

DIVISIBILITY BY 9

There is a well-known divisibility rule for the number **9**. In this piece, we explore that rule, and a slightly stronger version of it, and observe some consequences. Students will be able to extend the work here to a divisibility rule for **3** as well, and perhaps to a divisibility rule for **11** as well!

EXPLODING DOTS Topic:

Experience 5: Division in a $1 \leftarrow 10$ machine.

Suggested Grade Level:

Middle-school. High school.

See Also:

Divisibility by 9 again! Divisibility by 7

There are also links to puzzles from our partner BRILLIANT.





DIVISIBILITY BY 9

Many people know a rule for divisibility by nine.

A number is divisible by 9 only if the sum of its digits is divisible by 9.

For example, 387261 is divisible by 9—apparently—since 3+8+7+6+2+1=27 is. (And if we weren't sure about the number 27, we could test that it is divisible by 9 by noting that 2+7=9 certainly is.)

Check: $387261 \div 9 = 43029$ and indeed there is no remainder.

In fact, this rule can be made a little stronger.

A number leaves the same remainder upon division by 9 as does the sum of its digits.

For example, 40062 has sum of digits 12, which is 3 more than a multiple of 9, and indeed 40062 is 3 more than a multiple of 9: $40062 = 9 \times 4059 + 3$. Also, 77 is five more than a multiple of 9 just as 7 + 7 = 14, the sum of it digits, is.

Discuss the divisibility rule for 9 with your students, and its stronger version. Have they already heard of the first rule? The second one? Have them test the rules with some examples.

Now the real question is: Why do these rules work?

Some Things Students Might Notice or Ask

- 1. Examples do seem to suggest that the rules do hold true.
- 2. It is actually surprising that these rules work. If we jumble the digits of a number, the sum of digits does not change. Since 387261 is a multiple of 9, this means that 387216 and 783162 and 273613 and all other permutations 387261 are all multiples of 9 too. That's weird, and hard to believe!

Comment: You and the class might choose to dwell on this observation a bit. It is certainly not generally true that you can rearrange the digits of a number and maintain its divisibility by a given factor. For example, changing the order of the digits of 512 won't always keep the number divisible by 2, and changing the order of the digits of 2864 won't always keep it a multiple of 4. So it seems awfully strange that divisibility-by-9 doesn't "care" about the order of the digits. [Have your students ever noticed it is at least true for two-digit multiples 9? We have that 18 and 81, 27 and 72, 36 and 63, and, 45 and 54 are all multiples of 9.]



3. The second rule implies the first rule. A number is a multiple of 9 only if it leaves a remainder of zero upon division by 9. So, according to the second rule, if the sum of digits leaves a remainder of zero, that is, is a multiple of 9, then so is the original number.

Consequently, explaining the second rule automatically explains the first rule too.

SOLVING THE PUZZLE

See EXPERIENCE 5 of EXPLODING DOTS: Division in a $1 \leftarrow 10$ machine.

Let's look at division by 9 in a $1 \leftarrow 10$ machine.

To get a feel for what is going on, perhaps have students draw dots and boxes to compute $210 \div 9$. It will be tedious, as one must "unexplode" multiple times to find groups of 9, but the key is to notice that there will be 2+1 dots left over in the rightmost box as remainder. (Try it!)

This prepares us to see

```
Each dot in a 1 \leftarrow 10 machine leaves a remainder of 1 upon division by 9.
```

This picture illustrates why.





Thus we have:

If a number is represented by a total of N dots in a $1 \leftarrow 10$ machine, dividing by 9 leaves us with N dots in the rightmost box. And there might be some more groups of 9 we can circle there.

But let's think about what N is in this statement: it's the sum of the digits of the original number. And looking for groups of 9 in the rightmost box with N dots in it is the precisely the act of dividing N itself by 9.

The act of dividing a number by 9 in a $1 \leftarrow 10$ machine reduces to the equivalent act of dividing its sum of digits by 9.

Thus the original number and the sum of its digits leave the same remainder upon division by 9.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Many people know the rule: A number is divisible by 3 only if its sum of digits is. Is there a stronger version of this rule to consider? If so, can you prove it? (If not, can you at least prove the first version of the rule?)

Wild Exploration 2: Martians have three fingers on each of two hands and so naturally write all their numbers in base 6 using a $1 \leftarrow 6$ machine. Is there a divisibility rule for some special Martian number like the one we Earthlings have for 9 in our base-10 system?

Wild Exploration 3: What remainder(s) does a single dot in a $1 \leftarrow 10$ machine leave upon division by 11? Can you devise, and explain, a divisibility rule for 11? (Or look up a rule on the internet and see if you can explain it using a $1 \leftarrow 10$ machine.)

Wild Exploration 4: Which numbers k have the property that if N is divisible by k, then so are all the numbers obtained by rearranging the digits of N? (For example, k = 9 is one such number. So is k = 1.)

Further Reading:

Our partner BRILLIANT has a lovely series of puzzles based on the same ideas of this essay. Check out these two links.



https://brilliant.org/practice/divisibility-by-9-and-3/

https://brilliant.org/practice/last-digits-part-ii/?chapter=applications-of-number-bases