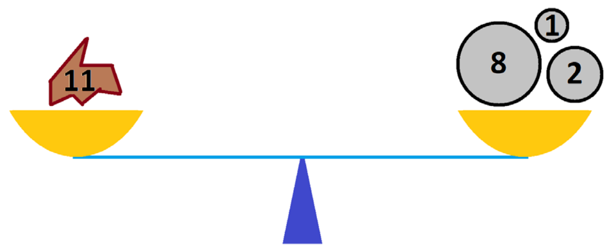
ANOTHER TWO-PAN BALANCE PUZZLE

Here's a surprising application of a $-1 \leftarrow 2$ machine and the *negabinary* codes that result from them. These codes have a natural representation in a two-pan balance setting.

EXPLODING DOTS Topic:
Experiences 2 and 9: Having fun in bases two and negative two!
Suggested Grade Level: High School, and All.

ANOTHER TWO-PAN BALANCE PUZZLE

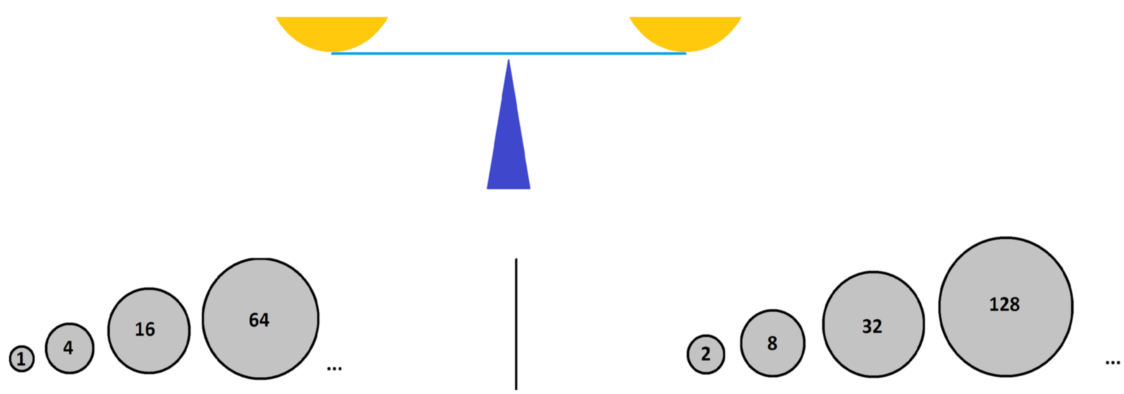
This puzzle assumes that students are familiar with the fact that every number can be written as sum of powers of two. Consequently, given a collection of rocks of weights running through the powers of two, any rock of integer weight can be placed on one side of a two-pan balance and be balance by a set of rocks from that collection



If this might not be evident to your students, present content of the "TWO-PAN SCALE PUZZLES" essay first. Then, when ready, try sharing this next challenge.

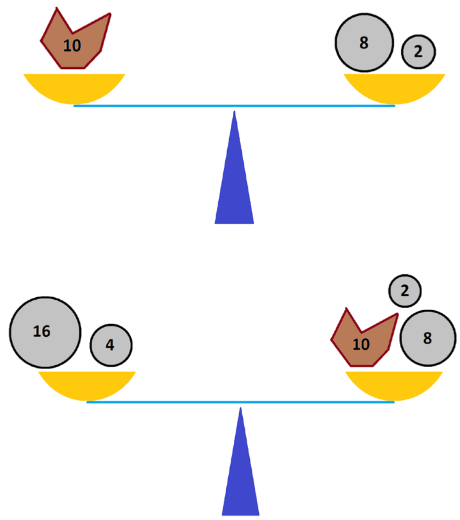
You have an infinite collection of rocks, one weighing 1 kilogram, one weighing 2 kilograms, one weighing 4 kilograms, one weighing 8 kilograms, and so on, with one rock matching each power of two in weight.

You also have a two-pan balance.



For reasons that really can't be explained you will place rocks of weights 1, 4, 16, 64, … kilograms on the left side of the balance (and never on the right) and rocks of weights 2, 8, 32, 128, … kilograms on the right side of the balance (and never on the left).

If someone places a rock of weight 10 kilograms on either side of the two-pan scale you realise that you can balance the system with your rocks while following these rules.



But now one wonders: Will this always be possible? Using the rocks of weights the powers of two, under the rules given, can one always create balanced system if a rock of random integer weight is placed in one of the two pans?

a) Suppose a rock of weight 20 kilograms is placed in the right pan. Can you use the rocks of weights the powers to two to balance matters? What if the rock is placed, instead, in the left pan?

b) Suppose a rock of weight 50 kilograms is placed in the left pan. Can you use the rocks of weights the powers to two to balance matters? What if the rock is placed, instead, in the right pan?

Let's just give the answer away.

c) Develop some convincing reasoning that if a rock of any integer weight is placed in either pan, then it is for certain possible to place rocks of weights the powers of two in the pans, while following the stated rules, to create a balanced system.

Some Things Students Might Notice or Question

1. Via trial-and-error we see that rocks of weights 4 and 16 in the left pan balance with the rock of weight 20 in the right pan. Also, rocks of weight 8 and 32 in the right pan balance with rocks of weight 20 and 4 and 16 in the left pan.

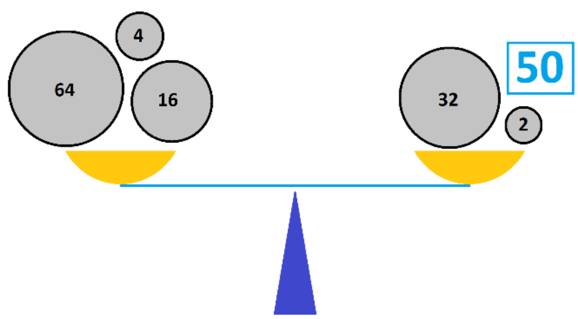
2. Dealing with a rock of weight 50 kilograms in either pan seems annoying and hard.

3. As a sum of power of two, $50 = 32 + 16 + 2$. But with the 16 rock on the left and the 2 and 32 rocks on the right, this doesn't seem to mean anything?

4. You can get the equivalent of "32 on the left" by placing the 64 rock on the left and the 32 rock on the right.

   Ooh! And "2 on the left" is equivalent to placing the 4 rock on the left and the 2 rock on the right.

   This leads to a balanced solution with a 50-kilogram weight on the right.



5. With persistent trial-and-error, we see that

   $$\mathbf{5}0 + 16 + 64 = 2 + 128 .$$

   This gives a balanced solution with the 50-kilogram weight on the left.

6. But this is all random trial-and-error. Is there a systematic approach?

7. The examples explored so far, 10 kilograms and 50 kilograms, are both weights an even number of kilograms. Might weights an odd number of kilograms be problematic?

8. Let's just read the solution and see if we can make sense of that!

THE PUZZLE EXPLAINED

EXPERIENCES 2 and 9 of EXPLODING DOTS: Having fun in bases two and negative two.

Digging into the number 50's "relationship" with the powers of two can be revealing. We certainly have

$$32+16+2=50$$

but the weights 2, 16 ,and 32 kilograms are not permitted to sit on the same one side of the balance. The equation, visually, has 50 on the right, and it also has 16 on the left, which is permitted in terms of placing weight on the balance pan, but 2 and 32 are also on the left and that is not permitted.

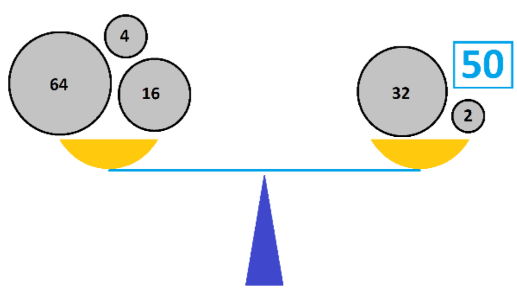
But adding 2 and 32 to each side of the equation gives

$$(32+32)+16+(2+2)=50+32+2$$

or

$$64+32+4=50+32+2,$$

which represents a balanced, permissible, equation.



Can we obtain a balanced solution with the weight 50 on the left?

Consider writing $50=32+16+2$. We need 16 on the left, and we can obtain this by adding 16 to each side of the equation.

$$50+16=32+(16+16)+2$$

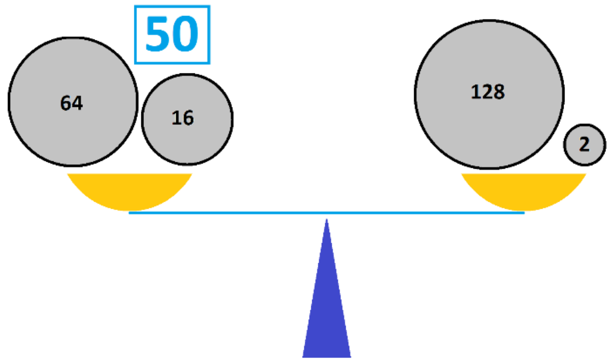
This reads

$$50+16=32+32+2=64+2.$$

But 64 cannot be on the right. So, let's try adding 64 to each side. This yields

$$50+64+16=128+2,$$

and we see we have a permissible balanced solution.



It now feels that it might be always be possible to balance a weight of any weight a whole number N of kilograms placed on either side of the balance scale with this sort of jiggling. Can we detail a procedure?

Given a weight of N kilograms

1. Write N as a sum of powers of two.

2. If we want the weight N to be in the left pan write the equation as "$N =$sum." If we want the weight N in the right pan, write the equation as "sum$=N$."

For example, with $N = 55$, we have $N = 1+2+4+16+32$.

The weight on the left.	**The weight on the right.**
$\boxed{55} = 1+2+4+16+32$	$1+2+4+16+32 = \boxed{55}$

3. Look at the smallest power of two in the sum. If is it on the "allowed" side of the equation, do nothing. If it is on the incorrect side of the equation, add that power of two to each side of the equation and perform any chain of computations that occur among the powers of two.

The weight on the left.	**The weight on the right.**
$\boxed{55} = 1+2+4+16+32$	$1+2+4+16+32 = \boxed{55}$
$\boxed{55}+1 = (1+1)+2+4+16+32$ $= (2+2)+4+16+32$ $= (4+4)+16+32$ $= 8+16+32$	

4. Repeat step 3 with the next smallest power of two that is on an incorrect side, and then the next smallest, and the next. Repeat until we have an expression with all powers of two in permissible positions.

The weight on the left.	**The weight on the right.**
$\boxed{55} = 1+2+4+16+32$	$1+2+4+16+32 = \boxed{55}$ $1+8+16+32 = 2+\boxed{55}$ $1+64 = 2+8+\boxed{55}$
$\boxed{55}+1 = (1+1)+2+4+16+32$ $= (2+2)+4+16+32$ $= (4+4)+16+32$ $= 8+16+32$	
$\boxed{55}+1+16 = 8+64$ $\boxed{55}+1+16+64 = 8+128$	

The only trouble with this reasoning is that it is not clear that algorithm we've designed will stop. How do we know that each step of fixing doesn't produce a larger power of two that needs fixing?

Hmm. This has me worried!

Let's look at the two equations we have for $N = 55$ in the example. One equation says that

$$55 = -1+8-16-64+128$$

and the other says

$$55 = 1-2-8+64\,.$$

Those minus signs have me thinking that maybe we should be working with the powers of negative two, not the powers of two. For example, the first of these equations can be rewritten as

$$-55 = 1 + (-2)^3 + (-2)^4 + (-2)^6 + (-2)^7,$$

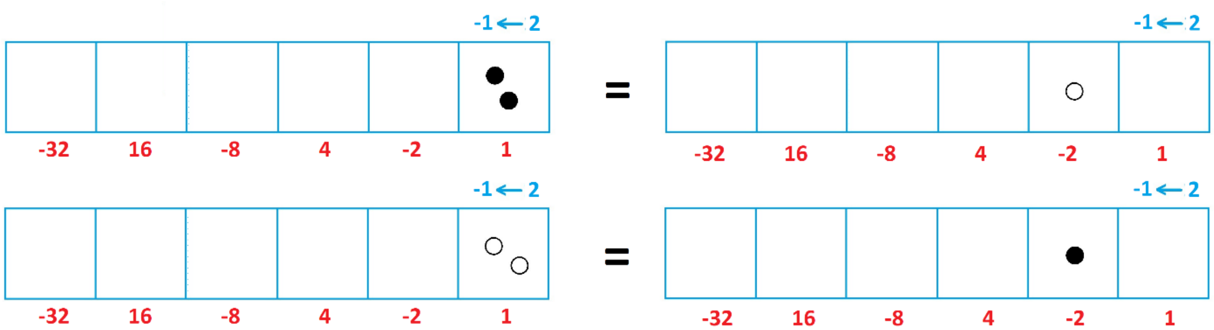
(I introduced a minus sign to look at -55 instead of positive 55 to align with these powers). And the second can be rewritten as

$$55 = 1 + (-2)^1 + (-2)^3 + (-2)^6.$$

So should we be looking at a $-1 \leftarrow 2$ machine rather than a $1 \leftarrow 2$ machine that works with the powers of two?

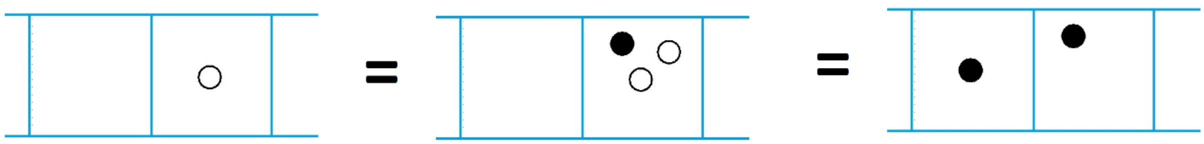
How does a $-1 \leftarrow 2$ machine work?

Well, in such a machine, two dots in one box explode away to be replaced by one antidot, one place to their left, and two antidots in one box explode away to be replaced by a dot, one place to their left.

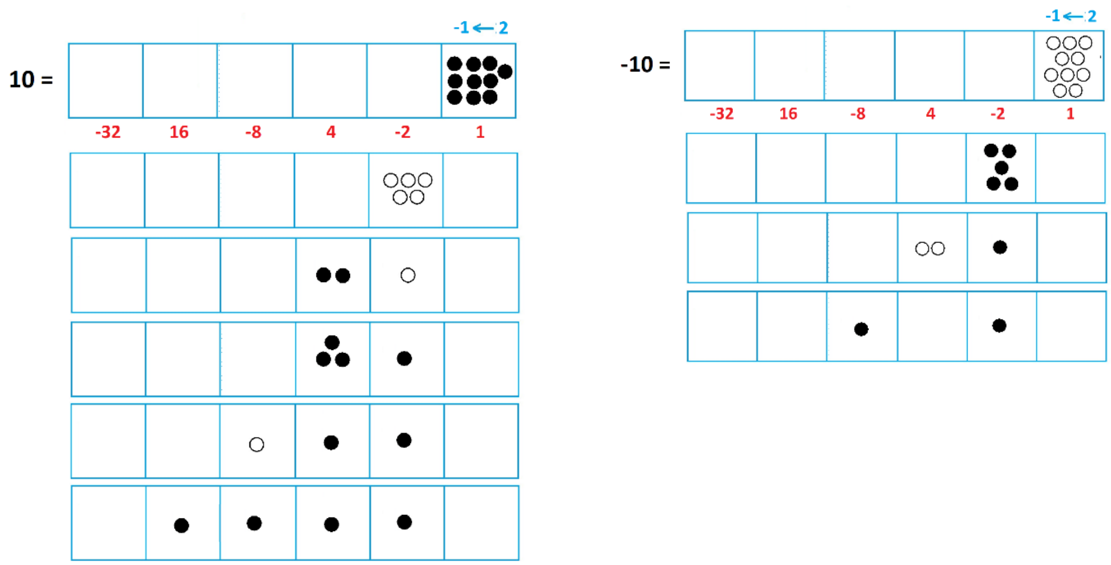


One sees that this is a machine giving codes of numbers in base negative two, with each box containing at most one dot or one antidot.

But we can go little bit further. Any box that contains an antidot, can be replaced by two single dots. We see this by adding a dot/antidot pair and performing one explosion.



This means that any number, positive or negative, can be represented as a code in a $-1 \leftarrow 2$ machine with nothing but dots with at most one dot per box. Either place N dots or N antidots in the rightmost box (for the positive integer N or the negative integer $-N$), explode away pairs of dots and antidots from the rightmost box. If an antidot is left behind, replace it with two dots as shown above, and then repeat this procedure for the second box from the right, then the third box from the right, and so on. What will be left behind is a representation of N or $-N$ with single dots in boxes, that is, a representation of N or $-N$ as a sum of single powers of -2.

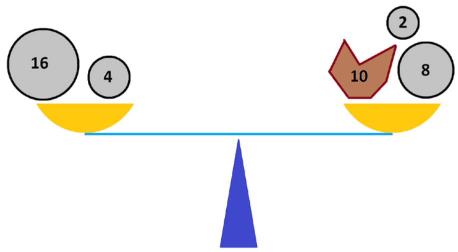


Positive ten is 11110 *in base negative two.*
Negative ten is 1010 *in base negative-two*

So this gives us a means to create balance. For example, from "positive ten $=11110$," that is, from

$$16+(-8)+4+(-2)=10$$

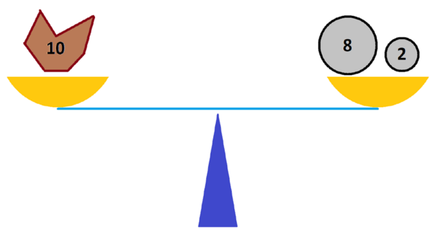
move the negative numbers to the other side of the equation to rewrite this as $4+16=10+2+8$, to be read off as balanced scenario!



From "negatative ten $=1010$," that is, from

$$(-2)+(-8)=-10$$

move the negative quantities to the other side of the equation to rewrite as $10=2+8$. Balance!



Base negative-two representations give balanced solutions for any given weight placed on either side of the scale!

Jargon: A representation of a number in base negative-two using the coefficients 0 and 1 is called is *negabinary* represention.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Are balanced solutions unique? Given a specific weight N could there be more than one way to create a balanced scenario with that weight in the left pan or more than one way with that weight in the right pan?

Is this the same question as: *Is the negabinary representation of a number sure to be unique?*

Wild Exploration 2: Is the algorithm we first described to in this essay sure to terminate? If so, how do you know? If it doesn't, find an example of a weight that has us repeating step 3 an infinite number of times.