A MANGO SHARING-AND-EATING PUZZLE

This piece is based on a USA Mathematical Talent Search question. It's a long and tricky question to read and make sense of, and this piece is chiefly about what to do when in such a challenging situation. The goal is not to solve the puzzle—though we do—but to learn how to honor one's emotional reactions and learn how to make a step of progress nonetheless. This is an important life skill!

EXPLODING DOTS Topic:

Experiences 1 and 2: Recognizing a $1 \leftarrow 2$ machine in disguise.

Suggested Grade Level:

High school.

A MANGO SHARING-AND-EATING PUZZLE

Consider sharing the following scary-looking problem with your students. Let them know that the point of this exercise is not to solve the puzzle—though, it will turn out we will—but to simply have an emotional reaction to it!

A group of 100 friends stands in a circle. Initially, one person has 2019 mangos, and no one else has mangos. The friends split the mangos according to the following rules:

- *sharing*: to share, a friend passes two mangos to the left and one mango to the right.
- *eating*: the mangos must also be eaten and enjoyed. However, no friend wants to be selfish and eat too many mangos. Every time a person eats a mango, they must also pass another mango to the right.

A person may only *share* if they have at least three mangos, and they may only *eat* if they have at least two mangos. The friends continue sharing and eating, until so many mangos have been eaten that no one is able to share or eat anymore.

Show that there are exactly eight people stuck with mangos, which can no longer be shared or eaten.

Source: USA Mathematical Talent Search USAMTS.
Year 31: Academic Year 2019-2020
Round 1: Problem 4/1/31

Some Things Students Might Say and Feel

This question is hard to understand: there is too much to take in!

The puzzle feels awfully contrived. Is it worth our while?

No one sits with 2019 mangos! There are never 100 people standing in a circle!

Ooh! I like puzzles like these.

A Discussion to Possibly Have

Mathematics is an intensely human subject. It was invented/created for humans, by humans, to be experienced, used, and enjoyed by humans. As such, we should each be our honest human selves when doing mathematics.

There are two fundamental steps to solving any problem in math – or life, for that matter.

Step 1: Have an emotional reaction to the problem. Openly acknowledge your reaction.

If a problem looks hard or scary, say so! If it seems artificial and contrived, say "Why should I care?" If you don't know what to do, acknowledged that you are flummoxed. And if you are intrigued, be intrigued. Whatever your reaction may me, consciously acknowledge your emotional response.

Then, when you are ready, take a deep breath and move to

Step 2: Do something. ANYTHING!

Turn the page upside down. Read the question again. Read the question backwards. Circle some words. Draw a picture. Draw a picture of a fish. Put the question aside, go for a walk, and come back later.

The point is to do something either relevant or irrelevant to the problem and work to get past an emotional impasse.

Many people just "shut down" when confronted with something confusing or scary. But by deliberately acknowledging one's emotions and taking a step of any kind to work with them, one often finds that a first step to handling the problem falls into place, even if your first step seemed irrelevant to the task at hand.

Discuss with your students some first steps they could take with this problem.

Some directly relevant ideas they suggest could include the following.

- Read the question again.
- Read the question again and pausing after each sentence to ask: Does that sentence seem important?
- Try the question starting with 1 mango, not 2019. Then maybe 2 mangos. Then 3.
- Draw a picture of some kind demonstrating how mangos "move."
- Start with answer 8 and somehow work backwards.
- Google the source of the problem and see if a solution was provided. (It was, but it too is very hard to parse!)

Listing such ideas is usually always fruitful. In this case, playing with small counts of mangos and drawing diagrams not only gives us a feel for what is going on, but proves to be enlightening!

By the way: Feel free to check out the interactive web app created by WildThinks Blog to enact this puzzle with small counts of mangos.

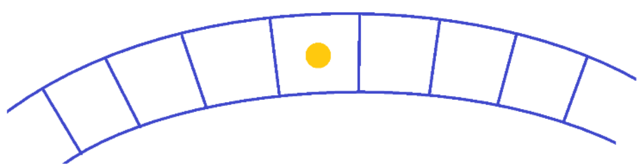
STUMBLING UPON A SOLUTION

See EXPERIENCES 1 and 2 of EXPLODING DOTS: Recognizing a $1 \leftarrow 2$ machine in disguise.

Okay … small counts of mangos.

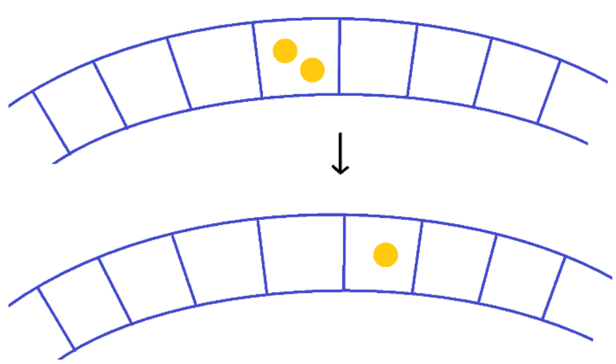
1 Mango

There isn't anything to do here. The situation stays with one person holding a mango.



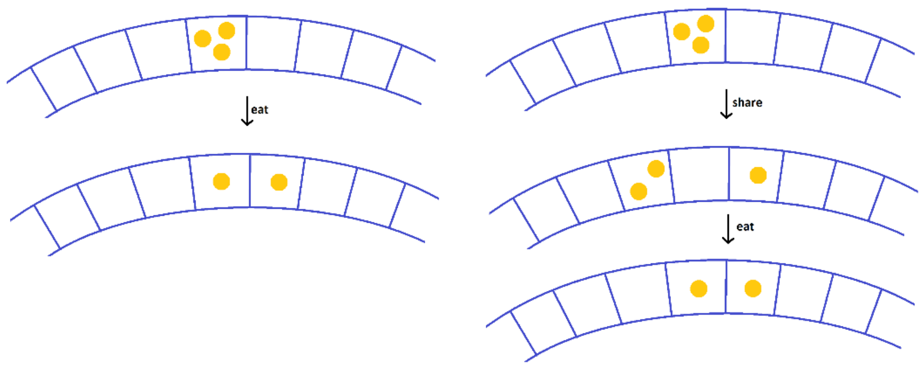
2 Mangos

The person holding the mangos can only eat one and pass one.



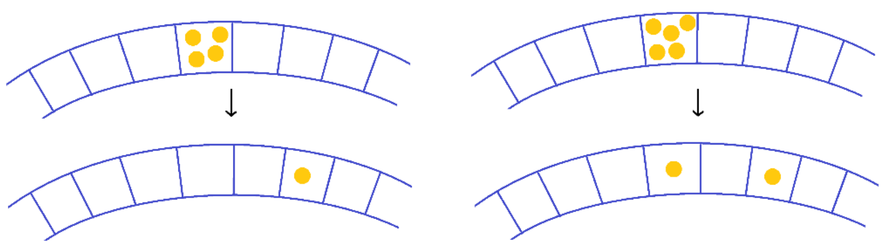
3 Mangos

Here the person can eat right away or share and then a neighbor eats.



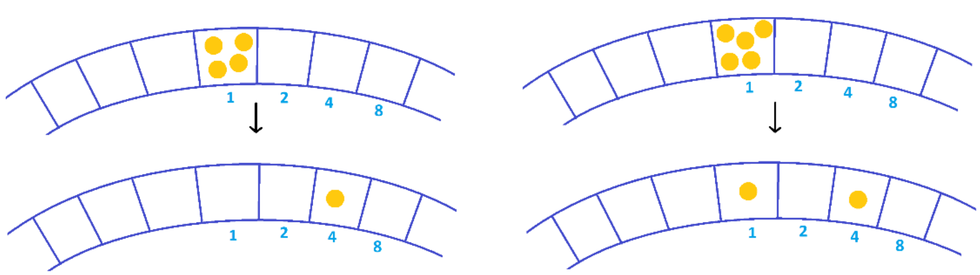
4 Mangos, 5 Mangos

There are more options available in these cases and one can check they all seem to lead to the same final configurations of mangos.

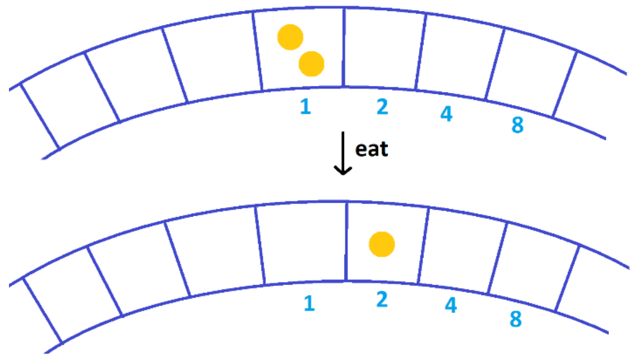


And after a while one can't help but notice that in an "eat" move, two mangos in any one cell of the diagram disappear and are replaced by one mango, one cell to the right.

Is this puzzle really about a $1 \leftarrow 2$ machine in disguise?

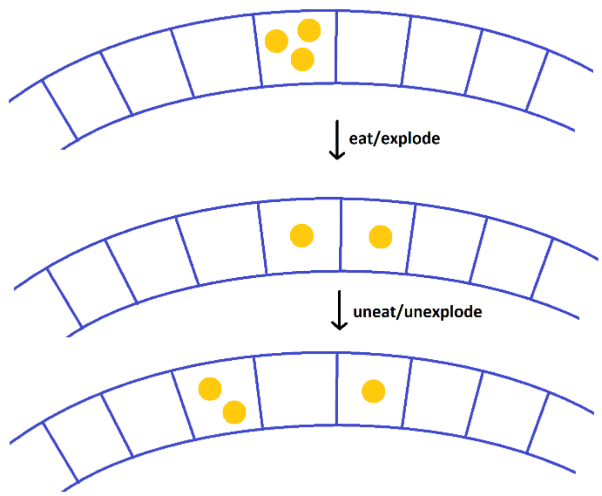


It looks like it, except that the explosions go in the opposite direction to what we are used to.



But there is the extra snag of a “share” move. Is that valid operation in a $1 \leftarrow 2$ machine?

Well … yes! Replacing three mangos in any one cell with two on one side and one on the other is just an explosion and unexplosion in a $1 \leftarrow 2$ machine.



So this puzzle is just about taking a set of mangos in the 1 position of a $1 \leftarrow 2$ machine and performing explosions and unexplosions until we settle on a pattern with at most one mango per cell.

We know that N dots in a $1 \leftarrow 2$ machine settle to the $1 \leftarrow 2$ machine code (binary code) of N, so N mangos in the hand of one person in this puzzle settle to the binary representation of N as mangos.

Since 2019 is 11111100011 in base two, we must settle to eight people holding a single mango, just as the puzzle suggests. And, just to be clear, since the base-two representation of 2019 is only eleven digits long, our mangos never “moved” around the entire circle of people. So it is indeed as though we are operating within a (non-circular) $1 \leftarrow 2$ machine.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: Is it "obvious" that every number has a *unique* $1 \leftarrow 2$ machine code? That is, could a number be represented in base two (using the digits 0 and 1) in more than one way? (If so, we have more work to do for this puzzle!)

Wild Exploration 2: What if there were 11 people standing in a circle with one person holding 2019 mangos? Is if the case that we again end up with eight people each holding one mango? What if, instead, it were just 10 people in a circle? 9 people in a circle?

Further Reading:

Here's the original problem (Problem 4/1/31) and published solution
https://www.usamts.org/Tests/Problems_31_1.pdf
https://www.usamts.org/Solutions/Solutions_31_1.pdf

Here, again, is WildThinks blog and web app about this problem.
https://wildthinkslaboratory.github.io/smartblog/usamts/2020/01/29/usamts2.html