## DIVISIBILITY BY 7 <br> 

Do you and your students know a divisibility rule for the number 7? This piece illustrates and explains one. And after exploring it, your students will also be able to create divisibility rules for 11, 13, 17, 19, and more!

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EXPLODING DOTS Topic:
Experience 6: Using dots and antidots for division in a 1 \leftarrow10 machine.
Suggested Grade Level:
Middle-school. High school.
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See Also:

Divisibility by 9
Divisibility by 9 again!

## DIVISIBILITY BY 7

Many people know divisibility rules for dividing by $2,3,4,5,8,9$ and 10 , and perhaps 11 (and, hence also for 6 by using the rules for 2 and 3 together, for 12 by using the rules for 3 and 4 together, and for 15 and the like). But do you and your students know this curious divisibility rule for the number 7?

To determine whether or not a number is divisible by 7 , delete its last digit and subtract double that deleted digit from what remains. The original number is divisible by 7 only if this new number is.

This procedure can be repeated until one has a number sufficiently small to be easily recognized as a multiple of 7 or not.

This is a bit hard to parse. An example helps. Let's check if 39872 is a multiple of 7 .


Delete the final digit, the 2, and subtract double this, 4, from the 3987 that remains. This gives 3983 . I can't readily tell if this is a multiple of 7 , so let's do this procedure again: delete the final digit 3 and subtract double this from the remaining 398 to yield 392 . It is still hard to tell whether or not we have a multiple of 7 . One more run through of the procedure yields 35 . This is a multiple of 7 , and this apparently means that our original number was too!

Check: $39872=5696 \times 7$. Yes!
By the way: Suppose I didn't know that 35 was a multiple of 7 . If I run the procedure through one more time, might I get a number I could surely recognize is divisible by 7 ?

Have your students practice the procedure a few times, with some beginning numbers that they know to be multiples of 7 and some known not to be.

The algorithm always seems to work. But the real question, of course, is: Why does it work?

## Some Things Students Might Notice or Ask

1. If you really want to know if a number is divisible by 7 or not, just get out a calculator!
2. This is just plain weird! Really ... why does it work?
3. This rule doesn't actually tell us what the answer to the division problem is. It just seems to be a YES/NO procedure.
4. If the beginning number is not a multiple of 7, does this procedure at least tell us what the remainder is upon division by 7 ? (Our divisibility rules for 9 did.)

## SOLVING THE PUZZLE

## See EXPERIENCE 6 of EXPLODING DOTS: Using Dots and AntiDots in Division

Let's look numbers represented in $1 \leftarrow 10$ machines.

We're interested in finding groups of 7 and seeing if there will be any remainders or not. The procedure doesn't claim to tell us what the answer to the division problem shall be nor what remainder we'll get, only whether or not well be able to find complete groups of seven.

Since we don't care about how many groups of 7 we can find, we are thus free to add extra groups of seven in any way we like to our pictures to help us answer our basic YES/NO question.

So let's be clever!
Suppose we have a single dot in the rightmost box of a number in a $1 \leftarrow 10$ machine. This picture shows the number 3251 .
$1 \leftarrow 10$


Let's add three groups of 7 to this picture, actually, three anti-groups of 7 ! That is, let's add 21 antidots.
$1 \leftarrow 10$

| $\bullet \bullet$ | $\bullet$ | $\bullet \ominus$ | $\bullet$ |
| :--- | :--- | :--- | :--- |
| 0 | 0 |  |  |

This has us now looking for whole groups of 7 in the left three boxes: we made the rightmost box empty. That is, we're now looking for groups of 7 in a picture of the number $325-2=323$.
$1 \longleftarrow 10$


Are there a whole number group of 7 s here? If the answer is YES, then there were a whole number of groups of 7 in the original picture too. If the answer is NO, then we have $1,2,3,4,5$, or 6 dots left over in that second-to-last box, which correspond to $10,20,30,40,50$, or 60 dots in the rightmost box. None of these counts yield whole groups of seven, so our original number also leaves a remainder upon division by 7. This shows that answering our YES/NO question for these three left boxes precisely matches answering the YES/NO question for the original number.

This reasoning extends.
If we have a number with $d$ dots in the rightmost box, add $d$ copies of 21 antidots. That leaves us answering the same YES/NO question for a number with the final digit $d$ deleted and $2 d$ subtracted from what remains.


There are a whole number of groups of 7 in this new smaller number only if the original number had a whole number of groups of 7 .

And this is it! This is the divisibility rule for 7 described!

## EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Here's a divisibility rule for the number 11. To determine if a number is divisible by 11, delete the final digit from the number and subtract that digit from what remains. Then the original number is divisible by 11 only if this new number is.
a) Can you make sense of the rule? Practice with some examples.
b) Can you explain the rule?

Extra: Use this rule to show that a number of the form $a|b| c \mid d$ is divisible by 11 only if the alternating sum of its digits, $a-b+c-d$, is. (There is nothing special about four-digit numbers here.)

Wild Exploration 2: Here's a divisibility rule for the number 17. To determine if a number is divisible by 17 , delete the final digit from the number and subtract five times that digit from what remains. Then the original number is divisible by 17 only if this new number is.
a) Can you make sense of the rule? Practice with some examples.
b) Can you explain the rule?

Wild Exploration 3: Here's a divisibility rule for the number 13. To determine if a number is divisible by 13, delete the final digit from the number and add four times that digit from what remains. Then the original number is divisible by 13 only if this new number is.
a) Can you make sense of the rule? Practice with some examples.
b) Can you explain the rule?

Wild Exploration 4: Here's yet another divisibility rule for the number 9. To determine if a number is divisible by 9, delete the final digit from the number and add that digit from what remains. Then the original number is divisible by 9 only if this new number is.
a) Can you make sense of the rule? Practice with some examples.
b) Can you explain the rule?

Wild Exploration 5: Make up a divisibility rule for the number 31. Make up ones for each of 19, 23, and 101 as well!

