## A STRATEGY FOR NIM 

The classic game of NIM is an ancient game played with piles of pebbles. Its origin is unknown. But it was only relatively recently in 1901 that the full mathematics behind the game was explored and explained. This was done by American mathematician Charles Bouton, who also coined the name "NIM."

A winning strategy for the game relies on writing counts of pebbles in binary and today's puzzle is to explain why this curious winning strategy works.

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EXPLODING DOTS Topic:
Experience 2: Understanding Binary
Suggested Grade Level:
Middle School, High School, and All.
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## A STRATEGY FOR NIM

Introduce the game of NIM to your students. Use counters, buttons, or some such for students to practice playing the game.

The game of NIM starts with three piles of counters on the tabletop between two seated players. One pile contains 3 counters, one 5 counters, and the third 9 counters.


Players take turns removing one or more counters from a single pile. (One is permitted to remove all the counters from a pile. One is required to take at least one counter.)

The person who takes the last counter-thereby leaving the tabletop empty-is the winner.

Play a few rounds of NIM with a partner just to get a feel for the game. Switch who has the first move a few times.

One can, of course, play this game with any number of piles, each containing any number of counters. To keep the conversation consistent, we'll just discuss this 3-5-9 game of NIM for now.

When the time is right, introduce material of this next section that describes a winning strategy.

In 1901, American Mathematician Charles Bouton found a curious winning strategy for playing NIM. We present his approach here in terms of codes in a $1 \leftarrow 2$ machine (aka binary codes).

Represent the count of pebbles in each pile in a $1 \leftarrow 2$ machine with the machines stacked on top of one another as shown.


At present, each column in this picture contains an odd number of dots.
A NIM move will change this diagram and change the even/odd-ness of some or of all of the columns. For example, removing two pebbles from the middle pile produces this picture.

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a) Find a NIM move from the original 3-5-9 diagram that the first player can take that, instead, gives a new diagram with each column having an even count of dots.
b) From your new diagram with each column possessing an even count of dots, explain why, in whatever NIM move the second player now takes, she is sure to create a diagram with at least one column containing an odd number of dots.
c) Prove, when presented with a diagram containing at least one column with an odd count of dots, it is always possible to make a single NIM move that gives a diagram with each column containing an even count of dots.
d) Prove, when presented with a diagram with all columns containing an even count of dots, every NIM move is sure to create a diagram with at least one column possessing an odd count of dots.
e) Explain why the first player of the 3-5-9 NIM game can be sure to win the game.

## Some Things Students Might Question or Notice

1. I don't know where to start. These questions seem hard!
2. So, what are we meant to do? Take this picture, change one of the rows (which means we're changing one of the piles), and get an even number of dots in each column? That doesn't seem possible!

3. Wouldn't any of these work, changing either the first, second, or third row as shown?


Oh! The first two have made piles bigger. The last one changes 3-5-9 to 3-5-6. It corresponds to taking three pebbles from the 9 pile.

So maybe it can always be done by removing pebbles from the biggest pile?
4. Part b) seems really hard: too many cases to check!

## SOLVING THE PUZZLE

## See EXPERIENCE 2 of EXPLODING DOTS: Understanding Binary.

No matter the number of piles or the size of the piles it is always possible to find a NIM move that turns a diagram with one or more columns containing an odd count of dots into a diagram with all columns containing an even count of dots.

Consider this diagram with three piles each containing a large number of pebbles.


Choose any dot in the leftmost "odd column" and delete it.


Next, either add or delete dots to the right of the deleted dot to give each column an even count of dots.


In this picture we have change the third pile from

to

that is, we changed the third pile from 92 pebbles to 74 pebbles, a smaller number. This corresponds to taking pebbles out from a pile and so is a valid NIM move.

But this begs the question:
In a $1 \leftarrow 2$ machine code, if we delete a dot and change some or all of the boxes to its right, is the result sure to be the code of a smaller number?

What's the worst possible case scenario? It would deleting just the one dot and adding the maximal number of dots possible to its right.

Does changing this picture
to this picture

give us a smaller number and so still represent the act of removing pebbles from a pile?

Well, yes!
Can you see that with unexplosions that the first of these pictures is equivalent to this picture?


And so it represents a number one larger than the second picture!

We have
Given a NIM game with a matching diagram with some columns containing an odd count of pebbles, there is always a valid NIM move that will yield a matching diagram with all columns containing an even count of pebbles.

Now, on the other hand, suppose you are handed a NIM scenario with matching diagram all of whose columns possess an even count of dots. (Call this an EVEN SCENARIO.) Whatever move you make changes the dots in one row of the diagram. In fact, we can be sure that the state of at least one box will change (if not, you haven't made a move). If a box loses a dot or if a box gains a dot, the column containing that box has turned to an "odd column."

> Given a NIM game with a matching diagram with all columns containing an even count of pebbles, any move is sure to produce a diagram with at least one column containing an odd count of pebbles.

Let's call a NIM scenario with at least one column containing an odd count of pebbles an ODD SCENARIO.

You, as a savvy player, now have a winning strategy if presented with an ODD SCENARIO: always play to give your opponent an EVEN SCENARIO. Your opponent will be forced to hand you back an ODD SCENARIO, which means there is at least one pebble remaining on the table. That is, your opponent simply cannot present you an empty set of piles. You have to be the one who does creates that, which means your win is certain with this strategy.

NEXT CHALLENGE: Can you create a mental schema so that you can do all the binary manipulations swiftly in your head while you play?

## EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

> Wild Exploration 1: Analyze misère NIM where the object of the game is NOT to win. Might there ever be a strategy for a player to ensure she never picks up the last pebble?

Wild Exploration 2: Every integer, positive or negative, can be uniquely represented in a $-2 \leftarrow 1$ with the digits 0 and 1 . For example, ten is represented as 11110 and negative ten as 1010 . These are called the negabinary codes of numbers.

Can one invent a NIM-like game with a winning strategy based on negabinary representations rather than binary representations?

