## A SUM OF FRACTIONS from BRILLIANT 

Our partner BRILLIANT (www.brilliant.org) has an astonishing library of general puzzles and innovative mathematics content. Here's a clever puzzle from Brilliant on converting fractions into decimals and making use of patterns. It's item 4 from this Practice Set: https://brilliant.org/practice/problem-solving-4/?chapter=divisibility-in-other-bases.


## BRILLIANT

## EXPLODING DOTS Topic:

Experience 8: Decimals in base 10.
Suggested Grade Level:
MIDDLE SCHOOL

## A SUM OF FRACTIONS from BRILLIANT

Have your students compute $1 \div 9$ in a $1 \leftarrow 10$ machine and thus rewrite the fraction $\frac{1}{9}$ as an infinite decimal. That might well be enough of a warm-up for this curious challenge from BRILLIANT.

Puzzle 4 from https://brilliant.org/practice/problem-solving-4/?chapter=divisibility-in-other-bases.

$$
S=\frac{1}{9}+\frac{1}{99}+\frac{1}{999}+\frac{1}{9999}+\frac{1}{99999}
$$

If $S$ is evaluated as a decimal, what is the $24^{\text {th }}$ digit to the right of the decimal point?

## Some Things Students Might Notice or Question

1. Can we just type this into a calculator?

The answer is, of course, YES! Have students try it. Does it prove to be helpful?
2. Should we work out the decimal expansions of the remaining four fractions in the sum?
3. Is there something special about the $24^{\text {th }}$ decimal place?
4. Why do we care?

## THE PUZZLE EXPLAINED

EXPERIENCE 8 of EXPLODING DOTS: Decimals in base 10.

Let's determine the decimal representation of each of the fractions in the sum.

From the warm-up we know

$$
\frac{1}{9}=.1111111 \cdots
$$

For $\frac{1}{99}$, we're looking for groups of 99 in this picture. (Let's write numbers rather than draw dots.)

$$
1 \leftarrow 10
$$



Two unexplosions allow us to see a group of 99 .


Another two unexplosions yields a next group of 99 .


And so on.
We're in a cycle and we see that

$$
\frac{1}{99}=.010101 \cdots
$$

It's fun to draw the picture for $\frac{1}{999}$ to see that sets of three unexplosions yield groups of 999 in the picture. We get

$$
\frac{1}{999}=.001001001 \cdots .
$$

And in one's mind eye, the pictures for

$$
\frac{1}{9999}=.000100010001 \cdots
$$

and

$$
\frac{1}{99999}=.0000100001000001 \cdots
$$

are also clear.
Adding the five decimals gives the decimal representation for the sum $\frac{1}{9}+\frac{1}{99}+\frac{1}{999}+\frac{1}{9999}+\frac{1}{99999}$.

$$
\begin{array}{r}
.11111111111 . . . \\
.0101010101 . . \\
.0010010010 . . \\
.0001000100 . . \\
+.0000100001 . . . \\
\hline=.1223231323 . .
\end{array}
$$

Each column will add together at least one, and at most five, 1 s . Thus, each decimal digit in the sum is either $1,2,3,4$, or 5 . We want the $24^{\text {th }}$ digit.

How many 1 s are in the $24^{\text {th }}$ column of this sum?

There is one from the fraction $\frac{1}{9}$.
There is one from the fraction $\frac{1}{99}$ as every second digit of this fraction is a 1 and 24 is a multiple of two.

There is one from the fraction $\frac{1}{999}$ as every third digit of this fraction is a 1 and 24 is a multiple of three.

There is one from the fraction $\frac{1}{9999}$ as every fourth digit of this fraction is a 1 and 24 is a multiple of four.

But there is not one from the fraction $\frac{1}{99999}$ as every fifth digit of this fraction is a 1 and number 24 is not a multiple of five.

Thus, the $24^{\text {th }}$ digit of $\frac{1}{9}+\frac{1}{99}+\frac{1}{999}+\frac{1}{9999}+\frac{1}{99999}$ is a 4.

## EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

## Wild Exploration 1:

a) Does the digit 5 ever appear in the decimal expansion of $\frac{1}{9}+\frac{1}{99}+\frac{1}{999}+\frac{1}{9999}+\frac{1}{99999}$ ? If so, where does a 5 first appear?
b) Here is Puzzle 5 from https://brilliant.org/practice/problem-solving-4/?chapter=divisibility-in-other-bases.

$$
S=\frac{1}{9}+\frac{1}{99}+\frac{1}{999}+\frac{1}{9999}+\frac{1}{99999}
$$

If $S$ is evaluated as a decimal, what is the length of its period of repetition?

Wild Exploration 2:
a) Enid computed $1 \div 11$ in a $1 \leftarrow 10$ machine and deduced that $\frac{1}{11}=.1|-1| 1|-1| 1|-1| \cdots$.


Can you make sense of what she did?
b) Enid also deduced that $\frac{1}{101}=.0|1| 0|-1| 0|1| 0|-1| \cdots$. Do you see how she concluded this?
c) When Enid was asked to find the $24^{\text {th }}$ digit to the right of the decimal point in the decimal expansion of $\frac{1}{11}+\frac{1}{101}+\frac{1}{1001}+\frac{1}{10001}+\frac{1}{100001}$ she wrote -4 . How did she come up with that answer do you think?

Let's take Enid's work and rewrite her decimals the way society expects to see decimals.
d) With unexplosions can you see that $.1|-1| 1|-1| 1|-1| \cdots$ equals $.090909 \cdots$ ? (Is $\frac{1}{11}$ indeed equal to $.090909 \cdots$ ?)
e) Convert $\frac{1}{101}=.0|1| 0|-1| 0|1| 0|-1| \cdots$ into a decimal society expects. Also find the expected decimal representations of $\frac{1}{1001}, \frac{1}{10001}$, and $\frac{1}{100001}$.
f) (OPTIONAL and HARD): What is the $24^{\text {th }}$ decimal digit of $\frac{1}{11}+\frac{1}{101}+\frac{1}{1001}+\frac{1}{10001}+\frac{1}{100001}$ as an answer society expects?

Wild Exploration 3: Try the remaining puzzles from https://brilliant.org/practice/problem-solving-4/?chapter=divisibility-in-other-bases.

