DIVIDING BY 101 PUZZLE

This piece is based on an American Mathematics Competition problem that asks us to determine when certain numbers are divisible by 101. It does require making use of dots and antidots within the $1 \leftarrow 10$ machine.

Upper high-school students could see a connection to the Factor Theorem in polynomial algebra.

EXPLODING DOTS Topic:

Experiences 5 and 6: Using dots and antidots for division in a $1 \leftarrow 10$ machine.

Suggested Grade Level:

Upper Middle-school. High school.

See Also:

Divisibility by 7

DIVIDING BY 101 PUZZLE

Here's a question that comes from the American Mathematics Competitions (the 2018, AMC 10B competition as question 13, in fact).

How many of the first 2018 numbers in the sequence 101, 1001, 10001, 100001, … are divisible by 101?

Perhaps write this question on the board and have a class discussion with your students about it.

Some Things Students Might Notice or Ask

1. Why the first 2018 numbers in the sequence? (If you don't mention the source of the question—that it comes from a competition that was held in the year 2018—this is indeed a strange number to pull out of the air!)

2. What is the 2018th number in the sequence?

3. Students will likely notice that the n th number in the sequence is the number created by a pair of 1s with n zeros between them.

4. The very first number in the list, at least, is divisible by 101. If we get out a calculator we can find other numbers in the list are divisible by 101 too. (Students working this way may well detect a pattern as to which ones are.)

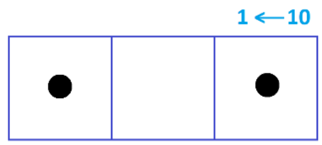
5. Our calculators can't handle numbers with 2018 zeros between a pair of ones.

SOLVING THE PUZZLE

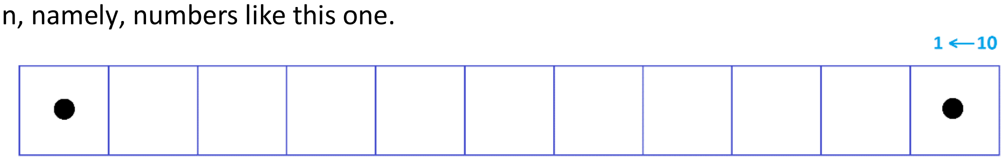
See EXPERIENCES 5 and 6 of EXPLODING DOTS: Using Dots and Antidots in base-ten division.

Let's look at division by the number 101 in a $1 \leftarrow 10$ machine.

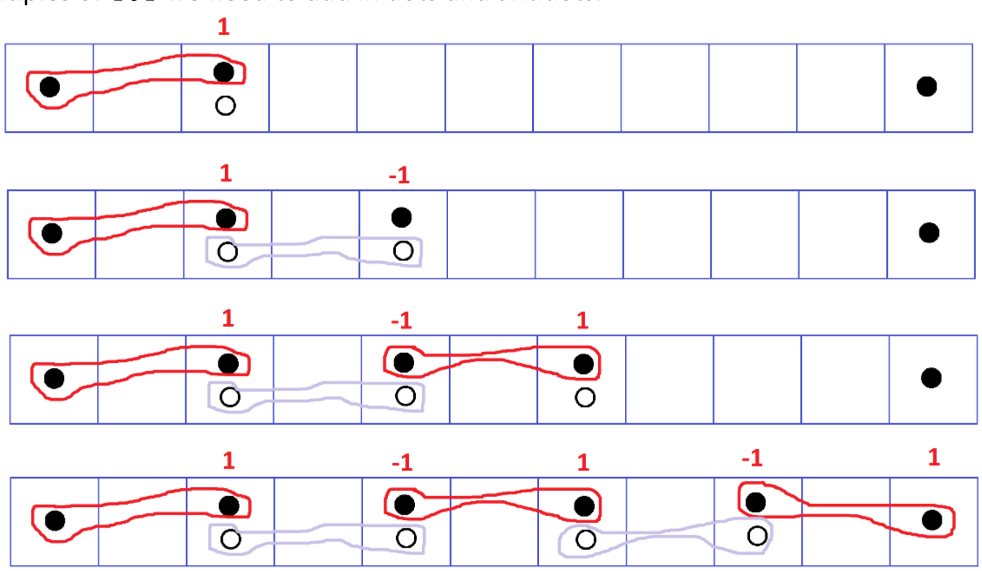
Now 101 looks like this

101 = 

and we looking to divide numbers by this that have single dots at their beginnings and ends and nothing in between, namely, numbers like this one.



To find multiples of 101 we need to add in dots and antidots.



This picture shows that 10000000001 is divisible by 101. The quotient is $1\,|\,0\,|\,-1\,|\,0\,|\,1\,|\,0\,|\,-1\,|\,0\,|\,1$. What number is that?

Some experimentation leads one to see that for a number of the form $100\cdots001$ to be divisible by 101 the number of zeros between the ones needs to be $1,\ 5,\ 9,\ 13,\ 17,\ \ldots$, a count that is one more than a multiple of four. (In the pictures we've drawn, we need N red loops and $N-1$ purple "anti-loops." Each red loop/purple loop pair requires four boxes new boxes, and the final red loop requires one extra box before the final dot.)

Okay, we're essentially done! Now it is a matter of fiddly thinking to determine how many of the first 2018 numbers in the list 101, 1001, 10001, 100001, … have a count of zeros that is one more than a multiple of four.

Let's focus on the list of numbers on more than a multiple of four 1, 5, 9, 13, 17, …. We see that 2017 is the largest number under 2018 that follows this form. Moreover, we have

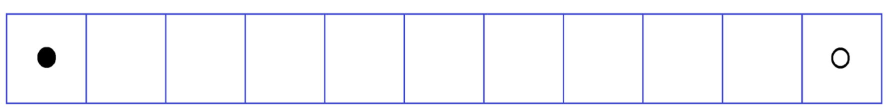
$$
\begin{aligned}
&1 = 0 \cdot 4 + 1 \\
&5 = 1 \cdot 4 + 1 \\
&9 = 2 \cdot 4 + 1 \\
&\vdots \\
&2017 = 504 \cdot 4 + 1
\end{aligned}
$$

So there are 505 numbers among the first 2018 numbers in the list 101, 1001, 10001, 100001, … with the count of zeros one more than a multiple of four. That is, **505** of the numbers of the first 2018 numbers in the sequence 101, 1001, 10001, 100001, … are divisible by 101!

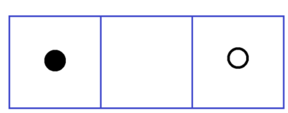
EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Consider the sequence of numbers 9, 99, 999, 9999, 99999, …. It is clear that every second number is divisible by 99. But to practice the ideas of essay, can you also see this is true by viewing each number in this sequence as a number that appears as follows in a $1 \leftarrow 10$ machine



with 99 represented as a dot-blank-antidot?



Wild Exploration 2: Explore which numbers in the list 11, 101, 1001, 10001, 100001, … are divisible by 11, and by 1001, and by 10001, and so on!

Wild Exploration 3: Here is a curious divisibility rule for the number 101: *To determine if a number is divisible by 101, delete its last digit and subtract that digit from the second-to-last digit of what remains. The original number is divisible by 101 only if this new (smaller) number is. One can repeat this procedure until one has a number small enough to readily recognize as a multiple of 101 or not.* Can you make sense of this procedure, and then explain why it works? (See the "Divisibility by 7" puzzle.)

EXTRA

Students having studied the factor theorem in advanced algebra might realise that y^N+1 is divisible by $y+1$ only if N is odd. (We need $y=-1$ to be a zero of y^N+1.) Writing $N=2k+1$ and setting $y=x^2$ this reads

$$\left(x^2\right)^{2k+1}+1 \text{ is divisible by } x^2+1,$$

that is, $x^{4k+2}+1$ is divisible by x^2+1.

Now setting $x=10$, we see that $10^{4k+2}+1$ is divisible by 101, and $10^{4k+2}+1$ is a number with $4k+1$ zeros between a pair of ones.