DIVISIBILITY BY 5 in base one-and-a-half

This puzzle is really "out there." It assumes some familiarity with the $2 \leftarrow 3$ machine and how the codes that arise from it are representations of numbers in base one-and-a-half with the digits 0, 1, and 2. (This is Experience 9 of the Exploding Dots story.)

Very little is known about the codes of numbers from this base, including basic divisibility rules. But one can, at least, give a curious divisibility rule for 5!

EXPLODING DOTS Topic:

Experience 9: Understanding the codes of the $2 \leftarrow 3$ machine.

Suggested Grade Level:

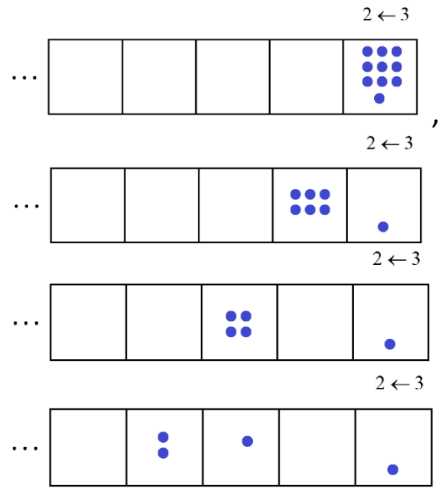
High-school and up.

DIVISIBILITY BY 5 in base one-and-a-half

If you and your students have played deeply with Exploding Dots and talked about the mysteries of the codes that arise in base one-and-a-half from a $2 \leftarrow 3$ machine, perhaps try presenting this puzzle.

In a $2 \leftarrow 3$ machine, **three** dots in any one box disappear – EXPLODE!—to be replaced with **two** dots just to their left.

For example, placing ten dots in the machine yields the code 2101 for ten.



Here are the codes for the number zero to forty from a $2 \leftarrow 3$ machine.

0			
1	**2102**	**21220**	**212021**
2	**2120**	**21221**	**212022**
20	**2121**	**21222**	**212210**
21	**2122**	**210110**	**212211**
22	**21010**	**210111**	**212212**
210	**21011**	**210112**	**2101100**
211	**21012**	**212000**	**2101101**
212	**21200**	**212001**	**2101102**
2100	**21201**	**212002**	**2101120**
2101	**21202**	**212020**	**2101121**

Prove that a number is divisible by 5 only if the alternating sum of the digitsof its $2 \leftarrow 3$ machine code is a multiple of five.

For example, twenty has code 21202 and $2-1+2-0+2=5$ is a multiple of five. And forty has code 2101121 and $2-1+0-1+1-2+1=0$ is a multiple of five. And eleven has code 2102 and $2-1+0=2=-1$ is not a multiple of five.

Some Things Students Might Notice, Say, or Ask

1. What??!!!

2. This feels like the divisibility rule for eleven in ordinary base ten.

3. The $2 \leftarrow 3$ machine code $a \mid b \mid c \mid \cdots \mid d \mid e$ (with each digit 0, 1, or 2) for a number N means we're writing N as

$$a\left(\frac{3}{2}\right)^{k} + b\left(\frac{3}{2}\right)^{k-1} + c\left(\frac{3}{2}\right)^{k-2} + \cdots + d\left(\frac{3}{2}\right) + e\,.$$

   Do we really have to mess with sums of fractions like these?

4. This is horrible!

SOLVING THE PUZZLE

See EXPERIENCES 2 and 4 of EXPLODING DOTS: Understanding the place-value machines.

This problem has three elements to consider:

- Things being multiple of fives
- Alternating sums of digits
- The mechanics of a $2 \leftarrow 3$ machine

I don't know how these ideas are meant to mesh together, but it does feel natural to consider what an explosion does in a $2 \leftarrow 3$ machine to the alternating sum of digits you have so far.

So we're either considering

$-a+b$ changing to $-(a+2)+(b-3)=-a+b-5$

or

$a-b$ changing to $(a+2)-(b-3)=a-b+5$.

Either way, an explosion in a $2 \leftarrow 3$ machine does not affect whether or not the alternating sum of digits you have so far is a multiple of five.

So if we put in N dots in the rightmost box of a $2 \leftarrow 3$ machine (with alternating sum $\cdots-0+0-0+N=N$) and perform explosions to get its $2 \leftarrow 3$ machine code, $a\,|\,b\,|\,c\,|\cdots|\,d\,|\,e$, say, the alternating sum of this code $a-b+c-\cdots-d+e$ differs from N by a multiple of five.

So N is a multiple of five precisely if the alternating sum is, just as the puzzle claims!

EXTENSION

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy this exploration?

Wild Exploration 1: Have we just proved that in any $b \leftarrow a$ machine, a number is divisible by $a+b$ precisely when the alternating sum of the digits of in this machine its code is? (Does this seem to fit the divisibility rule for eleven in base ten?)