THE JOSEPHUS PROBLEM

A horrific story dating back some 2000 year tells of a soldier Flavius Josephus (37 CE – ca.100 CE) being trapped in a cave with 40 fellow soldiers about to be captured by Roman forces. Rather than face the fate of being Roman slaves—or worse—they decided to commit collective suicide. They stood in a circle and counted off every third person, who was either to kill himself on the spot or, if incapable of conducting the act, be killed by his neighbors. A final soldier would be left standing who would have to commit suicide with no aid.

When the soldiers conducted this, Josephus and just one other man found themselves still standing. They decided they could not continue on and both surrendered to the Romans.

Horrors aside, this question leads to an interesting mathematical question. *Where should one stand in the circle to be the last person standing in this counting act?*

In this essay we take a less gruesome tact to this story and adjust the puzzle by counting off every second person.

EXPLODING DOTS Topic:
Experience 2: Understanding the binary codes of the $1 \leftarrow 2$ machine.
Suggested Grade Level:
High-school and up.

THE JOSEPHUS PROBLEM

Present the following problem to your students. (You, of course, have the option to explain the historic origin of this problem if you wish, but it could be best to leave that aside.)

A number of students, N of them, numbered 1 through N, sit in order in a circle. Walking around the circle many times the teacher taps each student on the shoulder alternately saying the words "in" and "out" as he does each tap. Any student tapped with the word "in" stays in place and those tapped with the word "out" must leave the circle and are out of the game. Even though the circle thins out as this game is played, the teacher keeps strolling around the chairs, tapping each remaining student on the shoulder—in, out, in, out, in, out, … —until one student remains. That lucky student wins a lifetime supply of really cool math books.

Each time the teacher plays this game with a group of students, he always starts by tapping the shoulder of student 1 with the word "in."

As practice, consider the situation with $N = 5$ students in a group. The teacher keeps student 1 in, sends student 2 out, keeps student 3 in, sends student 4 out, keeps student 5 in, sends student 1 out, keeps student 3 in, sends student 5 out to then leave student 3 as the winner.

Write $W(N)$ for the number of the winning student in a game played with N students.
We have $W(5) = 3$.

a) Check that $W(12) = 9$ and $W(16) = 1$.

b) If you like, complete the following table and look for patterns. Can you explain any patterns you see?

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
W(N)	1				3							9				

But here is the real question.

Write the number N in its binary $1 \leftarrow 2$ machine code.
Move the leading 1 from the front of the code to the back.
Then $W(N)$ is the number with that binary code.

For example, $N = 12$ has code 1100. Move the front 1 to the back and get 1001, which is the code for 9. And lo and behold, $W(12) = 9$! Also $N = 5$ has code 101, move the front 1 to the back and get 011 which is the code for 3. And indeed, $W(5) = 3$. Whoa!

c) Can you explain this mysterious connection to binary codes?

Some Things Students Might Notice, Say, or Ask

1. No!

   And this could be meant in one of two ways: that they can't explain what is going on, or that they simply reject the request to try to explain what is going on!

2. This feels like a really weird and hard puzzle.

3. "I don't get it."

   The thing to do here is to conduct some more practice examples. One can write out on paper what is going on in any particular example and collect data swiftly. For example, each line here shows which student is sent out in turn for $N = 5$ students. (In practice, one wouldn't draw a separate new line each time a student number is crossed out.)

   1 ~~2~~ 3 4 5
   1 ~~2~~ 3 ~~4~~ 5
   ~~1~~ ~~2~~ 3 ~~4~~ 5
   ~~1~~ ~~2~~ 3 ~~4~~ ~~5~~ 3 wins!

   It doesn't take too much effort to get some more data values.

N	1	2	3	4	5	6	7	8	9	10
W(N)	1	1	3	1	3	5	7	1	3	5

   One can see that eight, for example, is 1000 in binary and moving the front one to the back gives 0001, which is the code for 1. And indeed $W(8) = 1$.

   We can check that the claim of the puzzle keeps working!

4. Every student with an even number, 2, 4, 6, 8, 10, …, is eliminated right away. Only odd-numbered students have a hope of winning. (Indeed, it does look like that $W(N)$ always an odd value.)

5. In binary, the odd numbers are the ones with final digit 1.

6. "I am intrigued. But I don't know how or where to begin thinking about this!"

SOLVING THE PUZZLE

See EXPERIENCES 2: Understanding the binary codes of a $1 \leftarrow 2$ machine.

The first thing to note is that every student labeled with an even number is eliminated during the teacher's first round of the circle.

For instance, with $N = 12$ students, we are soon left playing a game with the six students numbered 1,3, 5, 7, 9, and 11, with student 1 acting as the first student, student 3 as the second student, student 5 as the third student, and so on.

$$\begin{array}{cccccccccccc} 1 & \cancel{2} & 3 & \cancel{4} & 5 & \cancel{6} & 7 & \cancel{8} & 9 & \cancel{10} & 11 & \cancel{12} \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 & \end{array}$$

Actually, we see that k th student among the odd-number labels is student number $2k-1$ in the original numbering system. The winnof the six-person game is student number $W(6)$, and this is student $2W(6)-1$ in the original numbering system. Thus we deduce that $W(12) = 2W(6)-1$.

In general, we have

$$W(2N) = 2W(N)-1.$$

Consider $N = 13$, an example with an odd number of students in the circle. Again, all the even-numbered students are immediately eliminated, and next eliminated is student 1.

$$\begin{array}{ccccccccccccc} \cancel{1} & \cancel{2} & 3 & \cancel{4} & 5 & \cancel{6} & 7 & \cancel{8} & 9 & \cancel{10} & 11 & \cancel{12} & 13 \\ & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

This leaves us playing a game with 6 students. The k th student in this game is student number $2k+1$ in the original numbering system. We see this time that $W(13) = 2W(6)+1$.

In general, we deduce

$$W(2N+1) = 2W(N)+1.$$

Actually, from $W(1)=1$ and these two relations we can now compute all the values $W(2)$ and $W(3)$, $W(4)$ and $W(5)$, $W(6)$ and $W(7)$, and so on.

Do our two relations explain the curious connection to binary codes? To be clear, we want to show that if $N = 1ab...cd$ in binary, then $W(N) = ab...cd1$ in binary. Hmm.

This claim is true for $N=1$, since $W(1)=1$.

The claim is true for the $N=2$ and $N=3$, the numbers with two-digit binary codes.

$$W(2) = W(10_2) = 01_2 = 1$$
$$W(3) = W(11_2) = 11_2 = 3$$

Comment: I guess we need to distinguish between base-two and base-ten codes. Let's use the subscript 2 to denote a binary code from now on.

The claim is true for the $N = 4, 5, 6, 7$, the numbers with three-digit binary codes.

$$W(4) = W(100_2) = 001_2 = 1$$
$$W(5) = W(101_2) = 011_2 = 3$$
$$W(6) = W(110_2) = 101_2 = 5$$
$$W(7) = W(111_2) = 111_2 = 7$$

Okay, let's generalize. Suppose we know that the claim is true for all numbers with k digits in their binary codes. Does it follow that it is also true for all numbers with $k+1$ digits in their binary codes?

Yes!

Here's why.

Suppose M has binary code $1ab...cd_2$ with $k+1$ digits.

If M is even, $d=0$ and $M=2N$ with $N=1ab..c_2$, which is a number with just k digits. Since N has k digits, we know $W(N)=ab....c1_2$.

But we also have

$$\begin{aligned} W(M)=W(2N)=2W(N)-1 \\ =2\times ab...c1_2-1 \\ =ab....c10_2-1 \\ =ab....c01_2 \end{aligned}.$$

which is $ab...cd1_2$, as hoped.

If M is odd, $d=1$ and $M=2N+1$ with $N=1ab..c_2$, which is a number with just k digits. Since N has k digits, we know $W(N)=ab....c1_2$.

But we also have

$$\begin{aligned} W(M)=W(2N+1)=2W(N)+1 \\ =2\times ab...c1_2+1 \\ =ab....c10_2+1 \\ =ab....c11_2 \end{aligned}.$$

which is $ab...cd1_2$, as hoped.

So from the claim being true for all numbers with three-digit binary codes, it follows that it is also true for all numbers with four-digit binary codes, and then for all numbers with five-digit binary codes, and so on, and so on.

The claim about binary codes is always true!

(My brain hurts!)

EXTENSION

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy this exploration?

Wild Exploration 1: Do the following observations make immediate sense?

a) If the initial count of students sitting in the circle is a power of two, then student 1 is sure to win.
b) If the initial count of students sitting in the circle is one less than a power of two, then the highest numbered student is sure to win.
c) If the initial count of student is one more than a power of two (and not two), then student 3 is sure to win.

Wild Exploration 2: Continuing the table of data values it seems there is a curious pattern.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
W(N)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

Between each power of two the values of the students who win run through the consecutive odd numbers, with the pattern being "reset to 1" at each power of two.

Is this pattern indeed true?

Wild Exploration 3:
a) Instead of counting in-out-in-out-in-out-… around the circle the teacher counts out-in-out-in-out-in-… instead. Care to analyse who wins this variation of the game?

b) Suppose the teacher deems every third student as "in" following the cycle out-out-in-out-out-in-out-out-in, and so on. Care to analyse who wins in this variation? Or more generally, what if the teacher followed the cycle setting every r th student as "in." (Part a) asks about the case $r=2$ and we just asked about the $r=3$ case.) Can you develop a general theory about winners in general games?

c) Who wins in games with the teacher following a repeated cycle of "in-out-in, in-out-in, in-out-in"?

Wild Exploration 4: Let's go back to the version of the game from this essay.
If $W(N)=ab...cd_2$, prove that $W(2N)=ab....c0d_2$ and $W(2N+1)=ab....c1d_2$.