A POURING PUZZLE

This puzzle leads on into a theory of infinitely long processes to be represented by (potentially) infinite decimals in unusual bases. The standard curriculum, of course, assume students have fully mastered thinking about decimals in early grades. But this puzzle demonstrates that there is no such thing as being "done" with a topic!

EXPLODING DOTS Topic:

Experience 8: Decimals in alternative bases

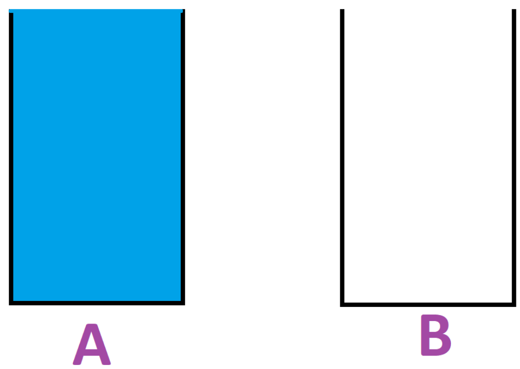
Suggested Grade Level:

High School.

A POURING PUZZLE

Here is a curious puzzle that leads into thinking about infinite processes. Share it with your students.

We have two identical containers A and B. Currently A is completely full of water and B is completely empty. We will be pouring water back and forth between the two containers.



In fact, we'll be performing very specific "pouring moves:" at any time we may either pour $\frac{2}{3}$ of the content of container A into container B, or pour $\frac{2}{3}$ of the content of container B into container A.

a) **After a finite sequence of pouring moves is it possible to end up with tank B precisely one-quarter full?**

If the answer is NO, then …

b) **Is it possible to see an amount of water in container B that is so close to one-quarter its volume that the human eye couldn't tell the difference?**

Some Things Students Might Question or Notice

1. The fractions two-thirds and a quarter just feel "incompatible." The answer to part a) is probably NO!

2. Given that the author of this question went on to offer part b) probably means that the answer to a) is indeed NO! (Going on pure psychology here!)

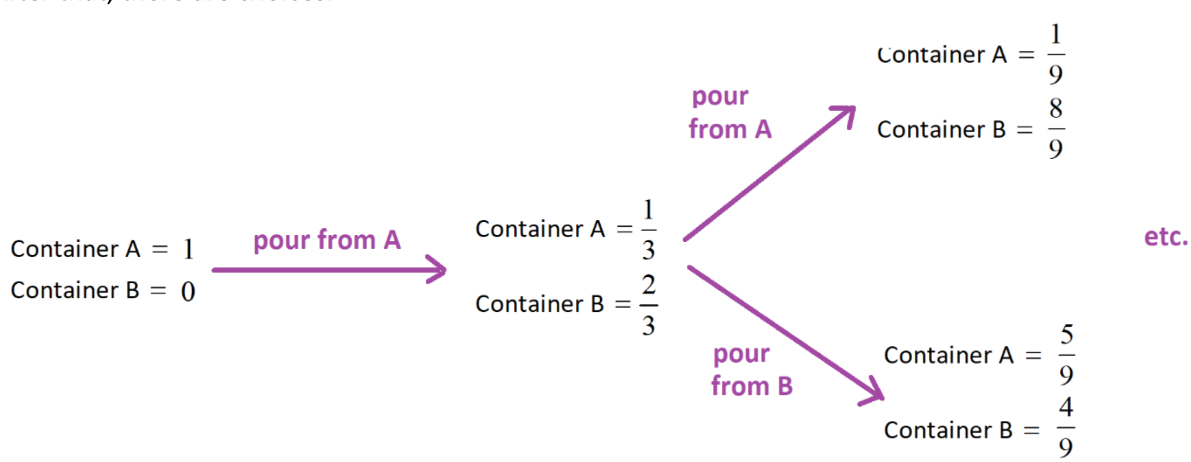
3. The author does not say one must alternate pouring directions. One could pour from the same container multiple times in a row.

4. The first move one makes must be to pour $\frac{2}{3}$ of the content from A into B to get

   Container A $= \frac{1}{3}$ full

   Container B $= \frac{2}{3}$ full.

   After that, there are choices.



   (In the diagram, the numbers how full each container is as a fraction of its volume.)

5. Since we're only ever taking thirds and two-thirds of fractions with denominators that are powers of three, we will never see the fraction $\frac{1}{4}$ arise. The answer to part a) is definitely NO!

SOLVING THE PUZZLE

See EXPERIENCE 8 of EXPLODING DOTS: Decimals in alternative bases.

There are two possible pouring moves: to pour from container A or to pour from container B.

Let's analyze what happens with each of these moves if we have container B, say, fraction x full. (We want to see $x = \frac{1}{4}$.) Container A is then fraction $1-x$ full. And to keep things simple, let's just focus on container B in this analysis.

$$\begin{tikzcd}
 & \text{Container B} = x + \frac{2}{3}(1-x) = \frac{2}{3} + \frac{x}{3} \\
\text{Container B} = x \arrow[ur, "\text{pour from A}"] \arrow[dr, "\text{pour from B}"'] & \\
 & \text{Container B} = \frac{x}{3}
\end{tikzcd}$$

With both pours we see that we are dividing the given quantity x by three. Perhaps it will be helpful, then, to think of our fractions as expressed in base three, that is, via an $1 \leftarrow 3$ machine: $x = .abcd\ldots$ for some digits a, b, c, d,….

To be clear:

An ordinary base-ten decimal $.abcd\ldots$ means the quantity $\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \frac{d}{10000} + \cdots$ with a, b, c, d, … any of the digits 0, 1, …, 9.

In base three, a "decimal" $.abcd\ldots$ instead means the quantity $\frac{a}{3} + \frac{b}{9} + \frac{c}{27} + \frac{d}{81} + \cdots$ with a, b, c, d,… any of the digits 0, 1, 2.

So, if $x = .abcd\ldots$ as a base-three decimal, then

$$\frac{x}{3} = \frac{1}{3}\left(\frac{a}{3}+\frac{b}{9}+\frac{c}{27}+\frac{d}{81}+\cdots\right) = \frac{a}{9}+\frac{b}{27}+\frac{c}{81}+\frac{d}{243}+\cdots = .0abcd\ldots$$

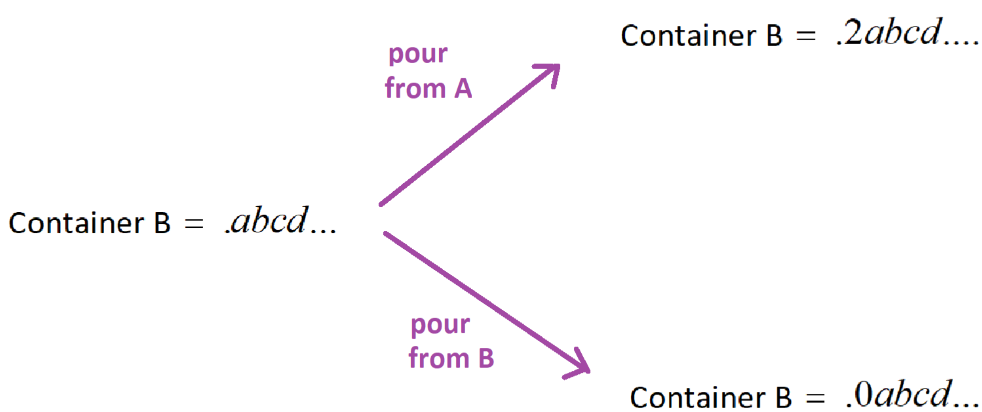
and

$$\frac{2}{3}+\frac{x}{3} = \frac{2}{3}+\frac{a}{9}+\frac{b}{27}+\frac{c}{81}+\frac{d}{243}+\cdots = .2abcd\ldots.$$

From now on, let's assume all our decimal expressions are base-three "decimals."

We have:

> If container B is fraction $.abcd\ldots$ full, then a "pour from A" move changes this to fraction $.2abcd\ldots$ (insert a 2 into the expression) and a "pour from B" move changes this to fraction $.0abcd\ldots$ (insert a 0 into the expression).



We start with fraction 0 of water in container B. We see that we can thus create the following fractions of water in that container

$$.2 = \frac{2}{3}$$

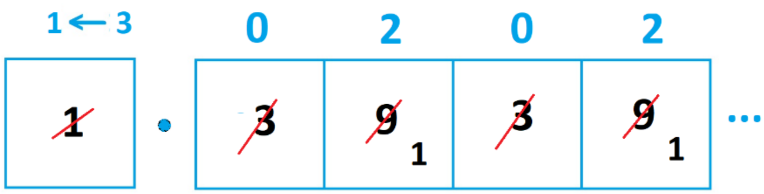
$$.02 = \frac{2}{9} \qquad .22 = \frac{8}{9}$$

$$.002 = \frac{2}{27} \qquad .202 = \frac{20}{27} \qquad .022 = \frac{8}{27} \qquad .222 = \frac{26}{27}$$

and so on.

We want fraction $\frac{1}{4}$ of water in container B. Can we create it?

To answer that we need to see what $\frac{1}{4}$ is as a base-three decimal. Let's compute $1 \div 4$ is a $1 \leftarrow 3$ machine.



(Ooh! If we're speaking base three, then $\frac{1}{4}$ should be written $\frac{1}{10}$, yes?)

We see that one-quarter is $.0202020202.....$

We can now answer the two questions.

a) With a finite number of pouring moves we can only create fractions that are expressed as finite decimals in base three. As one-quarter is an infinite decimal in this base, we'll never see container B precisely one-quarter full.

b) But we can create the fractions $.2$ and $.202$ and $.202020202$ and $.2020202020202020202020202020202$. That is, we can produce a fraction of water in container B as close to one-quarter as we like!

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1:

a) Could we ever see container B precisely one-third full? If not, could we be close?
b) Could we ever see container B precisely one-half full? If not, could we be close?

See the essay here that explores these thoughts.

Wild Exploration 2:
Draw on section of the number line one unit long.

a) Show on the picture the locations of all the fractions that have a 0 or a 2 as first digit in their base-three “decimal” representations.
b) Show on the picture the locations of all the fractions that have only 0s and 2s for the first two digits in their base-three “decimal” representations.
c) Show on the picture the locations of all the fractions that have only 0s and 2s for the first three digits in their base-three “decimal” representations.
d) How would you describe the locations of the all the fractions that could be fractions of volume in container B for our puzzle?

Comment: Look up the *Cantor Set* on the internet.

Wild Exploration 3: Suppose we changed the puzzle so that a “pouring move” consisted of pouring half the contents of one container into the other. Which fractions could we see, or come as close to seeing as we like, in conducting pouring moves?

Wild Exploration 4: Suppose we changed the puzzle so that a “pouring move” consisted of pouring a third of the contents of one container into the other. Which fractions could we see, or come as close to seeing as we like, in conducting pouring moves?

Wild Exploration 5: Develop a general theory about fractions of volume we could see in container B if a pouring move consisted of pouring proportion $r\%$ of the contents of one container into the other, even if r is an irrational number!