specific weight N could there be more than one way to create a balanced scenario with that weight in the left pan or more than one way with that weight in the right pan? Is this the same question as: Is the negabinary representation of a number sure to be unique?

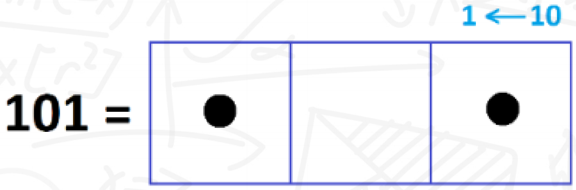
Wild Exploration 2: Is the algorithm we first described to in this essay sure to terminate? If so, how do you know? If it doesn’t, find an example of a weight that has us repeating step 3 an infinite number of times.

DIVIDING BY 101 PUZZLE

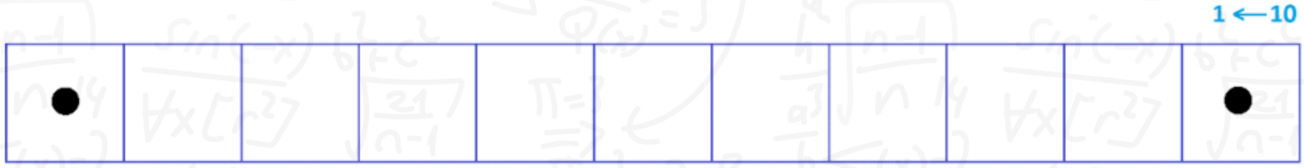
SOLUTION

Let's look at division by the number 101 in a 1←10 machine.

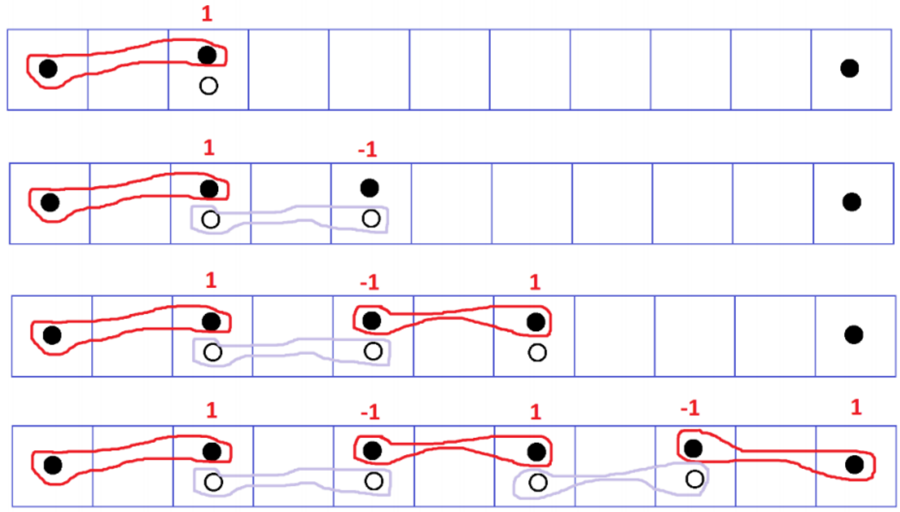
Now 101 looks like this



and we looking to divide numbers by this that have single dots at their beginnings and ends and nothing in between, namely, numbers like this one.



To find multiples of 101 we need to add in dots and antidots.



This picture shows that 10000000001is divisible by 101. The quotient is 1 | 0 | -1 | 0 | 1 | 0 | -1 | 0 | 1. What number is that? t

Some experimentation leads one to see that for a number of the form 100...001 to be divisible by 101 the number of zeros between the ones needs to be 1, 5, 9, 13, 17, ..., a count that is one more than a multiple of four. (In the pictures we've drawn, we need N red loops and N −1 purple "anti-loops." Each red loop/purple loop pair requires four boxes new boxes, and the final red loop requires one extra box before the final dot.)

Okay, we're essentially done! Now it is a matter of fiddly thinking to determine how many of the first 2018 numbers in the list 101, 1001, 10001, 100001, ... have a count of zeros that is one more than a multiple of four.

Let's focus on the list of numbers on more than a multiple of four 1, 5, 9, 13, 17, ... We see that 2017 is the largest number under 2018 that follows this form. Moreover, we have

1 = 0x4+1

5 = 1x4+1

9 = 2x4+1

⋮

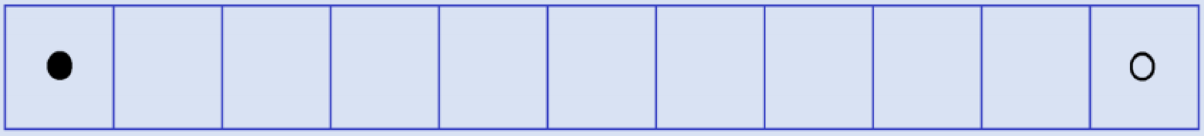
2017 = 504x4+1

So there are 505 numbers among the first 2018 numbers in the list 101, 1001, 10001, 100001, ... with the count of zeros one more than a multiple of four. That is, 505 of the numbers of the first 2018 numbers in the sequence 101, 1001, 10001, 100001, ... are divisible by 101!

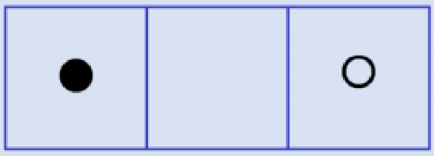
EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Consider the sequence of numbers 9, 99, 999, 9999, 99999, …. It is clear that every second number is divisible by 99. But to practice the ideas of essay, can you also see this is true by viewing each number in this sequence as a number that appears as follows in a 1←10 machine



with 99 represented as a dot-blank-antidot?



Wild Exploration 2: Explore which numbers in the list 11, 101, 1001, 10001, 100001, … are divisible by 11, and by 101, and by 1001, and so on.

Wild Exploration 3: Here is a curious divisibility rule for the number 101: To determine if a number is divisible by 101, delete its last digit and subtract that digit from the second-to-last digit of what remains. The original number is divisible by 101 only if this new (smaller) number is. One can repeat this procedure until one has a number small enough to readily recognize as a multiple of 101 or not. Can you make sense of this procedure, and then explain why it works? (See the “Divisibility by 7” puzzle.)

EXTRA

Students having studied the factor theorem in advanced algebra might realise that y^N+1 is divisible by $y+1$ only if N is odd. (We need $y=-1$ to be a zero of y^N+1.) Writing $N=2k+1$ and setting $y=x^2$ this reads

$$(x^2)^{2k+1}+1 \quad \text{is divisible by } x^2+1$$

that is, $x^{4k+2}+1$ is divisible by x^2+1.

Setting x = 10 can you now push forward to solve the challenge?

A POURING PUZZLE

SOLUTION

There are two possible pouring moves: to pour from container A or to pour from container B. Let's analyze what happens with each of these moves if we have container B, say, fraction x full. (We want to see x= $\frac{1}{4}$.) Container A is then fraction 1−x full.

And to keep things simple, let's just focus on container B in this analysis.

pour from A

Container B $= x + \frac{2}{3}(1-x) = \frac{2}{3} + \frac{x}{3}$

Container B $= x$

pour from B

Container B $= \frac{x}{3}$

With both pours we see that we are dividing the given quantity x by three. Perhaps it will be helpful, then, to think of our fractions as expressed in base three, that is, via an 1←3 machine: x=.abcd... for some digits a , b , c , d , ...

To be clear:

An ordinary base-ten decimal $.abcd\ldots$ means the quantity $\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}+\frac{d}{10000}+\cdots$ with a , b , c , d , ... any of the digits 0 , 1, ..., 9 .

In base three, a "decimal" $.abcd\ldots$ instead means the quantity $\frac{a}{3}+\frac{b}{9}+\frac{c}{27}+\frac{d}{81}+\cdots$ with a , b , c , d ,... any of the digits 0 , 1 , 2 .

So, if $x = .abcd\ldots$ as a base-three decimal, then

$$\frac{x}{3} = \frac{1}{3}\left(\frac{a}{3}+\frac{b}{9}+\frac{c}{27}+\frac{d}{81}+\cdots\right) = \frac{a}{9}+\frac{b}{27}+\frac{c}{81}+\frac{d}{243}+\cdots = .0abcd\ldots$$

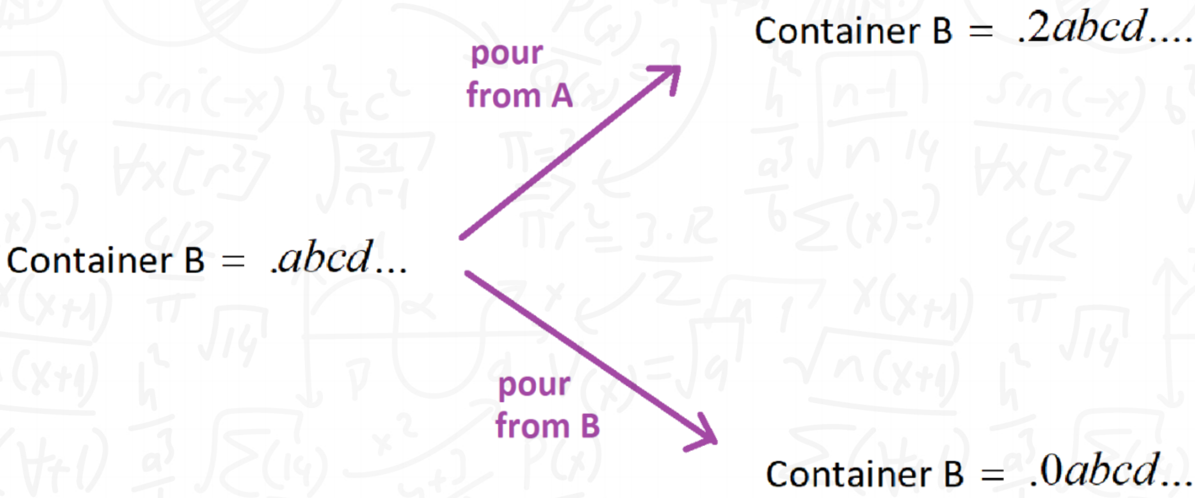
and

$$\frac{2}{3}+\frac{x}{3} = \frac{2}{3}+\frac{a}{9}+\frac{b}{27}+\frac{c}{81}+\frac{d}{243}+\cdots = .2abcd\ldots.$$

From now on, let's assume all our decimal expressions are base-three "decimals."

We have:

If container B is fraction .abcd... full, then a "pour from A" move changes this to fraction .2abcd ... (insert a 2 into the expression) and a "pour from B" move changes this to fraction .0abcd... (insert a 0 into the expression).



We start with fraction 0 of water in container B. We see that we can thus create the following fractions of water in that container

$$.2 = \frac{2}{3}$$

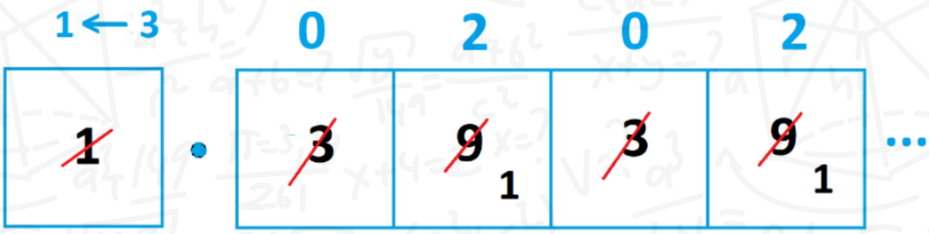
$$.02 = \frac{2}{9} \qquad .22 = \frac{8}{9}$$

$$.002 = \frac{2}{27} \qquad .202 = \frac{20}{27} \qquad .022 = \frac{8}{27} \qquad .222 = \frac{26}{27}$$

and so on.

We want fraction $\frac{1}{4}$ of water in container B. Can we create it?

To answer that we need to see what $\frac{1}{4}$ is as a base-three decimal. Let's compute $1 \div 4$ is a $1 \leftarrow 3$ machine.



(Ooh! If we're speaking base three, then $\frac{1}{4}$ should be written $\frac{1}{10}$, yes?)

We see that one-quarter is .0202020202.....

We can now answer the two questions.

a) With a finite number of pouring moves we can only create fractions that are expressed as finite decimals in base three. As one-quarter is an infinite decimal in this base, we'll never see container B precisely one-quarter full.

b) But we can create the fractions .2 and .202 and .202020202 and .2020202020202020202020202020202. That is, we can produce a fraction of water in container B as close to one-quarter as we like!

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1:

a) Could we ever see container B precisely one-third full?

If not, could we be close?

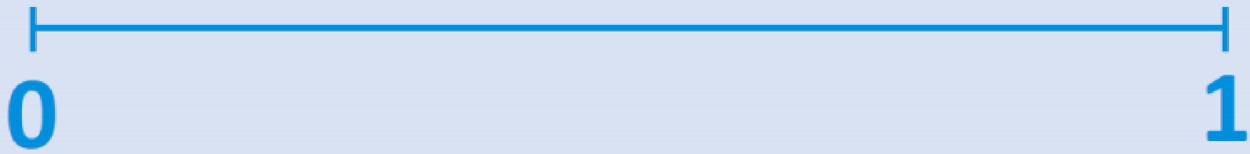
b) Could we ever see container B precisely one-half full? If not, could we be close?

See the essay at this webste to explores these thoughts.

http://www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_April-2020_Pouring-Puzzle.pdf

Wild Exploration 2:

Draw on section of the number line one unit long.



a) Show on the picture the locations of all the fractions that have a 0 or a 2 as first digit in their base-three "decimal" representations.

b) Show on the picture the locations of all the fractions that have only 0s and 2s for the first two digits in their base-three "decimal" representations.

c) Show on the picture the locations of all the fractions that have only 0s and 2s for the first three digits in their base-three "decimal" representations.

d) How would you describe the locations of the all the fractions that could be fractions of volume in container B for our puzzle?

Comment: Look up the Cantor Set on the internet.

Wild Exploration 3:

Suppose we changed the puzzle so that a "pouring move" consisted of pouring half the contents of one container into the other. Which fractions could we see, or come as close to seeing as we like, in conducting pouring moves?

Wild Exploration 4:

Suppose we changed the puzzle so that a "pouring move" consisted of pouring a third of the contents of one container into the other. Which fractions could we see, or come as close to seeing as we like, in conducting pouring moves?

Wild Exploration 5:

Develop a general theory about fractions of volume we could see in container B if a pouring move consisted of pouring proportion r% of the contents of one container into the other, even if r is an irrational number!

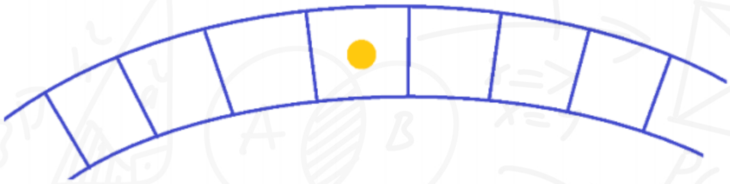
A MANGO SHARING-AND-EATING PUZZLE

SOLUTION

Okay. Let's start small. Consider a small counts of mangos.

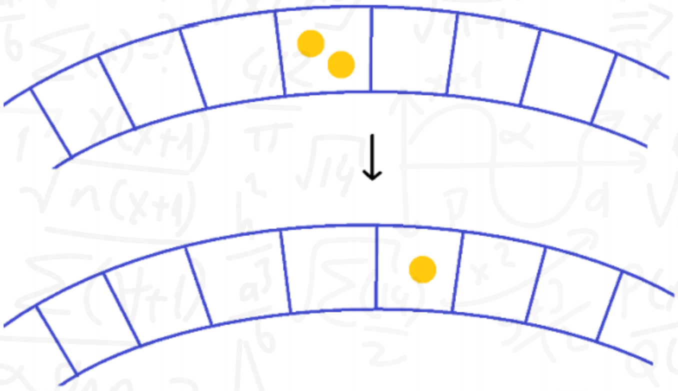
1 Mango

There isn't anything to do here. The situation stays with one person holding a mango.



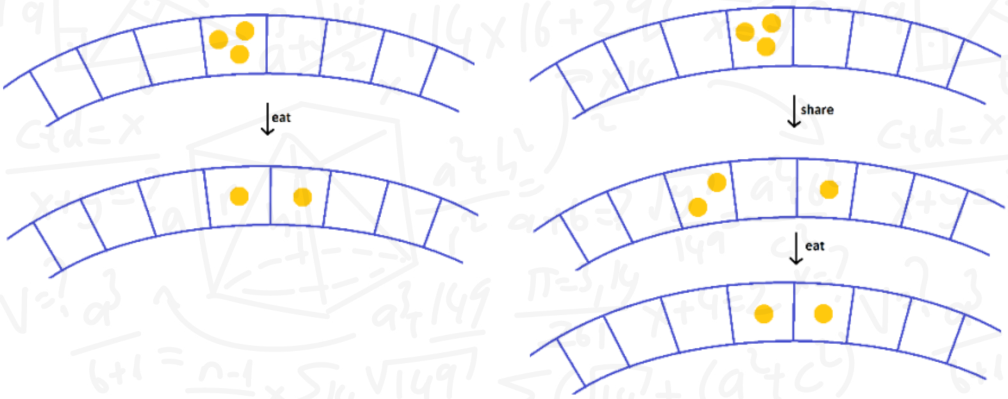
2 Mangos

The person holding the mangos can only eat one and pass one.



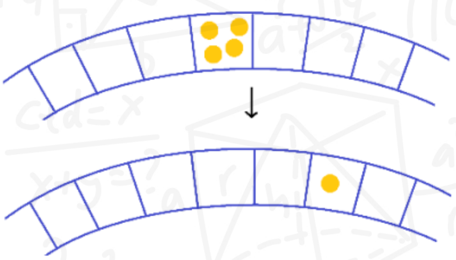
3 Mangos

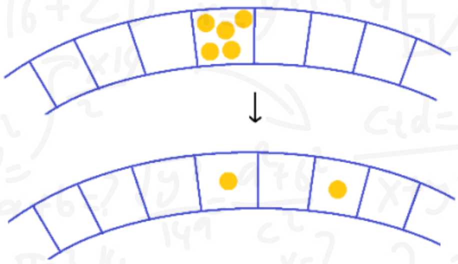
Here the person can eat right away or share and then a neighbor eats.



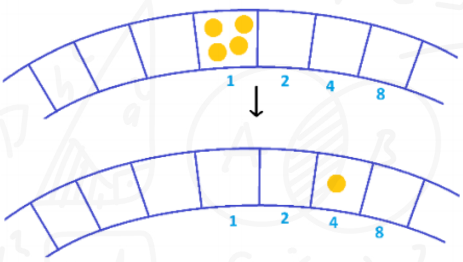
4 Mangos, 5 Mangos

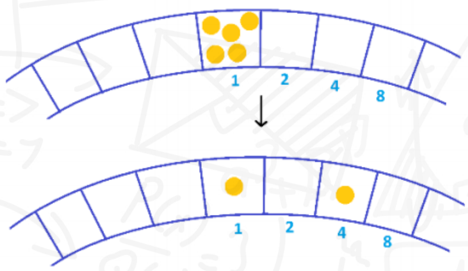
There are more options available in these cases and one can check they all seem to lead to the same final configurations of mangos.



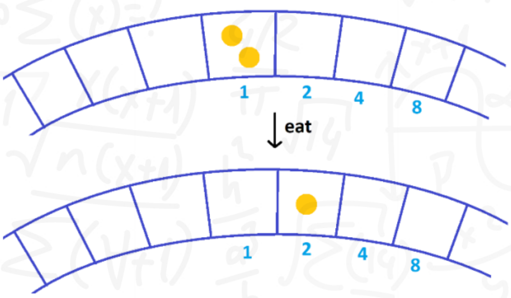


And after a while one can't help but notice that in an "eat" move, two mangos in any one cell of the diagram disappear and are replaced by one mango, one cell to the right. Is this puzzle really about a 1←2 machine in disguise?

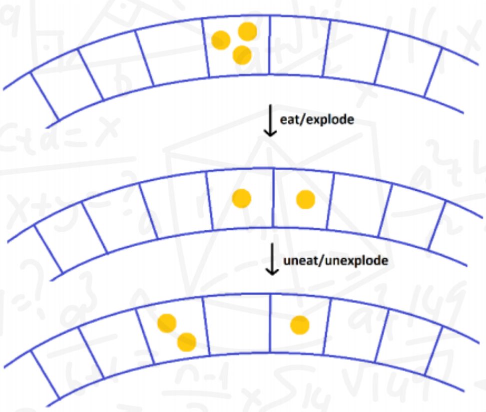




It looks like it, except that the explosions go in the opposite direction to what we are used to.



But there is the extra snag of a "share" move. Is that valid operation in a 1←2 machine? Well ... yes! Replacing three mangos in any one cell with two on one side and one on the other is just an explosion and unexplosion in a 1←2 machine.



So this puzzle is just about taking a set of mangos in the 1 position of a 1←2 machine and performing explosions and unexplosions until we settle on a pattern with at most one mango per cell.

We know that N dots in a 1←2 machine settle to the 1←2 machine code (binary code) of N , so N mangos in the hand of one person in this puzzle settle to the binary representation of N as mangos.

Since 2019 is 11111100011 in base two, we must settle to eight people holding a single mango, just as the puzzle suggests. And, just to be clear, since the base-two representation of 2019 is only eleven digits long, our mangos never "moved" around the entire circle of people. So it is indeed as though we are operating within a (non-circular) 1←2 machine.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Is it "obvious" that every number has a unique 1←2 machine code? That is, could a number be represented in base two (using the digits 0 and 1) in more than one way? (If so, we have more work to do for this puzzle!)

Wild Exploration 2: What if there were 11 people standing in a circle with one person holding 2019 mangos? Is if the case that we again end up with eight people each holding one mango? What if, instead, it were just 10 people in a circle? 9 people in a circle?

Further Reading:
Here's the original problem (Problem 4/1/31) and published solution
https://www.usamts.org/Tests/Problems_31_1.pdf
https://www.usamts.org/Solutions/Solutions_31_1.pdf

Here, again, is WildThinks blog and web app about this problem.
https://wildthinkslaboratory.github.io/smartblog/usamts/2020/01/29/usamts2.html

THE JOSEPHUS PROBLEM

SOLUTION

The first thing to note is that every student labeled with an even number is eliminated during the teacher's first round of the circle. For instance, with **N**=12 students, we are soon left playing a game with the six students numbered 1,3, 5, 7, 9, and 11, with student 1 acting as the first student, student 3 as the second student, student 5 as the third student, and so on.

1	~~2~~	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~	11	~~12~~
1		2		3		4		5		6	

Actually, we see that k th student among the odd-number labels is student number 2k-1 in the original numbering system. The winner of the six-person game is student number $W(6)$, and this is student $2W(6) - 1$ in the original numbering system. Thus we deduce that $W(12) = 2W(6) - 1$.

In general, we have

$$W(2N) = 2W(N) - 1.$$

Consider $N = 13$, an example with an odd number of students in the circle. Again, all the evennumbered students are immediately eliminated, and next eliminated is student 1.

~~1~~	~~2~~	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~	11	~~12~~	13
		1		2		3		4		5		6

This leaves us playing a game with 6 students. The k th student in this game is student number 2k + 1 in the original numbering system. We see this time that $W(13) = 2W(6) + 1$

In general, we deduce

$$W(2N + 1) = 2W(N) + 1$$

Actually, from $W(1) = 1$ and these two relations we can now compute all the values $W(2)$ and $W(3)$, $W(4)$ and $W(5)$, $W(6)$ and $W(7)$, and so on.

Do our two relations explain the curious connection to binary codes? To be clear, we want to show that if $N = 1ab...cd$ in binary, then $W(N) = ab...cd1$ in binary. Hmm.

This claim is true for $N = 1$, since $W(1) = 1$.

The claim is true for the $N = 2$ and $N = 3$, the numbers with two-digit binary codes.

$$W(2) = W(10_2) = 01_2 = 1$$
$$W(3) = W(11_2) = 11_2 = 3$$

Comment: I guess we need to distinguish between base-two and base-ten codes. Let's use the subscript 2 to denote a binary code from now on.

The claim is true for the $N = 4, 5, 6, 7$, the numbers with three-digit binary codes.

$$W(4) = W(100_2) = 001_2 = 1$$
$$W(5) = W(101_2) = 011_2 = 3$$
$$W(6) = W(110_2) = 101_2 = 5$$
$$W(7) = W(111_2) = 111_2 = 7$$

Okay, let's generalize. Suppose we know that the claim is true for all numbers with k digits in their binary codes. Does it follow that it is also true for all numbers with k+1 digits in their binary codes?

Yes!

Here's why.

Suppose ***M*** has binary code $1ab...cd_2$ with k+1 digits.

If ***M*** is even, d = 0 and ***M*** = 2***N*** with 2 ***N*** = $1ab..c_2$ which is a number with just k digits.

Since **N** has k digits, we know ***W*** (***N***) = $ab...c1_2$

But we also have

$$\begin{aligned} W(M) = W(2N) = 2W(N) - 1 & \\ &= 2 \times ab...c1_2 - 1 \\ &= ab....c10_2 - 1 \\ &= ab....c01_2 \end{aligned}.$$

which is $ab...cd1_2$ as hoped.

If ***M*** is odd, d = 1 and ***M*** = 2***N*** +1 with ***N*** = $1ab...c_2$, which is a number with just k digits.

Since **N** has k digits, we know (***W***) ***N*** = $ab...c1_2$

But we also have

$$\begin{aligned} W(M) = W(2N+1) = 2W(N) + 1 & \\ &= 2 \times ab...c1_2 + 1 \\ &= ab....c10_2 + 1 \\ &= ab....c11_2 \end{aligned}.$$

which is $ab...cd1_2$, as hoped.

So from the claim being true for all numbers with three-digit binary codes, it follows that it is also true for all numbers with four-digit binary codes, and then for all numbers with five-digit binary codes, and so on, and so on.

The claim about binary codes is always true!

(My brain hurts!)

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Do the following observations make immediate sense?

a) If the initial count of students sitting in the circle is a power of two, then student 1 is sure to win.

b) If the initial count of students sitting in the circle is one less than a power of two, then the highest numbered student is sure to win.

c) If the initial count of student is one more than a power of two (and not two), then student 3 is sure to win.

Wild Exploration 2: Continuing the table of data values it seems there is a curious pattern.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
W(N)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Between each power of two the values of the students who win run through the consecutive odd numbers, with the pattern being "reset to 1" at each power of two.

Is this pattern indeed true?

Wild Exploration 3:

a) Instead of counting in-out-in-out-in-out-… around the circle the teacher counts out-in-out-in-out-in- … instead. Care to analyse who wins this variation of the game?

b) Suppose the teacher deems every third student as "in" following the cycle out-out-in-out-out-inout-out-in, and so on. Care to analyse who wins in this variation? Or more generally, what if the teacher followed the cycle setting every r th student as "in." (Part a) asks about the case r = 2 and we just asked about the r = 3 case.) Can you develop a general theory about winners in general games?

c) Who wins in games with the teacher following a repeated cycle of "in-out-in, in-out-in, in-out-in"?

Wild Exploration 4:

Let's go back to the version of the game from this essay.

If $W(N) = ab...cd_2$, prove that $W(2N) = ab...c0d_2$ and $W(2N+1) = ab...c1d_2$

AMC 10A 2020 PROBLEM 21

SOLUTION

The numbers 289 and 17 are conspicuously choice. We notice that 289=17x17.

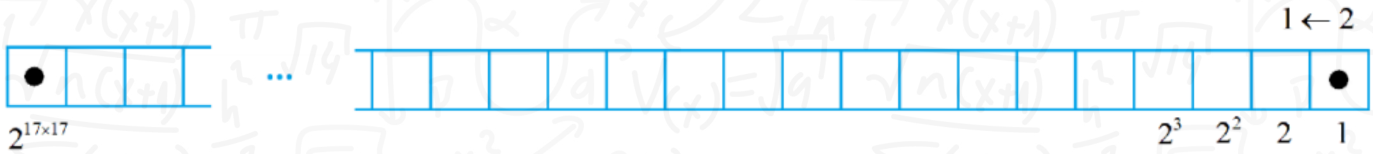
Even if I don't really get what the question is asking, I can at least see that we have to do something with

$$\frac{2^{17\times17}+1}{2^{17}+1}$$

That's too big a number to actually work out.

Maybe we can do the division to a limited degree? Since this question is all about the powers of two, maybe do the division in a 1←2 machine?

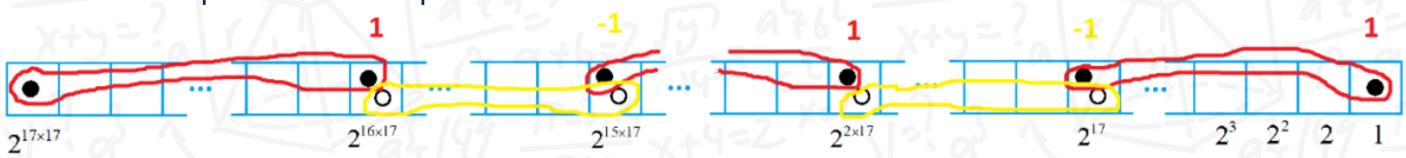
Here's $2^{17\times17}+1$ in the machine: two dots with a lot of empty boxes between them!



And we're looking for copies of $2^{17}+1$ which looks like this: two dots with 16 empty boxes between them.

Let's use dots and antidots to create copies of what we are looking for.

We can see copies and anti-copies of $2^{17}+1$



Ooh! Do we really end with a red loop?

We have a red loop from $2^{17\text{x}17}$ to $2^{16\text{x}17}$, and a red loop from $2^{15\text{x}17}$ to $2^{14\text{x}17}$,

and I guess all the way down to $2^{1\text{x}17}$ going to 1. Yes!

Okay, so

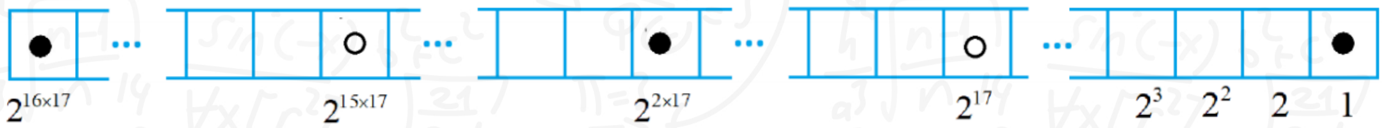
$$\frac{2^{17\times17}+1}{2^{17}+1} = 2^{16\times17} - 2^{15\times17} + 2^{14\times17} - 2^{13\times17} + \cdots + 2^{2\times17} - 2^{1\times17} + 1\,.$$

This feels good. But is it what we want?

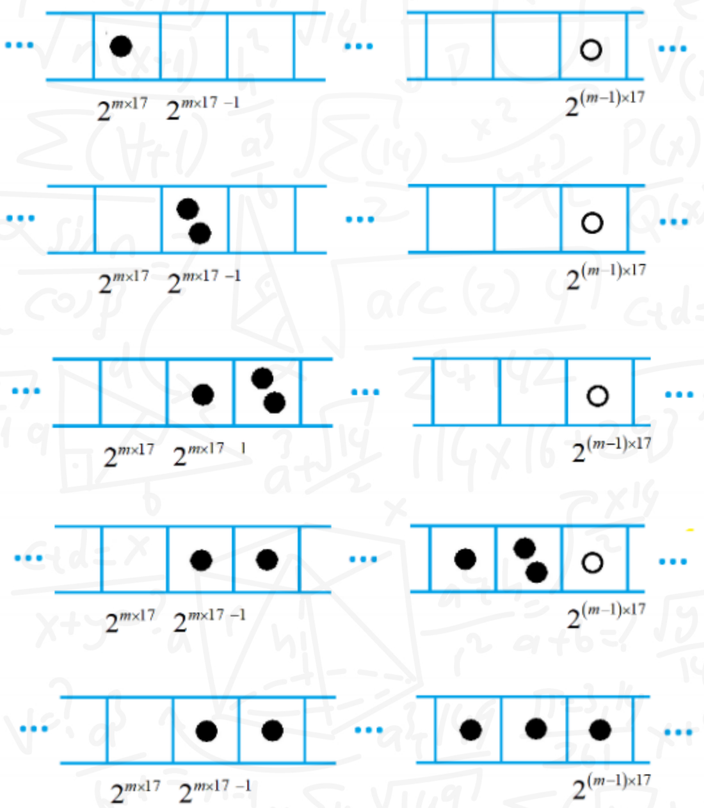
Rereading the question, we see it wants an answer in the form $2^{a_1} + 2^{a_2} + 2^{a_3} + \ldots + 2^{a_4}$ with all the exponents positive integers. More important, it wants one positive powers of two, no negatives!

Okay. We need to convert the number $2^{16\text{x}17}-2^{15\text{x}17}-2^{14\text{x}17}-2^{13\text{x}17}+\ldots+2^{2\text{x}17}-2^{1\text{x}17}+1$ into an equivalent expression with only positive sums of powers of two.

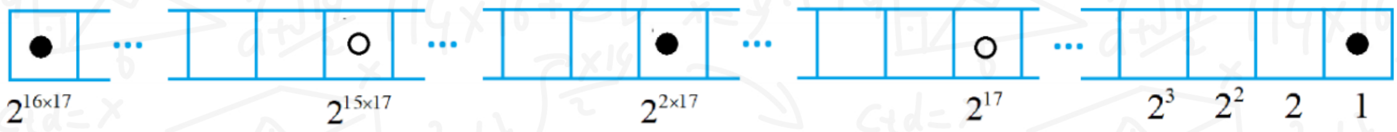
Well, in a 1←2 machine this number appears as follows.



And the way to "fix up" the appearance of antidots is to unexplode. And look what we can do!



Each “dot and anti-dot seventeen places to its right” pair is replaced with 17 dots. And there are eight such pairs. Ad when we do all the unexplosions we’ll have converted this picture with 8 such pairs and an extra dot at the end



into a picture with 8x17+1 = 137 dots, with a dot at largest power-of-two position $2^{16x17-1}$

Umm. What does the question want?

The question wants an answer in form $2^{a_1} + 2^{a_2} + 2^{a_3} + ... + 2^{a_4}$ (yep, got that) which is a sum of k powers of two, and it wants to know the value of k . Oh: k is the number of dots we have!

Answer: k = 137 .

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: What is the general value of $\dfrac{x^{n^2}+1}{x^n+1}$

Do we need n to be odd or to be even or to be prime or to be specially 17 for there to be no remainder?

Wild Exploration 2:

a) Show that each of the numbers 2^9+1 , $2^{121}+1$, and $2^{10201}+1$ are composite by finding a proper factor of each number. Is two raised to an odd square number, plus 1, always composite?

b) Is two raised to an even square power, plus 1, always prime?

DIVISIBILITY BY 5 IN BASE ONE-AND-A-HALF

SOLUTION

This problem has three elements to consider:

- Things being multiple of fives
- Alternating sums of digits
- The mechanics of a $2 \leftarrow 3$ machine

I don't know how these ideas are meant to mesh together, but it does feel natural to consider what an explosion does in a $2 \leftarrow 3$ machine to the alternating sum of digits you have so far.

So we're either considering

$-a+b$ changing to $-(a+2)+(b-3)=-a+b-5$

or

$a-b$ changing to $(a+2)-(b-3)=a-b+5$.

Either way, an explosion in a $2 \leftarrow 3$ machine does not affect whether or not the alternating sum of digits you have so far is a multiple of five.

So if we put in N dots in the rightmost box of a $2 \leftarrow 3$ machine (with alternating sum... $-0+0-0+N=N$ and perform explosions to get its $2 \leftarrow 3$ machine code, a | b | c | d | e say, the alternating sum of this code $a-b+c-\ldots-d+e$ differs from N by a multiple of five.

So N is a multiple of five precisely if the alternating sum is, just as the puzzle claims!

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Have we just proved that in any b←a machine, a number is divisible by a+b precisely when the alternating sum of the digits of in this machine its code is? (Does this seem to fit the divisibility rule for eleven in base ten?)