DIVISIBILITY BY 9 – AGAIN!

Here we explore a little-known technique for dividing by 9.

EXPLODING DOTS Topic:
Experience 5: Division in a $1 \leftarrow 10$ machine.
Suggested Grade Level:
Upper middle-school. High school.

See Also:

Divisibility by 9
Divisibility by 7

DIVISIBILITY BY 9 – AGAIN!

Here's a very strange way to divide a number by 9. We'll illustrate it with a specific example.

EXAMPLE: To divide 21023 by 9, write out the partial sums of its digits, computed from left to right

2 = 2
2 + 1 = 3
2 + 1 + 0 = 3
2 + 1 + 0 + 2 = 5
2 + 1 + 0 + 2 + 3 = 8

and then read off the answer:

$$21023 \div 9 = 2335 \quad R \quad 8.$$

(And indeed, $21023 = 9 \times 2335 + 8$**.)**

In the same way.

$$\begin{aligned} 1221 \div 9 &= 1 \mid 1+2 \mid 1+2+2 \quad R \quad 1+2+2+1 \\ &= 135 \;\; R \;\; 6 \end{aligned}$$

and

$$\begin{aligned} 20000 \div 9 &= 2 \mid 2+0 \mid 2+0+0 \mid 2+0+0+0 \quad R \quad 2+0+0+0+0 \\ &= 2222 \;\; R \;\; 2 \end{aligned}$$

One might have to perform some explosions along the way and deal with extra-large remainders. For instance,

$$\begin{aligned} 5623 \div 9 &= 5 \mid 5+6 \mid 5+6+2 \quad R \quad 5+6+2+3 \\ &= 5 \mid 11 \mid 13 \;\; R \;\; 16 \\ &= 623 \;\; R \;\; 16 \end{aligned}$$

and a remainder of "16" really corresponds to one extra group of 9 and a remainder of 7. So we actually have $5623 \div 9 = 624 \;\; R \;\; 7$.

Discuss this algorithm with your students. Help them make sense of the examples presented here and check more examples on their own. Then ask: *Why does this curious algorithm work?*

Some Things Students Might Notice or Ask

1. Examples do seem to suggest that this algorithm always works.

2. This is just plain weird! Really … why does it work?

3. What if we start with a multiple of 9. Shouldn't we get a remainder of zero? But this method will never give a remainder of zero!

SOLVING THE PUZZLE

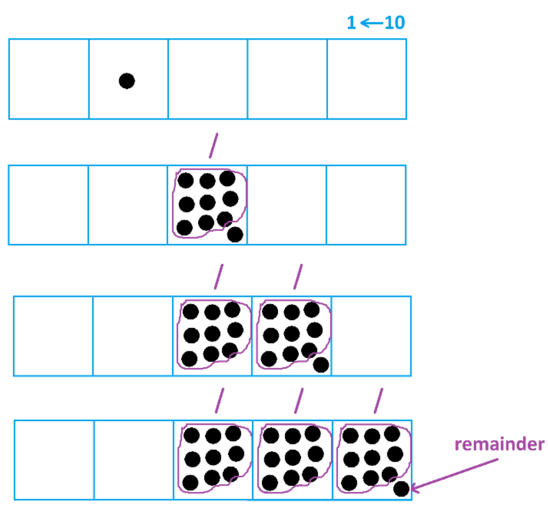
See EXPERIENCE 5 of EXPLODING DOTS: Division in a $1 \leftarrow 10$ machine.

Let's look at division by 9 in a $1 \leftarrow 10$ machine.

To get students going have them look at

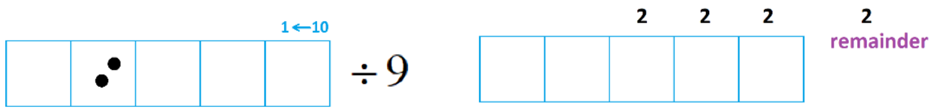
$$10 \div 9$$
$$100 \div 9$$
$$1000 \div 9$$
$$10000 \div 9$$

and so on in a $1 \leftarrow 10$ machine and see that they each given an answer of the form $1111...1\ R\ 1$.

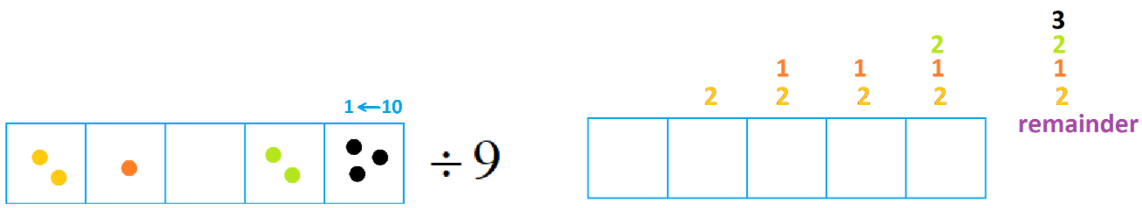


So each dot in a $1 \leftarrow 10$ machine, upon the act of dividing by 9, gives one tally mark in each column to its right and an extra tally mark in the "remainder section" at the end.

Two dots in a box will this give 2 tally marks in each of these positions; three dots in a box, three, and so on.



So dots representing the number 21023 in a $1 \leftarrow 10$ machine give tally marks as shown when divided by 9



and the answer $2 \mid 2+1 \mid 2+1 \mid 2+1+2 \;\; R \;\; 2+1+2+3$ emerges. This matches the opening example (except that we're not recording zeros here). We're also observing that 3 dots in the rightmost box match a remainder of 3 upon division by 9.

We could also take a purely arithmetical approach. Noticing, for example, that $10000 = 9999 + 1$, we readily see

$1 \div 9 = 0 \;\; R \; 1$

$10 \div 9 = 1 \;\; R \; 1$

$100 \div 9 = 11 \;\; R \; 1$

$1000 \div 9 = 111 \;\; R \; 1$

etc.

We deduce then, from $20000 = 10000 + 10000$ and $300 = 100 + 100 + 100$, for example, that

$$\begin{array}{ll} 2 \div 9 = 0\ R\ 2 & 3 \div 9 = 0\ R\ 3 \\ 20 \div 9 = 2\ R\ 2 & 30 \div 9 = 3\ R\ 3 \\ 200 \div 9 = 22\ R\ 2 & 300 \div 9 = 33\ R\ 3 \\ 2000 \div 9 = 222\ R\ 2 & 3000 \div 9 = 333\ R\ 3 \end{array} \quad \text{etc.}$$

Thus

$$2312 \div 9 = (2000 + 300 + 10 + 2) \div 9 = 222R2 + 33R3 + 1R1 + 0R2$$

and we see that this gives the answer $2 \mid 2+3 \mid 2+3+1\ R\ 2+3+1+2 = 256\ R\ 8$.

$$\begin{array}{rcccll} & 2 & 2 & 2 & R & 2 \\ & & 3 & 3 & R & 3 \\ & & & 1 & R & 1 \\ + & & & & R & 2 \\ \hline = & 2| & 2+3| & 2+3+1 & R & 2+3+1+2 \end{array}$$

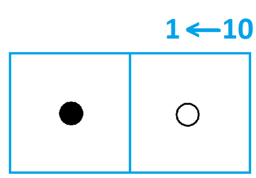
The partial sums of the digits naturally appear!

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Okay. What if we do start with a multiple of 9. How do we deduce from this algorithm that there is zero remainder? [Can you explain this true statement: *A number is divisible 9 only if the sum of its digits is a multiple of 9*?]

Wild Exploration 2: The number 9 can be also represented in a $1 \leftarrow 10$ machine by a dot and an anti-dot as shown.



Look at the number 21023 in a $1 \leftarrow 10$ machine and, starting at the left, make dot/anti-dot copies of 9 appear in your machine. (Add some antidots in your picture, and some dots to counteract them.) Do you see the computation $21023 \div 9 = 2 \mid 2+1 \mid 2+1+0 \mid 2+1+0+2\ \ R\ \ 2+1+0+2+3$ naturally emerging?

Wild Exploration 3: Here's a strange way to divide by 8.

Write down the first digit of the number. Double it and add it to the next digit. Double this answer and add it to the third digit. And so on. Keep doubling the previous obtained answer and add it to the next digit of the number.

You are now ready to read off the answer to your division problem!

$$\boxed{21023 \div 8}$$

2

1 + 2x2 = 5

0 + 2x5 = 10

2 + 2x10 = 22

3 + 2x22 = 47

= 2|5|10|22 R 47

= 2627 R 7

This is by no means an efficient way to divide by 8, but the interesting question is: *Why does this method work?*

Care to make an inefficient approach to dividing by 7?

Wild Exploration 4: Might you wish to explore decimals? Can this dividing-by-nine algorithm be used to compute $214.32 \div 9$, for instance? How? Do you get an infinitely long decimal result?