

TWO-PAN BALANCE PUZZLES



We start this classic weighing puzzle with a simple version that makes use of $1 \leftarrow 2$ machine codes (that is, binary codes) of numbers. This, in-and-of-itself, can serve as a lovely introduction to binary.

We then move to the main puzzle itself and some extensions.

EXPLODING DOTS Topic:

Experiences 2 and 9: Using a $1 \leftarrow 3$ machine but with the digits 0, 1, and -1 . (These are called “balanced ternary” codes.)

Suggested Grade Level:

High School

TWO-PAN BALANCE PUZZLES

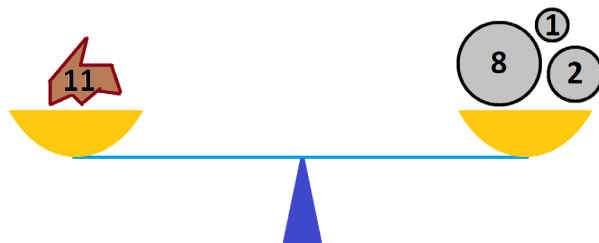
Here's a warm-up puzzle you can share with your students. (Or, for young students, this can be a lovely challenge in-and-of itself.)

Perhaps draw pictures on the board as you explain the puzzle.

You have four rocks. One weighs exactly 1 kilogram, one exactly 2 kilograms, one exactly 4 kilograms, and one exactly 8 kilograms. You also have a simple two-pan balance, with pans each big enough to hold up to four rocks.



If someone puts a rock of weight 11 kilograms, say, on the left pan, you could balance it perfectly by placing your 1 kg, 2 kg, and 8 kg rocks on the right pan.



Which rock weights in the left pan could you balance with some combination of your four rocks?

Your students will likely realize that you can balance a rock of whole number of kilograms weight from 1 kg all the way up to 15 kg with combinations of the four special rocks. From studying the codes for numbers from a $1 \leftarrow 2$ machine, we see that every number from 1 up to 15 can be represented as a sum of some collection of the numbers 1, 2, 4, and 8.

Now add to the story ...

You decide to open a rock-weighing business. You advertise:

Do you have a rock? And do you want to know its weight? If it weighs a whole number of kilograms between 1 kg and 15 kg, come to me! I'll figure out its weight for you.

With trial-and-error you can figure out which combination of your four rocks balance with the customer's rock, and hence determine its weight.

Business is good for a while, and then it suddenly halts! You notice down the road that Poindexter Farklesnark has just opened a competing business. He advertises:

Do you have a rock? And do you want to know its weight? If it weighs a whole number of kilograms between 1 kg and 40 kg, come to me! I'll figure out its weight for you.

Customers are flocking to his business.

You see that Poindexter is also using a two-pan balance and four special rocks, but he is placing combinations of his rocks on both sides of the balance with the customer's rock. And that is how he is balancing a larger range of weight values.

Is Poindexter using four rocks of weights 1 kg, 2 kg, 4 kg, and 8 kg? Or is he using a different set of four special rocks? (If so, what are the weights of his four special rocks?)

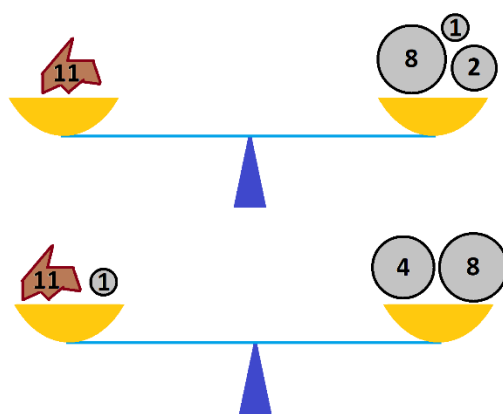
Check that Poindexter really is able to balance each of the weights 1kg up to 40 kg, inclusive, with four special rocks.

Some Things Students Might Notice or Question

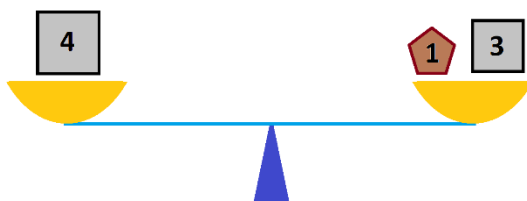
1. With rocks of weights 1 kg, 2kg, 4 kg, and 8 kg in the first business, students might wonder: Is it actually possible, "with trial-and-error," to figure out the weight of an unknown rock?

This might be worth dwelling on. Ask: How could you tell if a rock weighs more or less than 8 kg? How might you tell if a rock weighs more less than 12 kg? Could there be a general strategy to follow to determine the unknown weight of a rock?

2. With rocks of weights 1 kg, 2kg, 4 kg, and 8 kg, it is possible to balance scales in more than one way with a given customer's rock – if you allow use of both pans of the scale.



3. Students might notice that even with use of both pans, one cannot balance a rock of weight more than 15 kg using four rocks of weights 1 kg, 2 kg, 4 kg, and 8 kg. Poindexter has four special rocks of different weights.
4. Some students might argue that Poindexter must have at least one special rock of weight 1 kg: “How else can he balance a customer’s rock of 1 kg?” Other students might realize that realize that he need not! For example, with a 3 kg rock and a 4 kg rock, Poindexter can still detect a rock-weight of 1 kg.



5. Some students might decide to start with the assumption that Poindexter has a 1 kg rock, just for ease, and realize that he doesn’t actually need a 2 kg rock next if he works with a 3kg rock instead. With 1 kg and 3 kg at hand, Poindexter can balance all the weights 1 kg, 2 kg, 3 kg, and 4 kg. They might wonder what the next biggest rock weight could be to get 5 kg, 6kg, 7kg,
6. Students might say that having a rock in the same pan as the customer’s rock is like having an “anti-rock”: it subtracts from the effective weight of the customer’s rock.

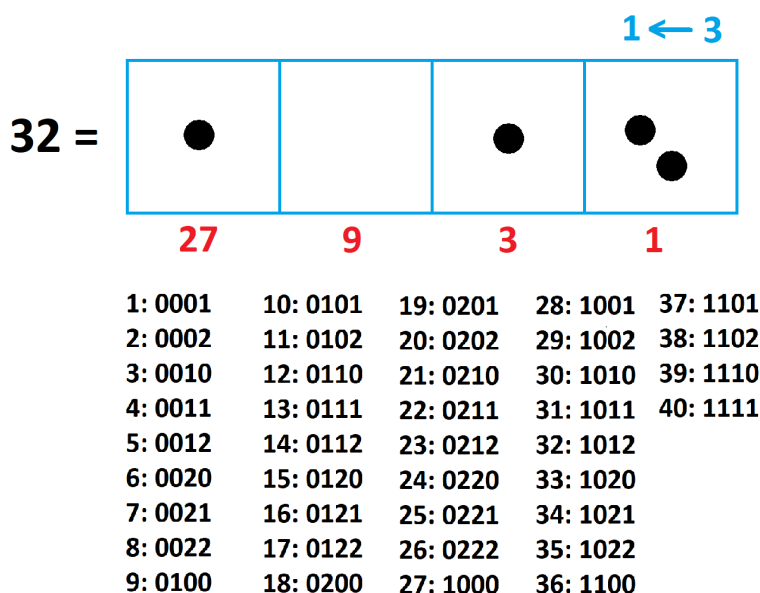
THE PUZZLE EXPLAINED

EXPERIENCES 2 and 9 of EXPLODING DOTS: Using a $1 \leftarrow 3$ machine but with the digits 0, 1, and -1 .

After some experimentation one comes to see that rocks of weights of 1 kg, 2 kg, 4 kg, and 8 kg—as for the binary codes of a $1 \leftarrow 2$ machine—don’t give us Poindexter’s results.

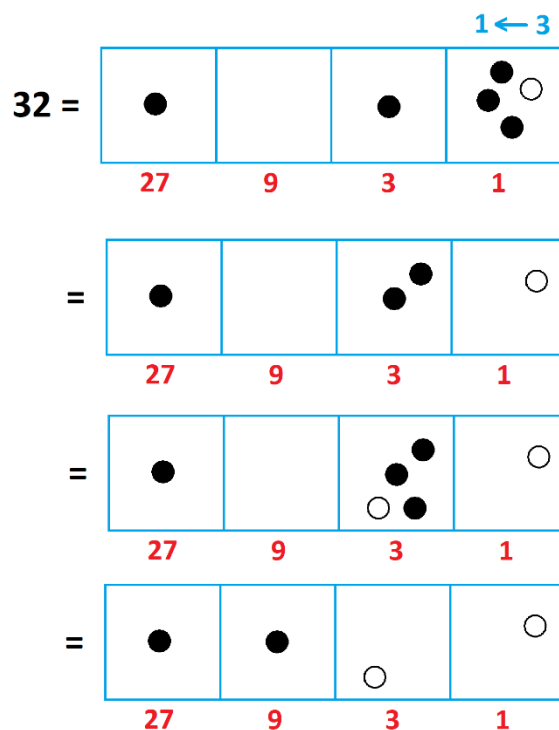
The key is to realize that placing rocks in a pan alongside the customer’s rock has the effect of “subtracting weight” from the customer’s rock: these rocks are behaving like “anti-rocks.” And so perhaps if we go to a bigger machine and allow anti-dots in our codes, we can encode a larger range of values.

Let’s not go too wild and just try the next sized machine, a $1 \leftarrow 3$ machine.

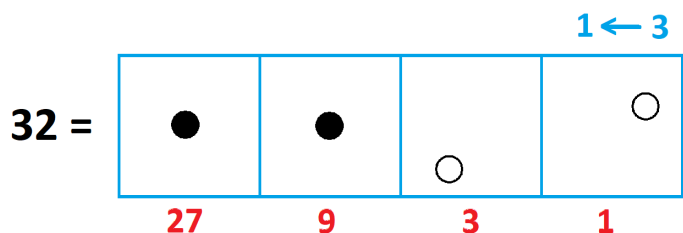


The code for 32, say, is 1012. So, with four rocks of weights 1 kg, 1kg, 3 kg, and 27 kg in the left pan, we can balance of rock of weight 32 kg in the right pan. The code for 35 is 1022 and requires five rocks to balance ... as it stands.

But if we do indeed allow for anti-rocks, that is, for antidots in our codes, then each 2 in a code can be “exploded away” by adding a dot and antidot to it.



This leaves us codes for numbers in a $1 \leftarrow 3$ with the digits 0, 1, and -1 . In particular, we see that each of the numbers 1 through 40 have codes with at most four digits. (What's the code for 41 with these digits?)

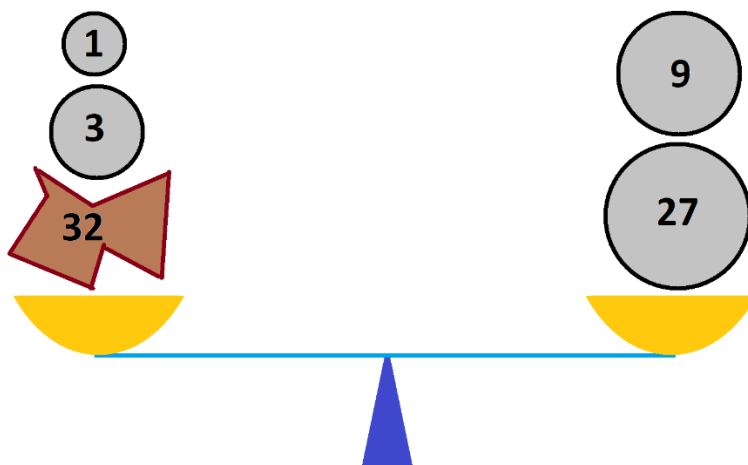


1: 0 0 0 1	10: 0 1 0 1	19: 1 -1 0 1	28: 1 0 0 1	37: 1 1 0 1
2: 0 0 1 -1	11: 0 1 1 -1	20: 1 -1 1 -1	29: 1 0 1 -1	38: 1 1 1 -1
3: 0 0 1 0	12: 0 1 1 0	21: 1 -1 1 0	30: 1 0 1 0	39: 1 1 1 0
4: 0 0 1 1	13: 0 1 1 1	22: 1 -1 1 1	31: 1 0 1 1	40: 1 1 1 1
5: 0 1 -1 -1	14: 1 -1 -1 -1	23: 1 0 -1 -1	32: 1 1 -1 -1	
6: 0 1 -1 0	15: 1 -1 -1 0	24: 1 0 -1 0	33: 1 1 -1 0	
7: 0 1 -1 1	16: 1 -1 -1 1	25: 1 0 -1 1	34: 1 1 -1 1	
8: 0 1 0 -1	17: 1 -1 0 -1	26: 1 0 0 -1	35: 1 1 0 -1	
9: 0 1 0 0	18: 1 -1 0 0	27: 1 0 0 0	36: 1 1 0 0	

Jargon: Base-three codes using the digits 0, 1, and -1 are called “balanced ternary” codes.

These codes explain the system Poindexter was using. He had four special rocks of weights 1 kg, 3 kg, 9 kg, and 27 kg, and, via trial-and-error, balanced a customer's rock with these four rocks. Each customer's rock will balance, and the balance is given by the balanced ternary code of its weight.

For example, 32 has balanced ternary code $1|1|-1|-1$. Each non-zero digit means "use the rock whose weight matches the place of that digit", with $+1$ specifically meaning "place that rock opposite the customer's rock" and -1 meaning "place that rock with the customer's rock."



EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Grizelda Bumblesnort later put Poindexter out of business. She too used a simple two-pan balance and four special rocks, but made the following claim:


Do you have a rock? And do you want to know its weight? If it weighs a whole number of kilograms between 1 kg and 81 kg, come to me! I'll figure out its weight for you.

What are the weights of her four special rocks?


[Hint: Grizelda never claims that she'll make her rocks balance with a customer's rock, only that she can deduce what the customer's rock-weight must be.

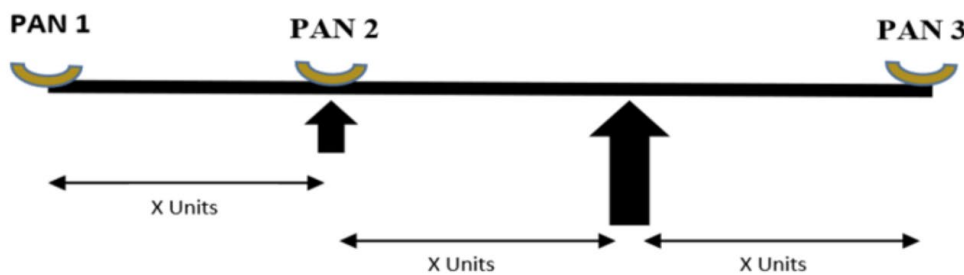
Wild Exploration 2: Global Math Project ambassador Kiran Bacche recently shared a puzzle on social media (@KiranABacche) he made up about weighing rocks. Can you solve his puzzle?

Can you make up your own special puzzles?



Lightweight Puzzle





What **two weights** are needed to weigh objects from **1KG** to **10KG** using the above **3-Pan** Balance?

Clue: Exploding Dots

How will you represent numbers from **1** to **10** in a **(1←4)** Exploding Dots Machine such that no box contains more than **2** dots?