MODULE 8

A REFRESHINGLY ACCESSIBLE APPROACH TO

QUADRATICS

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MODULE CONTENT

INTRODUCTION

Can you think of two numbers whose sum is 10 and product is 24?

Maybe your first response is: why? And that's a great response and I'm with you on it: Why indeed?

But here's the thing: Now that you've read this question, I bet you can't not think about it! It's now in your head and I bet you really are wondering if there are two numbers with sum 10, which multiply to 24.

After some while (and if you are like me, it will be a long while) you might think of 6 and 4. Great!

Now that I've lured you into this game—and it is just a game—here's a next puzzle:

Can you think of two numbers whose sum is 10 and product is 25?

After some while you might be wondering about 5 and 5 and asking: "Do the numbers have to be different?" I guess I didn't say anything about this, so 5 and 5 is fine. After all, why not? They work!

Okay then

Can you think of two numbers whose sum is 10 and product is 26?

Now things are tough!

We're playing a purely intellectual game for no purpose other than the fun of it. And humankind has been playing games like these for thousands of years. It's mathematic for the pure intellectual joy of it all. And the mathematics needed to fully command a puzzle like the ones able is the mathematics of this module.

But this, I know, is not enough to motivate this module.

So let me offer this:

Babylonian scholars of 2000 B.C.E. left clay tablet records of the mathematics they were thinking about. Some of the mathematics they described was motivated by practical problems such as working out areas of the rectangles with certain constraints on their side lengths—perhaps for field sizes—and some mathematics was motivated by pure number play. And both types of problems relied on the same mathematical technique of, as we would say today, "solving a quadratic equation." And these quadratic equations ever since have fascinated scholars across the globe and the centuries.

- Egyptian scholars of about the same time left record of solving "quadratic equations" too on their papyrus scrolls.
- Greek scholar Euclid (dubbed "the father of geometry") of 300 B.C.E. wrote about solving certain classes of quadratic equations in his famous tome, *The Elements.*

- Scholars of ancient China, as recorded in *The Nine Chapters on the Mathematical Art*, circa 200 B.C.E., solved quadratic equations too.
- Indian mathematician Brahmagupta (597 – 668 C.E.) wrote a passage that describes in words a general procedure for solving quadratic equations.
- Ninth-century Persian scholar al-Khwārizmī wrote special cases of a "quadratic formula" in algebraic symbols. (He avoided the use of negative numbers and hence his need for presenting equations in special forms.)
- In 1594, Belgian scholar Simon Stevin wrote down a general "quadratic formula" as we recognize it today.

And then at the turn of the 16^{th} century, Italian scientist Galileo logically deduced and experimentally verified that the mathematics needed to describe the projectile motion, that is, the motion of objects moving through the sky under the sole influence of gravity (ignoring air resistance and the broad curvature of the Earth) is precisely the mathematics of quadratics.

And so, at the base of some fundamental physic, sits the mathematics of this module. So be it for practical application, for fun intellectual play, or for some basic science, quadratics have been central to humankind's thinking for literally thousands of years. It is worth our while, as a member of the human clan, to at least be aware of what the fuss is all about.

So in this module we'll explore the mathematics of quadratics and attend to all the questions and ideas presented here in this short introduction.

8.1(A) THE KEY PLAYERS OF THE STORY

The story of quadratics is really the story utilizing area-thinking and coupling it with the power of symmetry. And by taking the time to understand the strength of these two key players in the story, one not only develops a profound understanding of this topic, but also develops the deep joy and the agency that comes from thinking like a mathematician.

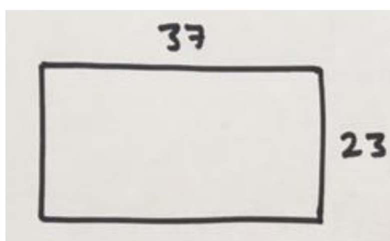
So, let's take our time and simply begin with the story of area.

AREA INFORMS BOTH ARITHMETIC AND ALGEBRA

Consider the arithmetic problem

Compute 37×23.

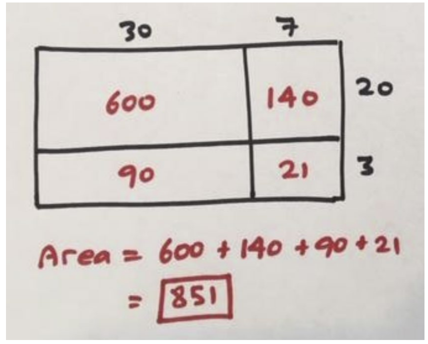
To me, this is really a geometry problem: we are being asked to compute the area of a 37-by-23 rectangle.



But the numbers are awkward! Can we make matters easier on ourselves in some way?

Principle: Mathematicians will always pause and think hard to avoid hard work!

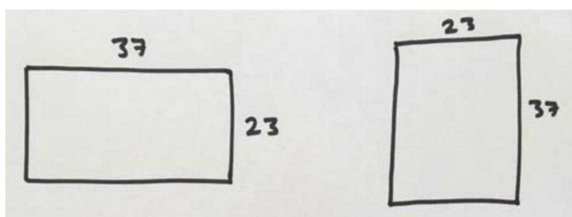
It seems natural to break each of the numbers 37 and 23 into friendlier numbers, say 30 and 7, and 20 and 3, and thus divide the rectangle into four pieces whose areas are simple to compute. The area of the rectangle—and thus the answer to 37×23—is $600 + 140 + 90 + 21 = 851$.



Practice 1: *Use the area model to compute* 3721×223. *(Into how many pieces might you divide your rectangle?)*

Practice 2: *Use the traditional long multiplication algorithm to compute* 845×387. *And use it again to compute* 387×845, *but as you do so this second time ask yourself: Is it obvious the algorithm will give the same final answer?*

Area-thinking informs us of the general behavior of numbers. For example, rotating a 37-by-23 rectangle ninety-degrees gives a 23-by-37 rectangle. The area of the rectangle does not change in this process, so it must be the case that 23×37 gives the same answer 851.



In general, rotating a picture of a rectangle 90-degrees shows that

$$a\times b=b\times a$$

for all numbers that can be the side-lengths of rectangles.

In the early grades, we know only of the positive counting numbers, and the area model shows that $a\times b=b\times a$ for all positive counting numbers. But as our schooling progresses, we learn of other types of numbers. We can imagine rectangles with fractional side lengths, and so we come to believe that $a\times b=b\times a$ holds true for all fractions as well.

Practice 3: *Use the area model to compute* $4\frac{1}{3}\times 10\frac{2}{5}$.

And we can imagine rectangles with irrational number side-lengths and so we believe that $a\times b=b\times a$ holds for irrational numbers as well.

And this belief, by this point, feels so natural and so right that we believe, surely, that $a \times b = b \times a$ should hold for ALL numbers, even ones that extend beyond geometry, namely, negative numbers.

Technically, one cannot have rectangles with negative side-lengths or with negative areas. But we like to believe that if we were to draw pictures of such rectangles they will still speak truth about arithmetic, that $a \times b = b \times a$ holds too for negative numbers, for instance.

> **Upshot:** *We like to believe that rectangle diagrams speak truth about arithmetic for all types of numbers, even if the numbers are technically not valid in geometry.*

This belief now answers an age-old-question.

Why negative times negative is positive

Just to be clear on the basics …

We all agree that positive times positive is positive. For example, 3×2, interpreted as "three groups of two," has answer $2+2+2=6$, positive six.

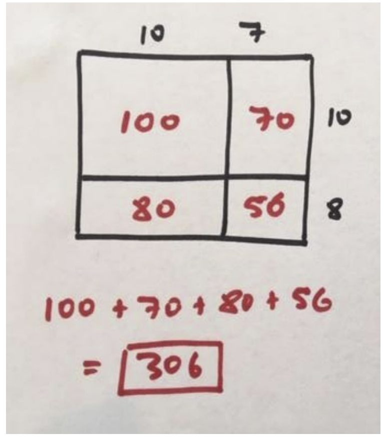
And most people are comfortable saying that positive times negative is negative. For instance, $3 \times (-2)$ is "three groups of negative two" and so has value $-2 + -2 + -2 = -6$, negative six.

Matters become trickier with negative times positive. For instance, reading $(-3) \times 2$ as "negative three groups of two" makes no sense! But armed with our belief that $a \times b = b \times a$ holds true for ALL types of numbers, we can argue then that $(-3) \times 2$ equals $2 \times (-3)$, and so is "two groups of negative three" and thus has value $-3 + -3 = -6$, negative six.

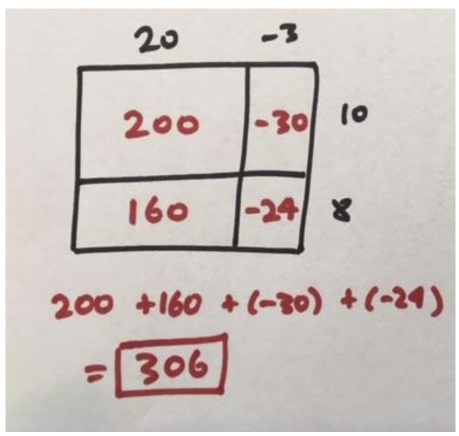
But trouble comes with negative times negative. What is $(-3) \times (-2)$? We've all been trained to say "positive six." But why? "Negative three groups of negative two" makes no sense. Nor does "negative two groups of negative three." Switching the order of the terms does not help.

To attend to this, let's come at it sideways by computing 17×18 four different ways!

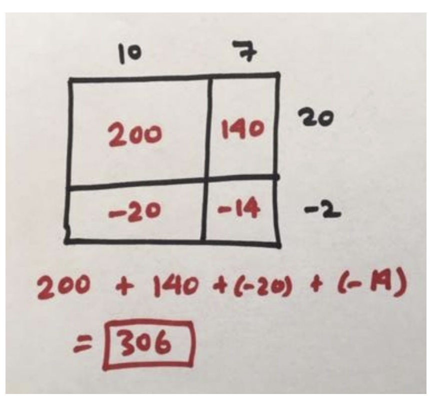
We can compute it be breaking up the numbers 17 and 18 in the expected ways to see the answer 306.



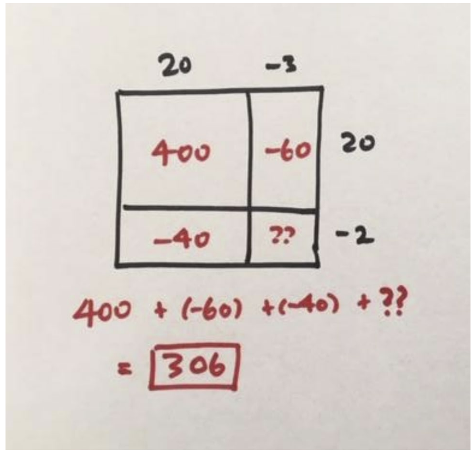
Or we try something unexpected with the number 17 and think of it as $20+(-3)$. Since we know *negative* $\times$ *positive* and *positive* $\times$ *negative* are both negative, we can complete the diagram and see the answer 306 appear again. (The diagram speaks arithmetic truth!)



We can see the answer 306 yet again in thinking of 18 as $20+(-2)$.



But let's now write both 17 and 18 in their unexpected ways. We see the diagram needs the answer to $(-3)\times(-2)$. But we know, from three computations now, that $17\times18=306$.



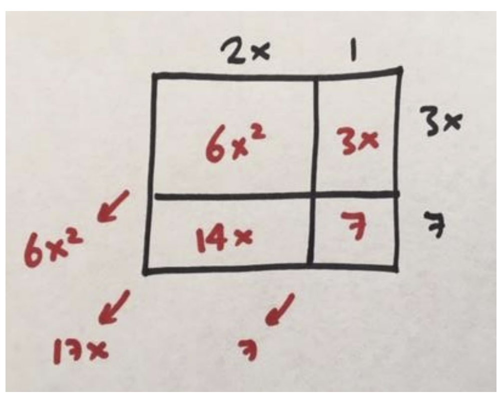
The mathematics shows that we have no choice but to set $(-3)\times(-2)=6$, positive six!

Upshot: *Our belief that all numbers obey the same rules of arithmetic inspired by geometry forces us to conclude that negative times negative is positive.*

Practice 4: *Compute* 16×15 *four different ways to conclude that* $(-4)\times(-5)$ *is positive* 20*.*

Since the area model speaks truth for all numbers we can use it too in algebra. For instance, the area model shows that $(2x+1)\times(3x+7)$ equals $6x^2+17x+7$.

And this is true no matter the value of x: positive or negative!



Practice 5:

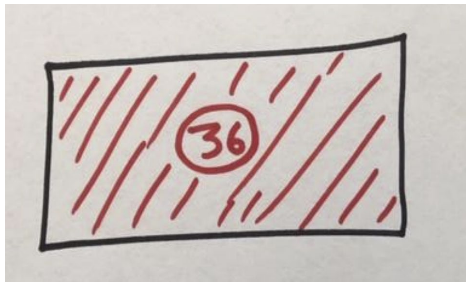
a) Use the area model to compute $(2x^2+x+1)(3x+2)$*.*

b) Use your answer to quickly see the value of 211×32*.*

c) Put $x=-10$ *into your answer from a). What multiplication problem is this the answer to?*

THE POWER OF SYMMETRY

Here's a rectangle. I tell you its area is 36 square units. What can you tell me about the rectangle?



Answer: Nothing! It might be a 4-by-9 rectangle, or a 2-by-18 rectangle, or a 4½-by-8. You don't know. You only know that it's area is 36 square units.

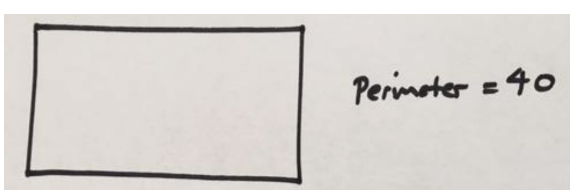
But suppose I now add one more piece of information about this rectangle, that it is *symmetrical*. Now you know <u>everything</u> about that rectangle. It must be a 6-by-6 square!

This illustrates the power of symmetry in mathematics. Often the introduction of symmetry in a scenario collapses unknown information into crystalline precise information.

> **Principle:** **Mathematicians recognize the power of symmetry. Symmetry is a mathematician's friend!**

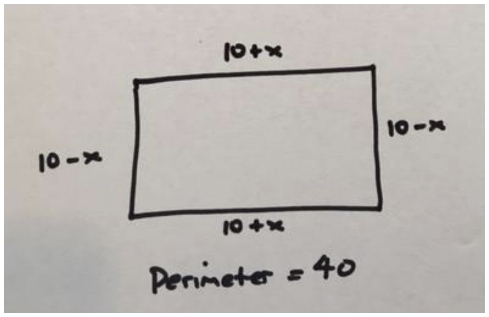
To illustrate the power of symmetry, let's end off with a classic textbook problem in the study of quadratics—even though we have not yet introduced what a quadratic is! We can solve the problem just with the joy of symmetry.

Example: *A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. What should the dimensions of the rectangle be to obtain a pen of maximal possible area?*



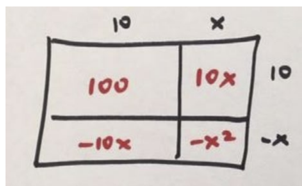
Answer: Let's consider the symmetrical situation first: a pen that is a square. Since the perimeter of the pen is 40 meters, this would be a 10-by-10 square.

But we have no reason to believe that a square pen is the one of maximal area. In general, a pen will have one side longer than 10 meters and the other shorter than 10 meters. Let's measure the dimensions of a general pen by recording the degree to which it deviates from being a symmetrical square pen. Let's consider a $10+x$-by-$10-x$ pen for some number x.



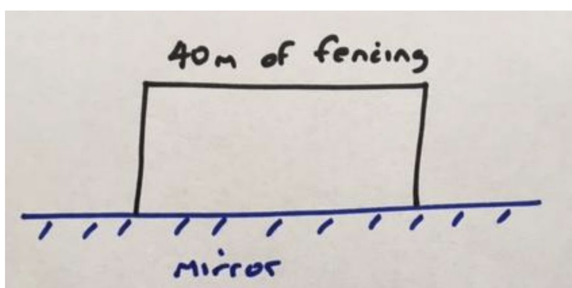
The area this pen is

$$\begin{aligned}(10+x)(10-x) &= 100+10x-10x-x^2\\ &= 100-x^2.\end{aligned}$$



And what value of x gives a maximal value for the area formula $100-x^2$? Clearly $x=0$ gives the maximal area. So the pen that does not deviate at all from being a square pen is the one of maximal area! The farmer should build a 10-by-10 pen.

Practice 6: (**CHALLENGE**) *A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. But she has huge mirrored wall in her field and wants to use the mirror as one side of her rectangular pen.*



What should the dimensions of the rectangle be in order to obtain a pen of maximal area?

Principle: Mathematicians will attempt to push the results of a previously solved problem to help with a new problem.

8.1(B) Equations that can be Solved by the Quadrus Method

Scholars of some 2000 years ago realized that there are certain types of equations that can be readily solved by drawing pictures of squares. And lo, and behold, these became known as equations that can be solved by the "square method." The Latin word for square is *quadrus*, and as Latin took on the role of being the official scientific language in the West, these equations became known as the "quadrus equations" or, as we call them today, *quadratic equations.*

> **Comment:** We regularly use the prefix *quad* in everyday life. It means "four." A quadrilateral, for instance, is a figure with four sides. A quadruped is an animal with four limbs. Students hold recess in the school quadrangle – a rectangular open area with four right angles.

So let's solve quadrus equations and see the power of area and symmetry in our play. That is, let's follow the *true mathematical story* of these equations.

We'll do this as a series of steps, starting with level 1 equations and building our way up to complex equations in level 6. Each step along the way will exercise our wits and teach us to how think like mathematicians!

LEVEL 1 QUADRATICS

Here's our first level 1 problem.

PROBLEM: *Solve* $x^2 = 100$.

If you read this out loud—"x squared equals one-hundred"—you notice we say the word *square*. This problem is asking for the unknown side length of a square whose area is 100.

As a geometry problem, there is only one answer, namely $x = 10$. But as we are extending our thinking beyond just literal geometry, we realise there is a second number in arithmetic whose square is also 100, namely, $x = -10$.

Thus the quadratic equation $x^2 = 100$ has two solutions:

$x = 10$ or -10.

PROBLEM: *Solve* $p^2 = 49$.

This problem is calling the unknown number p rather than x. Not a problem!
Something squared is 49. That something must be 7 or its negative version, -7.

$$p^2 = 49$$
$$p = 7 \text{ or } -7$$

And this is about it for level 1! Though there can be some slight trickiness.

PROBLEM: *Solve* $64x^2 = 25$.

It seems compelling to divide both sides of the equation by 64, to that is reads

$$x^2 = \frac{25}{64}.$$

We now see that x can have a fractional value.

$$x = \frac{5}{8} \text{ or } -\frac{5}{8}.$$

Counting Solutions: It seems that every level one equation we encounter will have 2 solutions: a number and its negative version. But this need not be the case. Consider the equation

$$x^2 = 0.$$

There is only one number whose square is zero, namely $x = 0$. So this is an example of a quadratic equation with just one solution.

> **Challenge:** *Some students might be tempted to write*
>
> $$x = 0 \text{ or } -0$$
>
> *not realising that -0 is the same as 0. How would you explain to a fellow student that "the opposite of zero" is again zero?*

It is possible for a quadratic equation to have no solutions at all! For example, there are no solutions to

$$x^2 = -5.$$

(Recall from the last section that positive times positive is positive and also negative times negative is positive. No number multiplied by itself will give a negative answer.)

It seems: Each quadratic equation will have either 0, 1, or 2 solutions.

We'll keep checking on the validity of this claim as we progress to more complicated equations.

PRACTICE 7: *Solve*

a) $2x^2 = 50$
b) $y^2 + 5 = 14$
c) $100x^2 = 1$
d) $4p^2 = 0.25$
e) $a^2 + 7 = 7$
f) $x^2 = 20$
g) $x^2 = -20$

LEVEL 2 QUADRATICS

Okay. Up a slight notch in difficulty.

PROBLEM: *Solve* $(x+3)^2 = 25$.

The key here is to "step back" from the problem and see what it is really asking: *Something squared is* 25. Oh! So the "something" better be 5 or -5. And what is the "something"? It is $x+3$.

We have

$$x+3=5 \text{ or } x+3=-5.$$

Subtracting 3 throughout gives

$$x=2 \text{ or } x=-8.$$

Great!

PROBLEM: *Solve* $(2x+4)^2 = 4$.

Something squared is 4 and so we have

$$2x+4=2 \quad \text{or } -2.$$

We want the value of x, so let's subtract 4 throughout

$$2x = -2 \text{ or } -6$$

and dividing through by 2 then gives

$$x = -1 \text{ or } -3.$$

Done!

PRACTICE 8: *Solve*

a) $(4x-6)^2 = -7$

b) $(4x-6)^2 = 0$

c) $(4x-6)^2 = 4$

d) $(4x-6)^2 = 5$

PRACTICE 9: *Solve*

a) $(y+1)^2 - 2 = 23$

b) $4(p-2)^2 - 16 = 0$

c) $9 + \left(34x - 77\frac{1}{2}\right)^2 = 0$

d) $\left(x - \sqrt{2}\right)^2 = 5$

LEVEL 3 QUADRATICS

Let's now go up another level. When one sees a level 3 problem for the very first time it is SHOCKING! It looks completely different from those in levels 1 and 2.

PROBLEM: *Solve* $x^2 + 6x + 9 = 25$.

This is an opportunity to exercise two key steps in solving a problem.

STEP 1: *Be your honest human self and acknowledge your human reaction to the problem!*

If a problem looks scary, go GULP! and say "This problem looks scary." If it looks cool and fun say "This looks fun." If it looks confusing and you don't know what to do say "I don't know what to do!"

Mathematics is a human enterprise, made by humans, for humans, so be your honest human self.

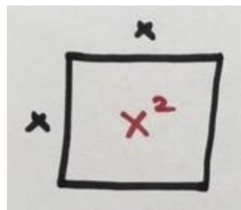
Next, take a deep breath and

STEP 2: *DO SOMETHING! Anything!*

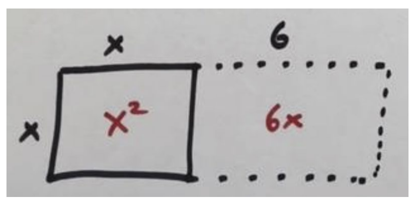
If you need to take a break to let your brain mull on the problem, take a quick break. Or maybe you can underline some words in the question, or draw a diagram, or read the question backwards. Just do something! You will be surprised how powerful taking a first piece of action—of any kind!—can be in getting going on a problem.

So what about our current problem, solving $x^2 + 6x + 9 = 25$?

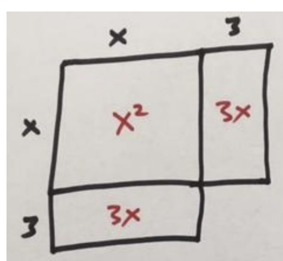
If I read it out loud, I do hear the word "square" again. I could, at least, draw a picture of a square to match the x^2 term. That's something!



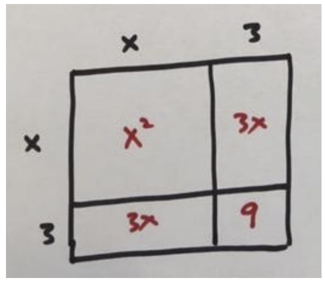
But there is also a $6x$ term. Could I add that to the picture? Sure, I can see how to add an area of $6x$.



But here I am drawing a picture of an unsymmetrical rectangle. And remember, from the last lesson we learned that symmetry is our friend. So can we add an area of $6x$ to our picture in a symmetrical way? Sure! Let's split that area and add on two three-by- x regions as shown.



Now we see we have a picture of an incomplete square. It feels absolutely compelling to complete the picture of the square! The final corner is a 3-by-3 square of area 9. And this is the precisely the number problem wants!



So our picture of areas x^2 and $6x$ and 9 is precisely the picture of an $x+3$ by $x+3$ square.

Now the original problem wants x^2+6x+9 to equal 25. So we want the area of our square to be 25. That is, we see we need to solve

$$(x+3)^2 = 25$$

and that is a level 2 problem we've already solved! We have

$$x+3 = 5 \text{ or } -5$$
$$x = 2 \text{ or } -8.$$

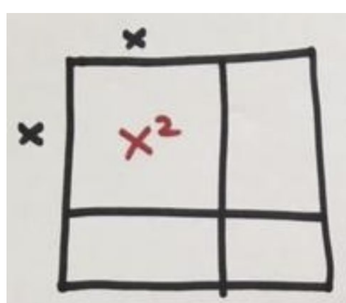
And one can double check that both $x=2$ and $x=-8$ really do make x^2+6x+9 equal to 25. So this is the challenge of level 3:

Recognise complicated-looking expressions as level 2 problems in disguise.

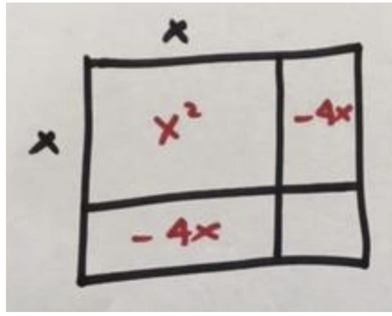
It's fun!

PROBLEM: *Solve* $x^2 - 8x + 16 = 17$.

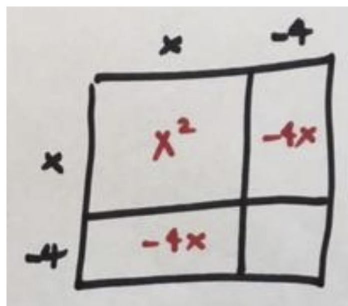
Okay, we have an x^2 term, so let's draw a square. In fact, since we know we are going to add on to this picture to make an even larger square, let's just go ahead and draw a big square divided into four symmetrical pieces right away.



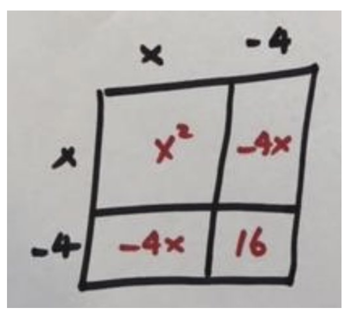
We need to include an "area" of $-8x$. (Remember, we are pushing our diagrams beyond geometry to speak general truth about all arithmetic.) We do it in a symmetrical way.



We must have side lengths of -4 in the picture. (Remember, even though the geometry is strange, these pictures speak arithmetic truth.)



And so the area of the final piece of the completed square is $(-4)\times(-4)=16$, which is precisely the value the problem wanted.



Thus we see that $x^2-8x+16$ is really an $x-4$ by $x-4$ square. We actually being asked to solve

$$(x-4)^2=17.$$

Now the number 17 is awkward, it doesn't have a nice square root. (Since $\sqrt{16}=4$ we have that $\sqrt{17}$ will be a number a little over four.) Be we can write our answers in terms of the square root of 17 and its negative version.

$$x-4=\sqrt{17} \text{ or } -\sqrt{17}$$

Adding four gives

$$x = \sqrt{17} + 4 \text{ or } -\sqrt{17} + 4.$$

Comment: Some people are fussy and prefer to write numbers like this with the integers mentioned first and the roots second. So some might present these answers as

$$x = 4 + \sqrt{17} \text{ or } 4 - \sqrt{17}.$$

Mathematics does not care, but your curriculum might!

PRACTICE 10: *Solve*

a) $p^2 - 6p + 9 = 9$
b) $x^2 - 4x + 4 = 1$
c) $x^2 - 20x + 100 = 7$
d) $r^2 - 16r + 64 = -2$
e) $x^2 + 2\sqrt{5}x + 5 = 36$
f) $x^2 - 2\sqrt{2}x + 2 = 19$

PRACTICE 11: *Solve for* x *giving your answer in terms of* A and B.

$$x^2 + 2Ax + A^2 = B^2$$

We're feeling good about our square method. By completing the picture of a square—literally!—we can solve some complicated quadratic equations. In fact, let's keep going and show how this square method solves *all* quadratic equations.

Just to be clear, any expression of the form $ax^2 + bx + c$ is called a **quadratic expression** and a **quadratic equation** is any equation that can be written in the form $ax^2 + bx + c = d$.

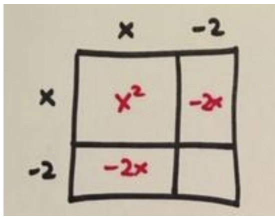
LEVEL 4 QUADRATICS

Consider this problem.

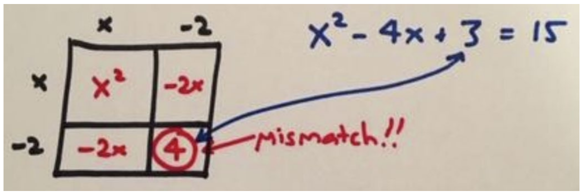
PROBLEM: *Solve* $x^2 - 4x + 3 = 15$.

This looks just like a level 3 problem. Is there some hidden difficulty? Let's find out!

Start by drawing the square. We have an x^2 piece and two pieces of "area" $-2x$. Thus we have two side lengths of -2.



Completing the picture of the square we see we have a final piece of area 4.



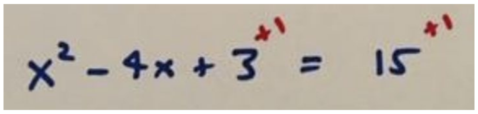
And that's the difficulty! The square wants the number 4 but the problem has only the number 3. Oh dear!

What can we do?

> **PRINCIPLE:** If there is something you want in math, just make it happen! (And deal with the consequences.)

Can we turn that "3" into a "4"?

Sure. Let's just add 1 to it! But there are consequences. If we add 1 to the left side of an equation, we must do the same to the right side as well.



Now we have the equation $x^2 - 4x + 4 = 16$ with the left side perfectly matching the pieces of the $x-2$ by $x-2$ square. So this is really the equation

$$(x-2)^2 = 16,$$

which is a level 2 problem!

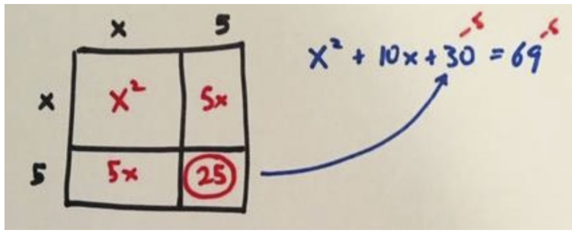
$$x-2 = 4 \text{ or } -4$$

$$x = 6 \text{ or } -2$$

Fabulous!

PROBLEM: *Solve* $x^2 + 10x + 30 = 69$.

Let's draw the square. We have an x^2 piece and two $5x$ pieces. Thus we have two side lengths of 5 meaning we want a final piece of area 25.



But we don't have "25" in the problem: instead it's 30. So let's subtract 5 and work with the equation $x^2 + 10x + 25 = 64$ instead.

And why do we like $x^2 + 10x + 25$? Because it is an $x+5$ by $x+5$ square. So, the problem we really wish to solve is

$$(x+5)^2 = 64.$$

And this is a level two problem.

$$x + 5 = 8 \text{ or } -8$$

$$x = 3 \text{ or } -13$$

Isn't this just grand?

PRACTICE 12: *Solve*

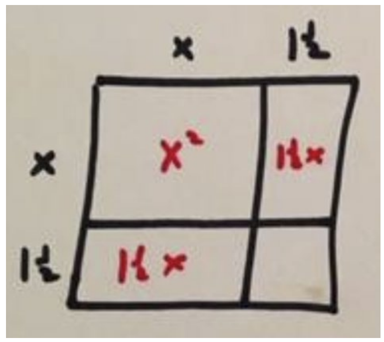
a) $f^2 + 8f + 15 = 80$
b) $w^2 + 90 = 22w - 31$
c) $x^2 - 6x = 3$.

LEVEL 5 QUADRATICS

We're ready for more!

PROBLEM: *Solve* $x^2 + 3x + 1 = 5$.

Let's go ahead and use the square method.



But we soon see that we are dealing with awkward fractions! They are not fun.

> **PRACTICE 13 (OPTIONAL):** *Push on with this problem and do work with fractions. Show that the square method eventually gives the solutions* $x = 1$ *or* $x = -4$. (The square method will <u>never</u> let you down!)

But is there a way we can avoid awkward fractions? That's is, can we be mathematicians and work to avoid hard work?

The problem lies with the middle coefficient, the number "3." It is odd, and so does not split into two nicely.

Is there a clever way then we can make that middle number even?

<u>IDEA 1</u>: *Add* x *to both sides of the equation.*

Let's solve instead

$$x^2 + 4x + 1 = 5 + x.$$

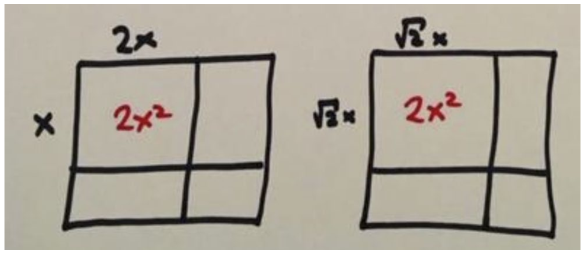
But then we'll have an answer for x that still involves x on the right, and so wouldn't be an actual number answer in the end. Hmmm.

<u>IDEA 2</u>: *Double everything.*

Why not solve instead $2x^2 + 6x + 2 = 10$?
We now have an even number in the middle.

Let's draw the square.

But then we see we have a problem: the piece of area $2x^2$.



We could think of this piece as x times $2x$, but then we are ruining our symmetry. (Remember, symmetry is our friend. We want to keep things square and not make rectangles.)

To keep it square could use $\sqrt{2}x$ times $\sqrt{2}x$. But if we didn't want to work with fractions, we probably don't want to with square roots either!

> **PRACTICE 14 (OPTIONAL):** *Push on with this problem and do work with the square roots. Show that the square method again eventually gives the solutions* $x=1$ *or* $x=-4$.

So, what can we do? Both our ideas were good. They just turned out not to be helpful.

> **Comment:** This the process of doing real mathematics! One often has brilliant ideas that, well, turn out not to be helpful. The thing to do then is to mull and think and wait for a next brilliant idea that just might work. This might take minutes, hours, days, even months or years for a particularly tough problem!

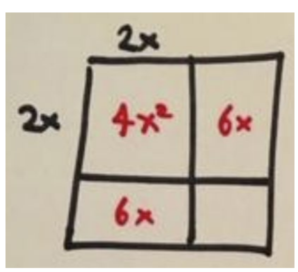
After some mulling and thinking it might occur to you to try the following.

IDEA 3: *Instead of doubling everything, try multiplying everything by four!*

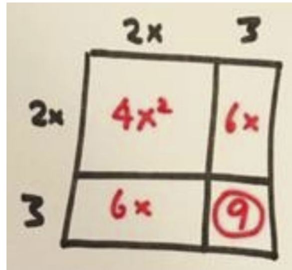
This then gives us the equation

$$4x^2+12x+4=20$$

and this is lovely. The first term, $4x^2$, is a nice perfect square and the middle number is even.



Something times $2x$ makes $6x$, so we must have side lengths of 3, and the final piece of the area is 9.



Adding 5 to both sides gives us the equation

$$4x^2 + 12x + 9 = 25$$

and the left side of this equation is precisely the $2x+3$ by $2x+3$ square. We are solving

$$(2x+3)^2 = 25$$

which is back to being a level two problem!

$2x+3=5$ or -5

$2x=2$ or -8

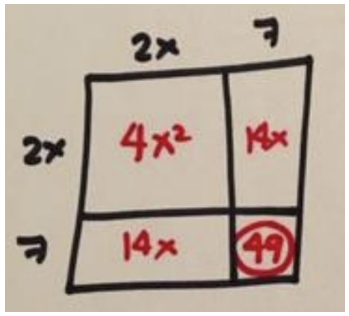
$x=1$ or -4

Wow!

Multiplying through by 4 unlocked the problem!

PROBLEM: *Solve* $x^2 + 7x - 2 = 5$.

We have an odd middle term. If we choose to avoid fractions we can multiply through by 4 and solve instead $4x^2 + 28x - 8 = 20$.



The square method shows we really want the number 49, not -8. So let's add 57 to each side and work with

$$4x^2 + 28x + 49 = 77.$$

The number on the right is awkward. Oh well!

We have

$$(2x+7)^2 = 77$$

$$2x + 7 = \sqrt{77} \text{ or } -\sqrt{77}$$

$$2x = \sqrt{77} - 7 \text{ or } -\sqrt{77} - 7$$

$$x = \frac{\sqrt{77}-7}{2} \text{ or } \frac{-\sqrt{77}-7}{2}.$$

PRACTICE 15: *Solve as many of these as you feel like doing.*

a) $w^2 - 5w + 6 = 2$
b) $x^2 + 9x + 1 = 11$
c) $p^2 + p + 1 = 0.75$
d) $x^2 = 10 - 3x$
e) $x^2 - x - 1 = 2\frac{3}{4}$
f) $x^2 + 3 = 9$

LEVEL 6 QUADRATICS

This is it, the final level!

Here's a next problem that has every possible difficulty that can ever occur when solving a quadratic equation.

PROBLEM: *Solve* $3x^2 + 5x + 1 = 9$.

The first issue is that we now have a number in front of the x^2 term. All other levels avoided this.

But in level four we did introduce a factor of four into our equations to work with $4x^2$, which we saw was a nice perfect square: it is $2x$ times $2x$.

Here we have $3x^2$, which is <u>not</u> a nice perfect square. Can we make it one?

Yes! Let's multiply the equation through by 3 and work with

$$9x^2 + 15x + 3 = 27.$$

Great! The first term is $9x^2 = (3x)\times(3x)$, a nice square.

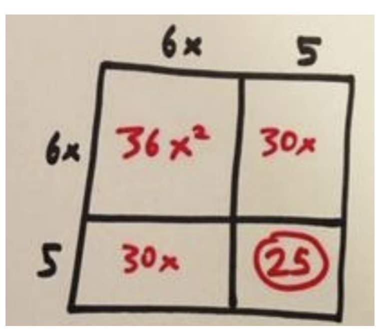
But we have an odd middle number, that 15!
So, let's multiply through by 4 to fix that.

A WORRY! Will doing so ruin our perfect square at the front?

We get

$$36x^2 + 60x + 12 = 108.$$

Phew! $36x^2 = (6x)\times(6x)$ is still a nice perfect square. In fact, multiplying a perfect square by 4 will never "ruin" a perfect square. (Can you see why?)
So now things look good for the square method.



We see the square wants the number 25, so let's add 13 to both sides and work with

$$36x^2 + 60x + 25 = 121.$$

And why did we do all this crazy work? To recognize $36x^2 + 60x + 25$ as a square: it is a $6x+5$ by $6x+5$ square. We have a level 2 problem!

$$(6x+5)^2 = 121$$

$6x+5 = 11$ or -11

$6x = 6$ or -16

$x = 1$ or $-\frac{8}{3}$

PROBLEM: *Solve* $5x^2 - 3x + 2 = 4$.

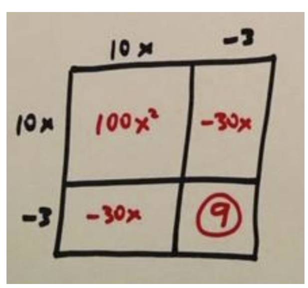
Let's multiply through by 5 to make a perfect square up front.

$$25x^2 - 15x + 10 = 20$$

Now let's multiply through by 4 to make the middle term even.

$$100x^2 - 60x + 40 = 80$$

The square method shows we want the number 49. Let's subtract 31 from each side.



$$100x^2 - 60x + 9 = 49$$

$$(10x-3)^2 = 49$$

$10x - 3 = 7$ or -7

$10x = 10$ or -4

$x = 1$ or -0.4

PRACTICE 16: *Solve as many of these you feel like doing.*

a) $2x^2 = 9$
b) $4 - 3x^2 = 2 - x$
c) $\alpha^2 - \alpha + 1 = \dfrac{7}{4}$
d) $3x^2 + 3x + 1 = 19$
e) $-3x^2 + 3x + 1 = 19$
f) $10k^2 = 1 + 10k$

PRACTICE 17: *Consider* $4x^2 + 6x + 3 = 1$. *Does it look like this quadratic equation will have problems when solving it? Does it have problems as you try to solve it? What can you do to obviate the difficulties you encounter?*

EVERYTHING IS LEVEL 2!

We have illustrated that every quadratic equation $ax^2 + bx + c = d$ is really just a level 2 question in disguise.

$$(something)^2 = number$$

And as we noted in the last essay, each such equation has either 0, 1, or 2 solutions. As such, any quadratic equation $ax^2 + bx + c = d$ has either 0, 1, or 2 solutions.

PRACTICE 18:
a) Design a quadratic equation that has two negative solutions.
b) Design a quadratic equation with just one solution, namely, $x = 4$.
c) Design a quadratic equation with $x = 2$ and $x = 10$ as solutions.

PRACTICE 19:
a) A rectangle is twice as long as it is wide. Its area is 30 square meters. What are the dimensions of the rectangle?
b) A rectangle has one side 4 meters longer than the other. Its area is 30 square meters. What are the dimensions of the rectangle?

PRACTICE 20:
a) Solve $\tau^2 - 5\tau + 7 = 1$.

The symbol $\sqrt{\ }$ means the <u>positive</u> root of a number. (For instance, $\sqrt{9}=3$, and not -3, even though there are two numbers whose squares are 9.) This is a mathematics convention, and it can be confusing as it is ignoring symmetry.

But we can say that if x is a positive number, then we have $x=\left(\sqrt{x}\right)^2$.

b) Solve $x-5\sqrt{x}+7=1$.
   HINT: Look at part a).

c) Solve $x-2\sqrt{x}=-1$

d) Solve $x+2\sqrt{x}-5=10$ and be clear why this equation has only one solution!

e) Solve $2u^4+8u^2+7.5=0$.

PRACTICE 21: *Consider $y=2(x-4)^2+6$. What value for x produces the smallest possible value for y? Why?*

PRACTICE 22: *Find one solution to $(x+1)^3=27$.*

PRACTICE 23 (TOUGH!): *In the following equation, solve for x in terms of a and b.*

$$x^2-(a+b)x+ab=0.$$

<u>HINT 1</u>: Multiply through by 4 just in case $a+b$ is odd.
<u>HINT 2</u>: $(a+b)^2-4ab$ equals $a^2-2ab+b^2$, which happens to equal $(a-b)^2$. (Check these claims.)

PRACTICE 24 (OPTIONAL): *This problem will require you to multiplying through by 4 many times!*

a) Solve $x^2+x=2$.
b) Solve $2x^2+x=3$.
c) Solve $4x^2+x=5$.
d) Solve $8x^2+x=9$.
e) Solve $16x^2+x=17$.

If you are game …

f) Find the solutions to $2^N x^2+x=2^N+1$.

8.1(C) THE FAMOUS QUADRATIC FORMULA

You may well be aware that there is famous formula one can use to solve all quadratic equations. (Indian 7th-century scholar Brahmagupta was the first to describe it.) Most curricula want students to know this formula, memorise it, and use the formula over and over again.

But here's the secret: We have actually been doing the quadratic formula all along! Our square method, the *quadrus* method, is the quadratic formula in disguise.

The goal of this section is to explain this.

THE QUADRATIC FORMULA

Recall how we solve a level 6 quadratic equation such as

$$5x^2 - 3x + 2 = 4.$$

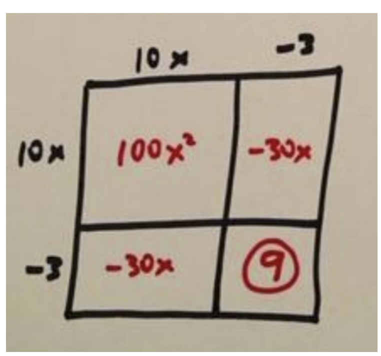
We start by multiplying through by 5 to make the first term a nice square.

$$25x^2 - 15x + 10 = 20$$

Now we notice an odd middle term, so we multiply through by 4 to make that term even and, at the same time, ensure we still have a perfect square out front.

$$100x^2 - 60x + 40 = 80$$

The numbers are large, but they are the perfect numbers to create a lovely square.



But we see we need to adjust the number 40 and make a 9. So we subtract 31 from both sides of the equation.

$$100x^2 - 60x + 9 = 49$$

And we like this equation because the left side is a perfect $10x-3$ by $10x-3$ square. We have

$$(10x-3)^2 = 49$$

which we can readily solve.

$$10x-3=7 \text{ or } -7$$

$$10x=10 \text{ or } -4$$

$$x=1 \text{ or } -\frac{2}{5}$$

Beautiful!

Now let's do the same work again, not on an equation with specific numbers, but on an abstract quadratic equation.

A CURRICULUM CONVENTION

Almost every curriculum insists that quadratic equations be written with some left side set **equal to zero** on the right. For example, most curricula would prefer to rewrite

$$5x^2-3x+2=4$$

as

$$5x^2-3x-2=0$$

by subtracting 4 from each side.

We've never bothered to do that! With the square method we knew we were likely going to change the constant term in a quadratic expression, so we just waited until we could see what number would be best for it. Most curricula insist, however, that you make a change right away and rewrite the equation so that it equals zero, even if you decide to change the number again later on.

So, let's follow the curriculum convention and assume we are solving a quadratic equation of the form

$$\boxed{ax^2+bx+c=0\,.}$$

Example: For $5x^2-3x-2=0$ we have

$$a=5,$$
$$b=-3,$$
$$c=-2.$$

Let’s follow our square method and solve this equation.

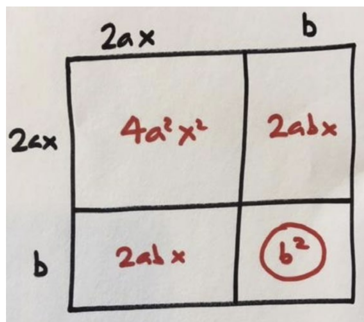
Multiply through by a to make the first term a nice square.

$$a^2x^2 + abx + ac = 0$$

Just in case the middle term is odd (we don’t know), let’s cover ourselves and multiply through by 4.

$$4a^2x^2 + 4abx + 4ac = 0$$

We’re now ready to draw the square.



Splitting “$4abx$” into two gives regions of area $2abx$. If one side length of each is $2ax$, then the remaining side length is b. This means we need a final piece of the square of area b^2.
But we have $4ac$ instead of b^2 in our equation $4a^2x^2 + 4abx + 4ac = 0$.

Let’s subtract $4ac$ from both sides

$$4a^2x^2 + 4abx = -4ac$$

and then add b^2 to each side

$$4a^2x^2 + 4abx + b^2 = -4ac + b^2.$$

Most people choose to rewrite the right-hand side.

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

Now $4a^2x^2 + 4abx + b^2$ is a perfect $2ax + b$ by $2ax + b$ square. We have

$$(2ax + b)^2 = b^2 - 4ac$$

which is a level 2 problem.

Something squared is $b^2 - 4ac$ and so we get

$$2ax + b = \sqrt{b^2 - 4ac} \text{ or } -\sqrt{b^2 - 4ac} .$$

Adding $-b$ throughout gives

$$2ax = -b + \sqrt{b^2 - 4ac} \text{ or } -b - \sqrt{b^2 - 4ac} .$$

Dividing through by $2a$ then gives our final solutions.

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Most people prefer to combine these two solutions by using a funny $\pm$ symbol in the middle to indicate we can have a $+$ sign or a $-$ sign.

The Famous Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example: For we have $a = 5$, $b = -3$, and $c = -2$. The quadratic formula then gives

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot (5) \cdot (-2)}}{2 \cdot 5}.$$

This is

$$x = \frac{3 \pm \sqrt{49}}{10}$$

which is

$$x = \frac{3+7}{10} \text{ or } \frac{3-7}{10}$$

giving $x = 1$ or $x = -\dfrac{2}{5}$, just as before!

> **PRACTICE 25:** *Solve* $3x^2 + 5x + 1 = 9$ *by using the quadratic formula and then again by using the square method. (Of course, you should get the same answers each time!)*

I personally prefer the square method for solving quadratic equations: I understand what to do and there is nothing for me to memorize, I adjust numbers as I go along and don't worry about making my sure the equation "equals zero," and I think it is fun! But it is a slower enterprise.

Many people prefer to use the quadratic formula because it is speedier.

But let's be honest: If your goal really is just to get an answer and to do so as fast as possible for some reason, then the most intelligent thing is not to follow either approach and simply use a free algebra system on the internet instead! (We live in the 21st century after all!)

And this explains why I really like the square method. It is the story of the *THINKING*, and mathematical thinking brings me joy. Getting the answers to specific problems is not the point of the story. It is not the important part.

HONESTY MOMENT: Most every quadratic equation presented to students in a curriculum uses "nice" numbers. But if I had to solve an awkward quadratic equation like

$$1.3x^2 + \frac{\pi}{3}x - \frac{17}{\sqrt{2\frac{1}{2}}} = 0$$

by hand, the square method would be miserable. I'd probably use the quadratic formula. (Though that would likely be miserable too!)

PRACTICE 26 (OPTIONAL): *Solve*

$$1.3x^2 + \frac{\pi}{3}x - \frac{17}{\sqrt{2\frac{1}{2}}} = 0.$$

(Did you notice that this question is optional?)

PRACTICE 27: *Solve whichever of these quadratic equations you feel like doing, using whatever method you like. (Or solve some twice with two different methods!)*

a) $6x^2 - x + 10 = 11$
b) $30x^2 - 17x = 2$
c) $x^2 - 4x + 4 = 0$
d) $2x^2 + 5 = 11x$
e) $93x^2 - 117 = 0$
f) $x^2 + x + 1 = 2$
g) $x^2 + x + 1 = 1$
h) $x^2 + x + 1 = 0$

PRACTICE 28: *Solve for* x *in terms of* a *and* b*.*

$x^2 - (a+b)x + ab = 0$.

PRACTICE 29: *Find a number* p *so that* $1+\dfrac{1}{p}$ *equals* p *.*

PRACTICE 30: *A quadratic equation* $ax^2 + bx + c = 0$ *has precisely one solution. What is the value of* $b^2 - 4ac$ *?*

PRACTICE 31: *The list of* oblong numbers *begins* $2,\ 6,\ 12,\ 20,\ 30,\ 42,\ 56,\ \dots$. *Here each number is the product of two consecutive integers.*

$2 = 1\times 2$
$6 = 2\times 3$
$12 = 3\times 4$
etc.

a) *What is the one-hundredth number in the list?*
b) 5402 *is an oblong number. At which position in the list does it sit?*

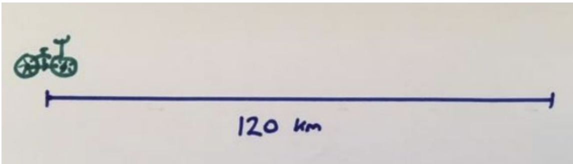
8.1(D) UNUSUAL QUADRATIC QUESTIONS

A standard textbook is full of problems to practice solving quadratic equations. Many of the problems will look like the ones we've presented in the previous sections, to simply examine and solve a stated equation. But some might look quite different and, on first appearance, might seem to have nothing to do with quadratic equations. Let me now present some unusual-looking quadratic equation practice questions.

PRACTICE 32: *I ride my bike along a straight stretch of road. The road is 120 km long and I ride at a constant speed.*

I then ride my bike back along the same stretch of road, again at a constant speed, but this time 10 km/hr faster than I did before.

I was one hour quicker on my return journey.
What were my two riding speeds?



Do you remember the two key steps to problem solving?

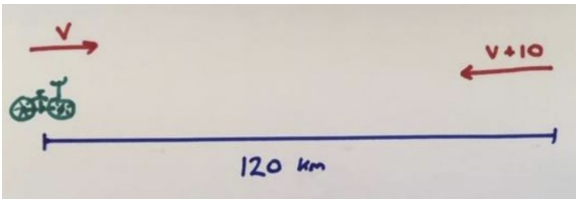
Step 1: *Have an emotional reaction.*

Take a deep breath and then

Step 2: *DO SOMETHING! Anything!*

This question does look a bit scary. Where are the numbers? Where's the equation? All I see is the distance 120 km and a time difference of 1 hour.

Just to DO SOMETHING, we could indicate on the picture one constant speed of v km/hr, say, and then a second constant speed of $v+10$ km/hr for the way back. That's a start!



What next?

We can get some times!

Going 120 km at a constant speed of v km/hr will take $\frac{120}{v}$ hours. The return journey, 120 km at a speed of $v+10$ km/hr, will take $\frac{120}{v+10}$ hours. And this is 1 hour quicker.

So we must have

$$\frac{120}{v+10}=\frac{120}{v}-1.$$

I bet we can rework this equation to get a quadratic equation in v which we can then solve!

Your job: *Finish this question.*

PRACTICE 33: *Xavier, in a speed banana-eating contest, ate his first set of six bananas in t seconds, but took 5 seconds longer eating his second set of six bananas. His banana-eating rate was $\frac{6}{t}$ bananas-per-second for his first set. His rate was 0.1 bananas-per-second less for his second set.*

How long did Xavier spend eating his first six bananas?

WHAT IS THE VARIABLE? CHANGING PERSPECTIVE

Equations that don't look quadratic at first just might be, if your change your perspective. But there might be issues!

PROBLEM: *Solve* $\sqrt{x}+2=x$.

Right away there is something curious to note about this equation. In order for the quantity $\sqrt{x}$ to make sense, x cannot be negative. This question must then have the hidden assumption that x *is greater than or equal to zero.* That assumption not explicitly stated, but it must be there.

Next: Is it a quadratic equation? If we regard "x" as the variable, then certainly not: it is not an equation of the form $ax^2+bx+c=0$. But if we write $x=\left(\sqrt{x}\right)^2$ (remember, we must be assuming x is a non-negative number), then the equation is

$$\sqrt{x}+2=\left(\sqrt{x}\right)^2$$

which is a quadratic equation in the variable $\sqrt{x}$. It's

$$\left(\sqrt{x}\right)^2 - \left(\sqrt{x}\right) - 2 = 0.$$

Challenge: *What does the quadratic formula give for the solutions of this equation? Are they actually solutions?*

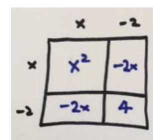
Let me attempt to solve the original equation $\sqrt{x} + 2 = x$ a different way. Let's write

$$\sqrt{x} = x - 2$$

and then square both sides of the equation to get

$$x = (x-2)^2.$$

The right-hand side is an $x-2$ by $x-2$ square, which expands as $x^2 - 4x + 4$, so we have the equation



$$x = x^2 - 4x + 4.$$

This is the quadratic equation

$$x^2 - 5x + 4 = 0.$$

To practice the quadratic formula, we see that this has solutions

$$x = \frac{5 \pm \sqrt{25-16}}{2} = \frac{5 \pm 3}{2} = 4 \text{ or } 1.$$

So it looks like $x = 4$ and $x = 1$ are the solutions to the equation $\sqrt{x} + 2 = x$.

But look what happens when we check!

For $x = 4$ we have

$$\sqrt{4} + 2 = 4$$

which is true and correct. So yes, $x = 4$ really is a solution.

For $x = 1$ we have

$$\sqrt{1} + 2 = 1$$

which is FALSE! We see that $x=1$ is <u>not</u> actually a solution!

What's going on?

We mentioned in a previous exercise that the $\sqrt{\ }$ symbol is a symbol from geometry and is an asymmetrical symbol: it only allows the positive square root of a number. We started with the equation $\sqrt{x}=x-2$, which is a restricted equation as it will allow only for one type of square root, the positive one. And then we squared the equation to play with $x=x^2-4x+4$ instead. But this new equation has no $\sqrt{\ }$ symbol mentioned in it and so allows for <u>all</u> possible roots of numbers. It is not surprising then that it might give more "solutions" than the original equation will allow.

UPSHOT: Watch out if you manipulate an unusual equation focused on an unusual variable (such as $\sqrt{x}$) to obtain a more familiar type of equation: you might obtain candidate solutions that are not actually allowed for the original, more restrictive equation. Always check which of your final candidate solutions are actually solutions.

PRACTICE 34: *Solve* $w+3\sqrt{w}+2=0$.

PRACTICE 35: *Find at least two solutions to* $x^4-3x^2-4=0$.

PRACTICE 36: *Find all possible solutions to* $\frac{4}{x^2}+\frac{4}{x}+1=0$.

THE OPENING PUZZLE

Recall the opening puzzle of the introduction asks for two numbers which sum to 10 and have product 26.

$$sum=10$$
$$product=26$$

Finding such a pair of numbers is tricky.

Let's call the numbers we're looking for p and q. They must satisfy

$$p+q=10$$
$$pq=26\ .$$

Actually, we see that once we find one number, say p, then the first equation gives us q: it must be $10-p$. So, let's focus on the number p.

The second equation, with this focus on p, thus reads

$$p(10-p)=26.$$

Expanding the left side, we get $10p - p^2 = 26$, which can be written in the familiar form of a quadratic equation

$$p^2 - 10p + 26 = 0.$$

Ooh! And this is interesting! I've become pretty adept at the square method and I can see I want to rewrite this as

$$p^2 - 10p + 25 = -1.$$

(Do you see why my brain wanted this? If not, quickly draw the square.)

This now reads as

$$(p-5)^2 = -1,$$

which has no solutions!

Exercise: *If you prefer, use the quadratic formula to show that* $p^2 - 10p + 26 = 0$ *has no solutions.*

There are no two numbers that sum to 10 and have product 26. This was a trick question!

PRACTICE 37: *Find all values* k *for which there is a pair of numbers that sum to* 10 *and have product* k.

$$sum = 10$$
$$product = k$$

(For instance, we saw that $k = 24$ *and* $k = 25$ *are both possible values for* k*, and* $k = 26$ *is not.)*

PRACTICE 38: *Let* S *be a fixed number. Find, in terms of* S*, the largest possible number* k *for which there is a pair of numbers that sum to* S *and have product* k.

$$sum = S$$
$$product = k$$

PRACTICE 39: *Find the smallest possible value* k *for which the equation* $x+\frac{1}{x}=k$ *has a solution for a positive value* x.

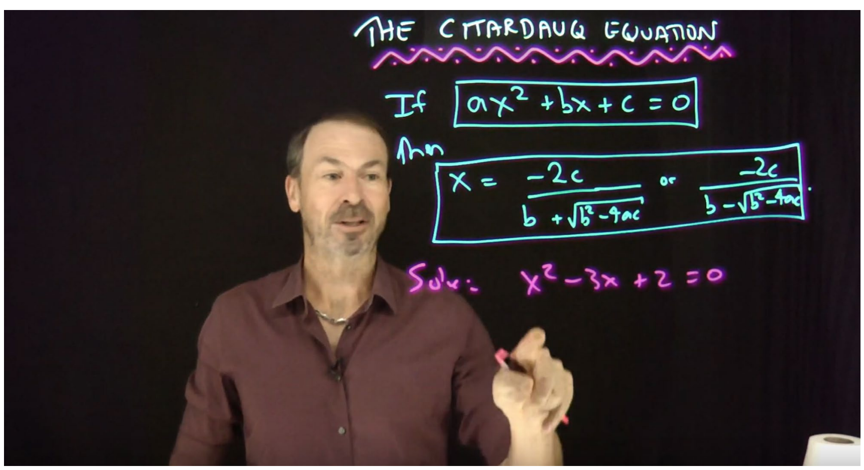
PRACTICE 40: *a) Find two numbers that differ by* 100 *and have product 5069.*

b) When Anu thought about this question she first said to herself: "Symmetry is my friend. A symmetrical solution would have the two numbers the same. So let me represent the numbers by how different they are from being the same." She decided to write the two numbers as $n-50$ *and* $n+50$. *How do you think she then proceeded with the problem?*

c) *Find two numbers whose sum is* 100 *and whose product is 2491. (What is a "symmetrical" way to set up this problem?)*

SOMETHING SILLY: Do you know the not-famous **Citardauq Formula** for solving quadratics?

If $ax^2+bx+c=0$, then $x=\dfrac{-2c}{b\pm\sqrt{b^2-4ac}}$.



You can learn all about it here, if you want.

Warning: This is silly! It is here to point out that one need not choose to deem important each and every thing a text book could mention. You can decide for yourself the key things to keep close in your mind, which to simply be aware of (and rely on your common sense and wits instead), and which to essentially ignore.

8.2(A) STEPS TOWARDS GRAPHING

So far we've talked about the algebra of quadratics and saw the power of symmetry throughout that story. Now we'll discuss the graphing of quadratics and see the power of symmetry at play yet again. Symmetry is such a mighty good friend!

Before we move to the opening puzzle, let's have a brief discussion about what mathematics is.

MATHEMATICS AS A LANGUAGE

Many people say that mathematics is a language. And this is true. Since this essay and the video accompanying it are being portrayed in English, the language of mathematics is, right now, English! (And if I were writing in Hindi or in Korean, the language of mathematics would be Hindi or Korean.)

Every mathematical statement is a sentence.
For example, the statement

$$5 = 2 + 3$$

has a noun (the quantity "5"), a verb ("equals"), and an object (the quantity "$2+3$"). As such the sentence should come with proper English punctation: It needs a full stop (period) at its end.

$$5 = 2 + 3.$$

The statement $7 > 4 + 9$ is also a sentence.

Our first sentence happened to be a true sentence about numbers and this second one a false sentence about numbers. As mathematics tends to focus on truth, it is interested in sentences that represent true statements about numbers.

But matters are a little curious in an algebra class. We write sentences about numbers all the time, but the numbers are not specified. For example

$$w^2 = p$$

is a sentence. It has noun "an unspecified quantity w squared," verb "equals," and object "an unspecified number p". But without knowing what specific numbers one has in mind for w and p we cannot determine whether this a true or a false sentence. (It is like saying: "Karthik is over 6 feet tall." Without knowing which particular Karthik of the world the speaker is referring to, we do not know whether this sentence is true or false.)

Since mathematics tends to focus on truth it seems natural to collect from an equation, like $w^2 = p$, all the data values that make this a true sentence about numbers. People usually collate such data in a table.

w	p
2	4
1	1
0	0
-3	9
$\frac{1}{2}$	$\frac{1}{4}$
$\sqrt{\pi}$	π

For instance, $w = 2$, $p = 4$ is a data pair that leads to a true number sentence, namely $(2)^2 = 4$, and so could appear in the table. The data pair $w = 5$, $p = 16$ does not yield a true number sentence, $(5)^2 = 16$ is a false number sentence, and so would not appear in the table.

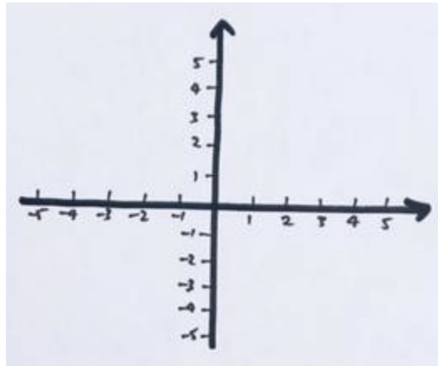
VISUALISING DATA

After four lectures with me you can tell that I am a very visual person. In fact, most people find the visual representation of ideas and data compelling, useful, and insightful. In particular, it is good to give a visual representation to data.

A *graph* is a visual representation of data values.

From our equation $w^2 = p$ it was natural to collect data values that yield true number sentences.

In our early grades we represented numbers visually on a number line. We here have two sets of possible number values and so it seems we'll need two number lines: one for the w values and one for the p values. Mathematics does not care how and where we place these two number lines on the page, but it has become the social convention to place one number line horizontally, with the positive values heading to the right, the other vertically, with positive values heading upwards, having these two lines cross each other at their zeros.



Now comes the question: Which number line should be for the w values and which for the p values? Again, mathematics does not care which number line is labeled which. But society does!

Most people find me writing $w^2 = p$ strange, and would prefer me to write instead

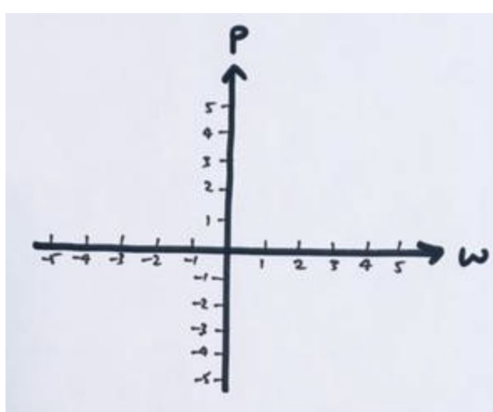
$$p = w^2.$$

Here it looks like we have a formula for the p-value that matches any given w value: when $w = 8$, we have $p = 8^2 = 64$; when $w = -0.2$, we have $p = (-0.2)^2 = 0.04$; and so on. These are data pairs giving true number sentences.

So it feels like the chosen value for w is the "driving force" here, with the matching value of p depending on the w-value chosen. People might say that w is the <u>independent variable</u> here with p the <u>dependent variable</u>.

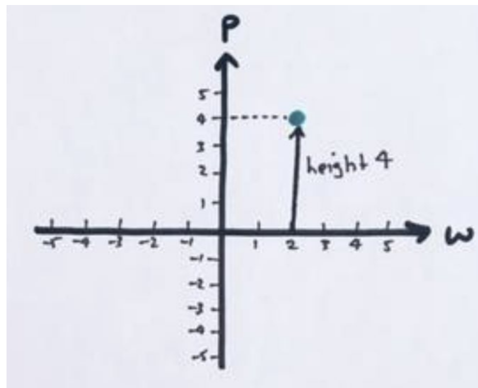
If you happen to be thinking this way, then social convention (society again!) wants to set the horizontal number line for the independent variable and the vertical one for the dependent variable.

Jargon: People tend to call the two number lines in our picture <u>axes</u>. The point where the two axes cross is called the <u>origin</u> of this system of "coordinate axes."

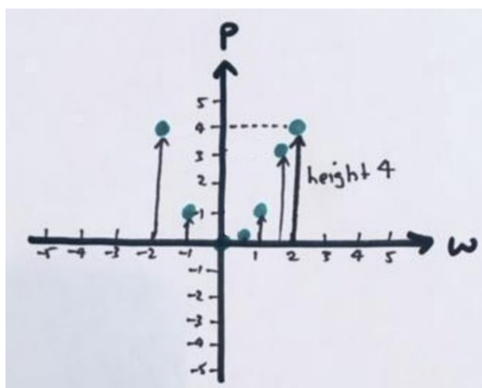


Now: how does one represent data on this set of crossing number lines?

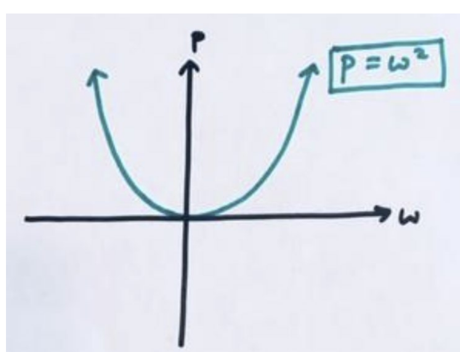
To represent the first data pair in our table, $w = 2$, $p = 4$, find the number 2 on the w axis and move upwards 4 units. Mark that point at height 4. Because the p axis is vertical, we can see that the height of this point lines up with the value $p = 4$ on the vertical axis.



In this way we can depict each data pair in our table with a point in the picture.



And if we could plot every single integer and fractional and irrational pair of data values possible it seems the set of all points depicting our data trace out a single U-shaped curve.



People will label this curve with the equation that generated the data values for that curve.

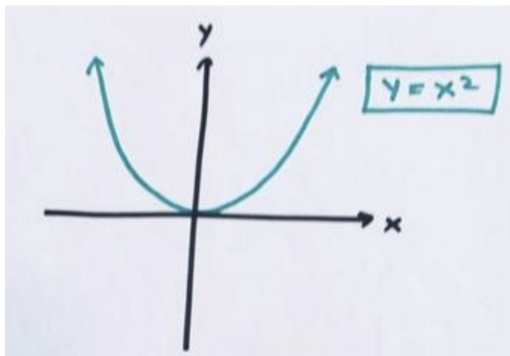
And notice that in this special example we have a lovely symmetrical curve!

A CHANGE OF NOTATION

In upper-school mathematics people tend to use the same letters of the alphabet over and over again for unknown values in algebra class. Instead of calling the independent variable w and the dependent variable p it has become convention to use:

x for the independent variable
y for the dependent variable.

So to follow convention, let's relabel our graph as shown. We have the graph of the equation $y = x^2$.



And this graph really is symmetrical about the vertical axis. For instance,

- for $x = 2$ we have $y = (2)^2 = 4$ and for $x = -2$ we have $y = (-2)^2 = 4$, the same.
- for $x = 10$ we have $y = (10)^2 = 100$ and for $x = -10$ we have $y = (-10)^2 = 100$, the same.

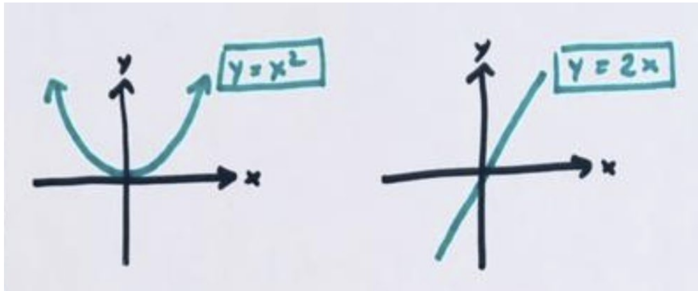
And so on.

As we progress with our story of graphing, we'll make powerful use of this symmetry.

A GRAPHING PUZZLER

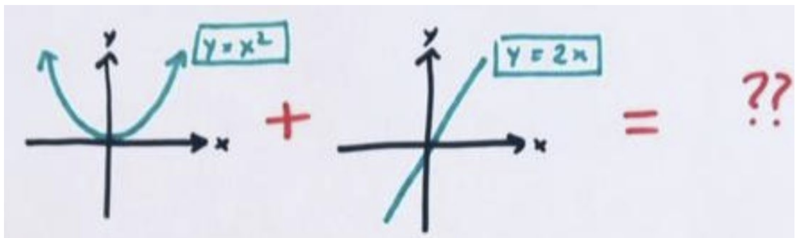
We're now ready for a puzzle.

We just graphed the equation $y = x^2$ and saw a lovely symmetrical U-shaped graph. And if you graph the equation $y = 2x$ (make a table of all the data values that give a true number sentence and then plot those data values), you'll see the picture of a line of slope (gradient) 2 through the origin.



Here's a weird question: *What picture would we obtain if we "added" these two graphs?*

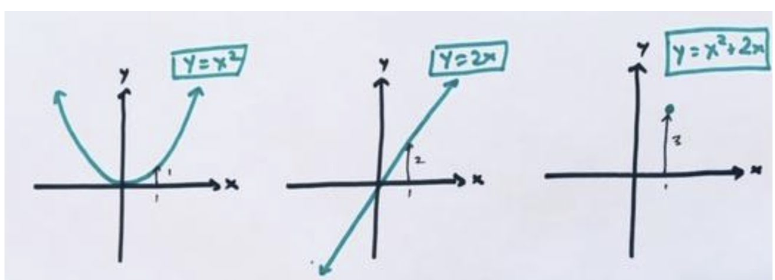
What could I mean by that?



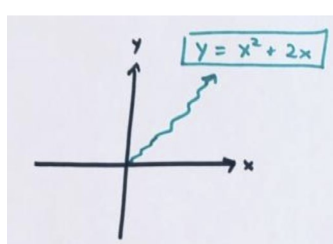
Let's look at each x value and add their matching y values. For example:

For $x=1$, we'll add $(1)^2$ and $2(1)$ and get 3.
For $x=2$, we'll add $(2)^2$ and $2(2)$ and get 8.
For $x=10$ we'll add $(10)^2$ and $2(10)$ and get 120.

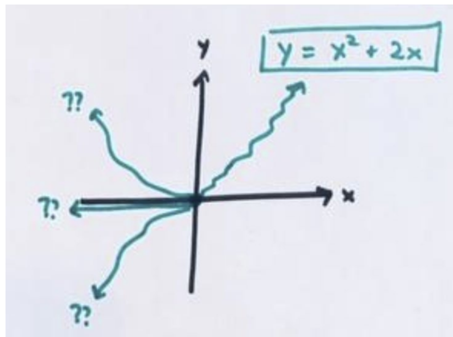
That that is, we are plotting the graph of the equation $y=x^2+2x$.



It is clear as that as we put in larger and larger positive x values, we are getting higher and higher points. The graph rises upwards as we move to the right. (Maybe in a straight line? Maybe in some curved way?)



Things are more interesting to the left for negative x values: the graph of $y = x^2$ has data points with positive heights in this region, but the graph of $y = 2x$ has points of negative heights here. We'll be adding together positive and negative heights. Will they cancel out and give zero heights? Will the positives heights "beat" the negative ones and give an overall graph of positive height? Or will the negative heights "win"?



Do plot the graph of $y = x^2 + 2x$. Make a table of data values that give true number sentences and plot those data points. Consider enough of them to get a good sense of the shape of the graph. You are in for a surprise!

We'll come back to this later on.

Insight: The graph of $y = x^2$ has beautiful vertical symmetry. The graph of $y = 2x$ has no vertical symmetry. In fact, it has "anti-symmetry": the heights to the left of the vertical axis are the exact opposites of the heights on the right. So there is no reason to believe that "adding" these two graphs will give a symmetrical image.

PRACTICE 41:

a) Sketch a graph of the one-variable equation $x^2 = 4$. *(So it's graph will require only one number line, one for* x *values.)*

b) Sketch a graph of the one-variable inequality $x^2 \geq 4$.

PRACTICE 42: *Sketch a graph of the two-variable equation* $x^2 = y^2$.

PRACTICE 43:

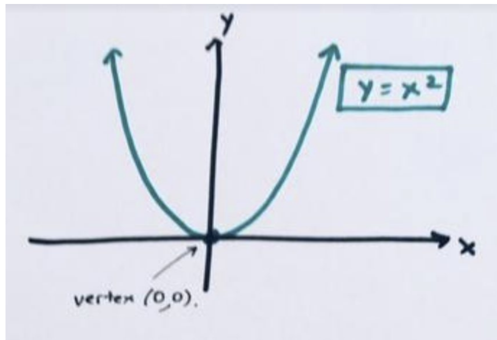
a) *Sketch a graph of the one-variable equation* $x = 3$.

b) *Sketch a graph of* $x = 3$ *thinking of it as a two-variable equation. (Imagine it as* $x + 0 \cdot y = 3$ *if you like.)*

c) *Sketch a graph of* $x = 3$ *thinking of it as a three-variable equation. (Imagine it as* $x + 0 \cdot y + 0 \cdot z = 3$ *if you like.) How will you draw your three number lines?*

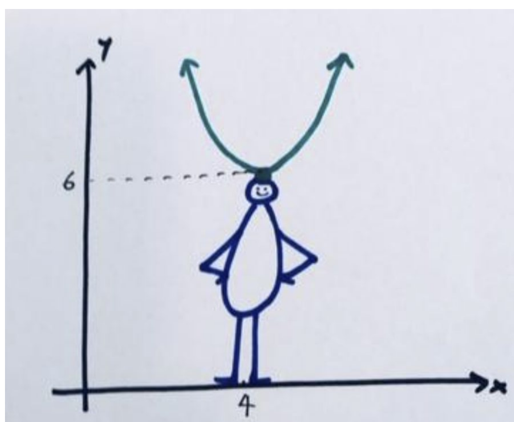
ANOTHER GRAPHING CHALLENGE

Consider the lovely symmetrical U-shaped graph from last time.



I like this graph so much that I would like to have one balancing on my head please!

Just to be clear, I am six feet tall and I will stand at position 4 on the x-axis. So this means I would like to have one of these symmetrical curves balanced at the point $(4,6)$ (that is, at the point with $x=4$, $y=6$) rather than at the point $(0,0)$.



> **Jargon:** Given a U-shaped curve like the one we see, the point at which it curve dips down to its lowest value is called the *vertex* of the curve. The vertex of $y=x^2$ is the point $(0,0)$.

So, we want a new U-shaped curve with vertex $(4,6)$. And that is the challenge of this essay.

> *Find an equation* $y=$ _______ *whose graph is a symmetrical U-shaped curve with vertex on the top of my head.*

And this challenge is hard! It takes hours, days, even weeks of deep thinking and trying different things to eventually find success with it. (Such is the nature of doing mathematics.)

Please feel free to stop reading this essay and sit with this challenge for a good long while. Try graphing different variations of the $y = x^2$ equation, say, $y = x^2 + 2$, $y = 4x^2$, $4y = x^2$, $6y = 4x^2 + 2$ and the like, observe the results you get, and see if any helpful ideas come your way. Unfortunately, I am going to give all the answers away on the next few page and so take away your joy in playing and figuring things out for yourself. (Sorry!)

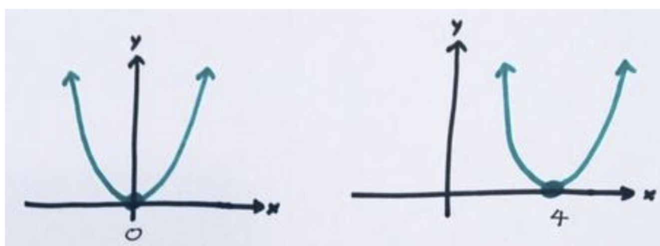
SHIFTING CURVES

We need to apply two motions to the picture of the $y = x^2$ graph: shift it rightwards four units and shift it upwards six units.

Let's think of each of these motions in turn.

1. *A Horizontal Shift*

What equation would give the same graph as $y = x^2$ but shifted four units over?



The key is to realise that in the $y = x^2$ graph all the "interesting action" is happening at $x = 0$. We want all that action to happen at $x = 4$ instead. So let's adjust the equation $y = x^2$ so that the number 4 now "behaves like zero."

How?

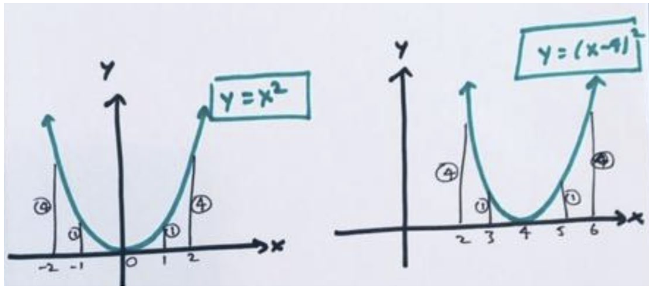
It takes a while for a flash of insight to come, but eventually one thinks of writing this equation.

$$y = (x-4)^2$$

Put in $x = 4$ and we get $y = 0^2$ and so, in some sense, 4 is behaving like 0. Wow!

And we see that putting in $x = 5$ and $x = 3$ we get $y = (1)^2$ and $y = (-1)^2$, and putting in $x = 6$ and $x = 2$, we get $y = (2)^2$ and $y = (-2)^2$, and so on. The values to the right and left of 4 in $y = (x-4)^2$ are matching the values to the left and right of 0 in $y = x^2$.

The graph of $y=(x-4)^2$ really is the graph of $y=x^2$ shifted horizontally so that 4 behaves like zero.



To shift the graph of an equation horizontally just ask "Which number would I like to behave like zero?" and then adjust the equation to make that happen!

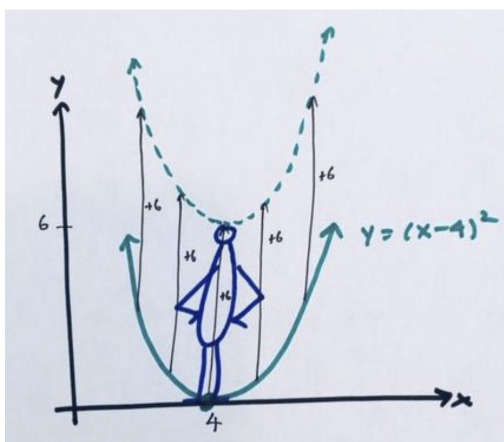
PRACTICE 44:

a) *Sketch the graph of* $y=(x-10)^2$.

b) *Sketch the graph of* $y=(x+5)^2$.

2. *A Vertical Shift*

We now want to shift the graph of $y=(x-4)^2$ vertically upwards 6 values.



Instead of being $0^2=0$ units high at $x=4$, we want it to be $0^2+6=6$ units high.

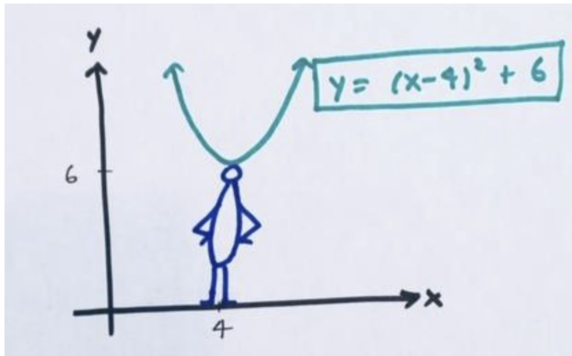
Instead of being $1^2=1$ unit high at $x=5$, we want it to be $1^2+6=7$ units high.

Instead of being $(-2)^2=4$ units high at $x=2$, we want it to be $(-2)^2+6=10$ units high.

And so on.

This suggests we work with the equation

$$y = (x-4)^2 + 6.$$



And one can check with a table of data values that this equation does indeed do the trick.

And when we look at the equation $y = (x-4)^2 + 6$ we can see that is it is basically the $y = x^2$ equation adjusted two ways.

1. The value $x = 4$ has been made to "behave like 0" (and so there is a horizontal shift of four units),
2. 6 has been added to the formula for the heights (and so there is a vertical shift six units upwards).

Magical!

PRACTICE 45:

a) *Sketch the graph of* $y = (x-5)^2 + 2$.

b) *Sketch the graph of* $y = (x+5)^2 - 2$.

PRACTICE 46: *When we say that the graph of* $y = x^2$ *is a U-shaped graph, is that a correct analogy? The two sides of the letter "U" are vertical. Does the graph of* $y = x^2$ *possess vertical lines?*

We've played with the equation $y = x^2$ and adjust it to create new equations with the same symmetrical U-shaped graph but shifted to different positions in the plane. But as we know from our algebra work, quadratic equations could also possess a coefficient attached to the x^2 term. How do the graphs of the equations $y = ax^2$ appear for different values of a?

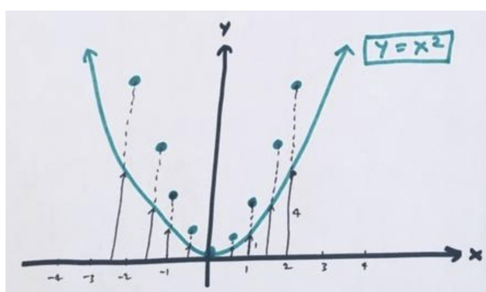
We'll explore that issue in this essay, and see that there are an infinitude of U-shaped curves I can choose from to have balance on my head!

STEEPNESS

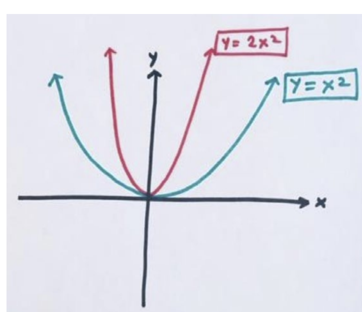
Consider the equation $y = 2x^2$.

At $x = 1$, we have $y = 2(1)^2$, double the value we had for the $y = x^2$ equation.
At $x = 2$, we have $y = 2(2)^2$, double the value we had for the $y = x^2$ equation.
At $x = -33$, we have $y = 2(-33)^2$, double the value we had for the $y = x^2$ equation.

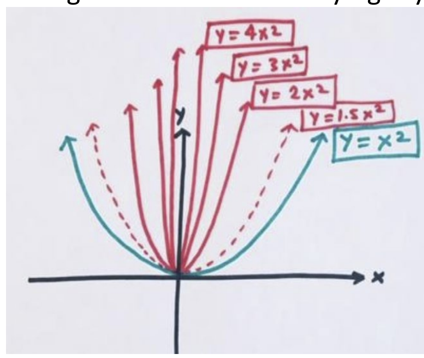
We see that the heights of all our data points for the equation $y = 2x^2$ are double the heights of the data points for $y = x^2$.



The point at $x = 0$ is still at height zero.
The graph of $y = 2x^2$ is again a symmetrical U-shaped graph, still centered with vertex at the origin, but it is a steeper curve.



We can make an even steeper graph by playing with $y=3x^2$ or $y=20x^2$ or $y=10000000x^2$! (This third equation will have a graph that hugs the vertical axis very tightly if we tried to sketch it accurately.)



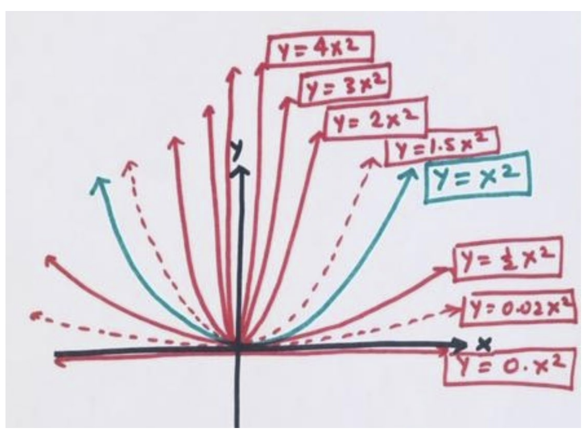
We can make a shallower U-shaped graph by using a coefficient small than 1. For instance, the equation

$$y=\frac{1}{2}x^2$$

has data points at half the heights of those of $y=x^2$, and

$$y=0.02x^2$$

has data points at one-fiftieth the heights.
All the data points for $y=0\cdot x^2$ have zero height and so this is U-shaped graph that is so shallow that it is flat!



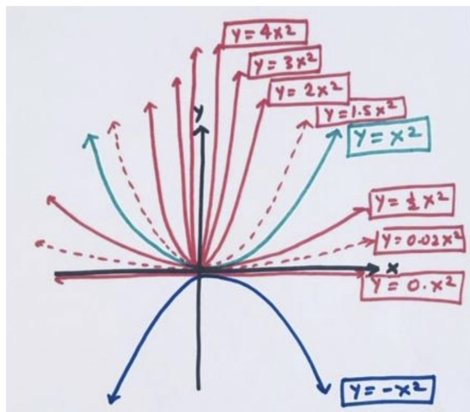
What if we choose a coefficient that is even lower that zero? That is, what if we worked with a negative coefficient?

Consider, for instance, $y = -x^2$.

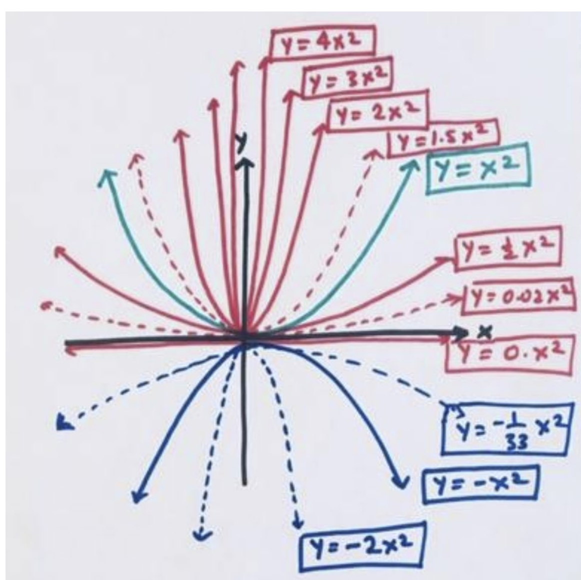
At $x = 1$, we have $y = -(1)^2$, the opposite value we had for the $y = x^2$ equation.
At $x = 2$, we have $y = -(2)^2$, the opposite value we had for the $y = x^2$ equation.
At $x = -33$, we have $y = -(-33)^2$, the opposite value we had for the $y = x^2$ equation.

And at $x = 0$ we still have $y = -(0)^2 = 0$.

The graph is the same as the original $y = x^2$ but now pointing in the negatives.



And we can see now that $y = -2x^2$, for instance, would give a steeper U-shaped graph pointing downwards, and $y = -\frac{1}{33}x^2$ would give a shallow downward pointing graph.

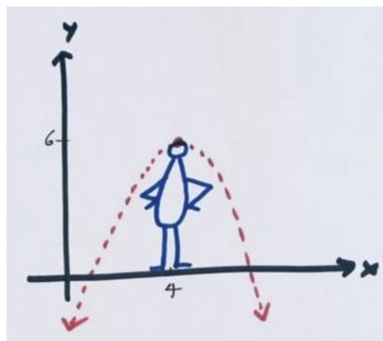


So we have

The graph of $y = ax^2$ is a symmetrical U-shaped graph based at the origin with the value a affecting the steepness of the graph.

If a is positive, the U-shape is upward pointing.
If a is negative, the U-shape is downward pointing.

PRACTICE 47: *Find three different equations that give U-shaped graphs that balance on my head this way.*



PRACTICE 48: *Draw, on the same sets of axes, rough sketches of each the following equations.*

$y = x^2$	$y = 1.1x^2$	$y = 0.9x^2$
$y = -x^2$	$y = -1.1x^2$	$y = -0.9x^2$

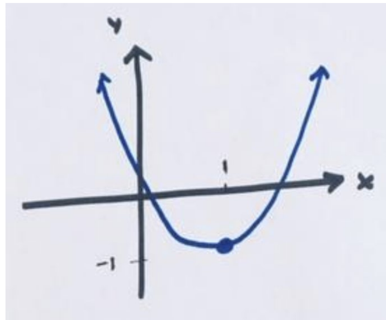
PRACTICE 49: *Sketch graphs of*

a) $y = 3(x-5)^2$

b) $y = 3(x-5)^2 + 4$

c) $y = -2(x+4)^2 + 40$.

PRACTICE 50: *Which of the following equations could have the graph shown?*



a) $y = \frac{4}{3}(x+1)^2 + 1$

b) $y = -\frac{4}{3}(x+1)^2 + 1$

c) $y = \frac{4}{3}(x-1)^2 + 1$

d) $y = -\frac{4}{3}(x-1)^2 + 1$

e) $y = \frac{4}{3}(x+1)^2 - 1$

f) $y = -\frac{4}{3}(x+1)^2 - 1$

g) $y = \frac{4}{3}(x-1)^2 - 1$

h) $y = -\frac{4}{3}(x-1)^2 - 1$

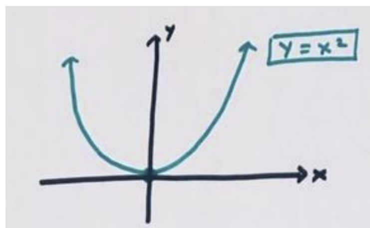
PRACTICING GRAPHING

We've been adjusting the equation $y = x^2$ to give new equations whose graphs are the same U-shaped graph we see from $y = x^2$ but shifted horizontally, shifted vertically, and made steeper or broader and upward or downward facing.

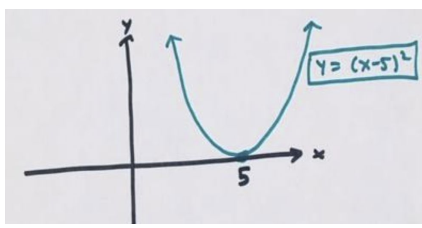
For example, if asked to

Sketch $y = 3(x-5)^2 + 10$,

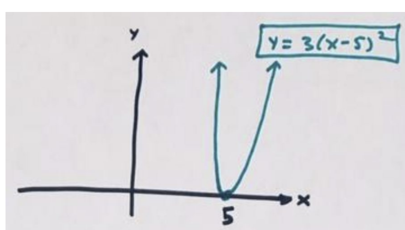
we can first recognize this as the basic $y = x^2$ equation



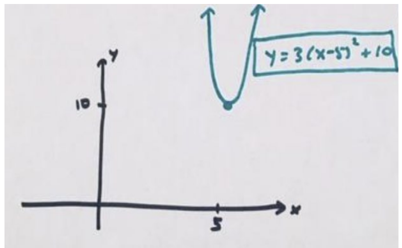
but with $x = 5$ behaving like zero



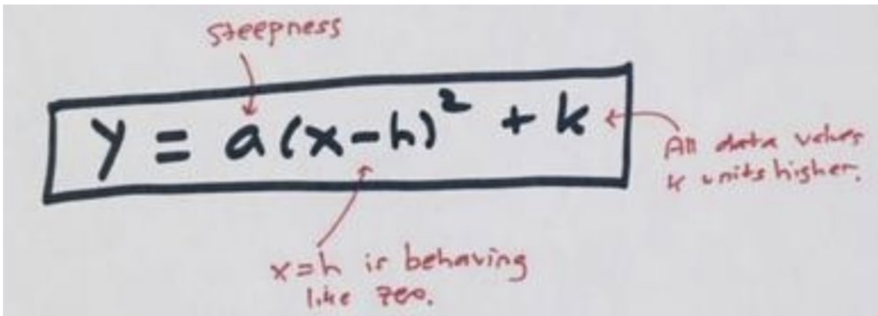
and made steeper with a steepness factor of 3



and with all data points shifted 10 units higher.



The following picture summarises all that we just saw!



We have a steepness factor a, we have $x = h$ behaving like zero, and we have all data values shifted k units higher.

PRACTICE 51: *Sketch the graph of* $y = -2(x+10)^2 - 7$.

PRACTICE 52: *The graph of a quadratic equation has a vertical line of symmetry at* $x = 3$*, and has highest value* $y = 17$*. Which of the following could be an equation for that quadratic?*

a) $y = 200(x-3)^2 + 17$

b) $y = -200(x-3)^2 + 17$

c) $y = 200(x-3)^2 - 17$

d) $y = -200(x-3)^2 - 17$

PRACTICE 53: *Sketch a graph for each of the following equations.*

a) $y = 2 - x^2$

b) $y = \frac{1}{3}\left(x - \frac{1}{2}\right)^2 - 4$

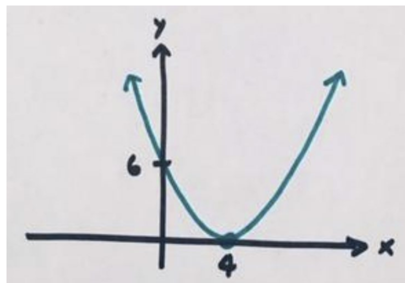
c) $y = 0.0034(x+0.276)^2 + 0.778$

d) $y = 200000(x-200000)^2 - 200000$

PRACTICE 54: *If* $y = a(x+b)^2 + c$ *has a graph passing through the origin and with* $(2,3)$ *as the vertex, then what is the value of* $a+b+c$ *?*

a) $\frac{1}{4}$ b) $1\frac{3}{4}$ c) $4\frac{1}{4}$ d) $5\frac{1}{4}$

PRACTICE 55: *Write a quadratic equation that fits this graph.*



PRACTICE 56: *Write down a quadratic equation whose graph passes through the points* $(3,18)$ *and* $(17,18)$ *and has lowest value* 5*.*

PRACTICE 57: *Write down a quadratic equation whose graph passes through the* x *axis at* $x=-2$ *and at* $x=10$ *and passes through the* y *axis at* $y=-6$*.*

PRACTICE 58: *Write down quadratic equations with symmetrical U-shaped graphs possessing the following properties:*

a) *Crosses the* x*-axis at* 3 *and* 5 *and the* y*-axis at* 1000*.*

b) *Passes through* $(4,10)$ *,* $(6,10)$ *and* $(8,13)$*.*

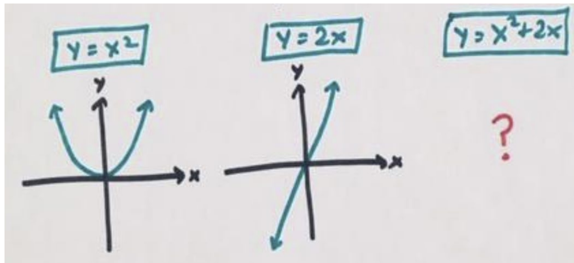
c) *Has vertex* $(5,5)$ *and passes through* $(4,4)$*.*

d) *Has vertex the origin and passes through the point* $\left(\sqrt{2},\pi\right)$*.*

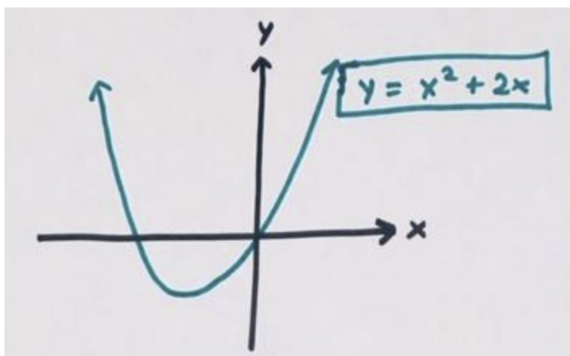
DIFFERENT-LOOKING EQUATIONS

Let's go back to the opening graphing puzzle.

What graph results if we add together the matching data heights of the $y = x^2$ *and* $y = 2x$ *graphs?*



If you actually made a table of data values that make $y = x^2 + 2x$ a true number sentence and plotted those points, then you may have been surprised to see another symmetrical U-shaped curve appear.



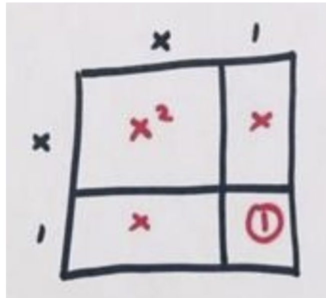
And this seems shocking! The graph of $y = x^2$ has perfect vertical symmetry, the graph of $y = 2x$ has perfect vertical anti-symmetry, and so there is no reason whatsoever to believe that a graph with perfect vertical symmetry would appear.

Let's prove that the curve we see really is a perfectly symmetric U-shaped curve. We can use our algebra of quadratics for this.

Consider $y = x^2 + 2x$.

Let's analyse the "$x^2 + 2x$" part of this equation using our quadrus method, back in lectures 1 and 2.

Let's draw a square with one piece of area x^2.



We're keeping things symmetrical, so we can say that this comes from $x \times x$. We also split the $2x$ piece into two equal areas. And then we see we need an additional area of 1.
Adding 1 through our original equation gives

$$y+1=x^2+2x+1$$

which we now recognize as

$$y+1=(x+1)^2.$$

Subtracting 1 shows that the equation

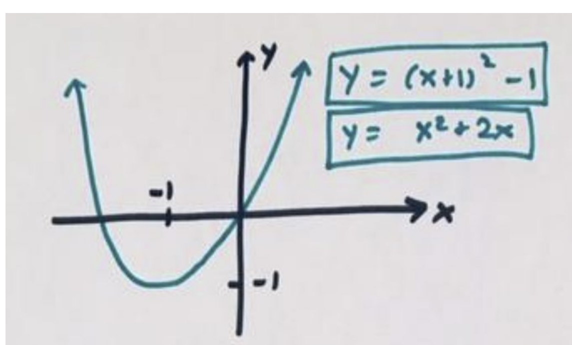
$$y=x^2+2x$$

is really the equation

$$y=(x+1)^2-1$$

in disguise.

And this is fabulous! We can now see that the graph of our equation is indeed the perfectly symmetrical U-shaped graph of $y=x^2$ with $x=-1$ behaving like zero for the x-values and with all heights shifted down by 1.



PRACTICE 59: *Use algebra to prove that the graph of* $y = x^2 - 6x + 10$ *is sure to be a symmetrical U-shaped graph.*

PRACTICE 60: *Use algebra to prove that the graph of* $y = x^2 + 8x - 7$ *is sure to be a symmetrical U-shaped graph.*

Even if there is a coefficient in front of the x^2 term in a quadratic equation, we can use the quadrus method to rewrite quadratic equations.

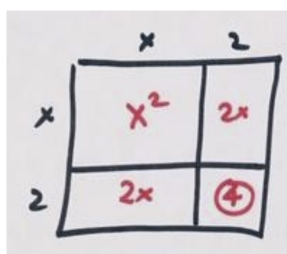
PROBLEM: *Use algebra to prove that the graph of* $y = 2x^2 + 8x + 6$ *is sure to be a symmetrical U-shaped graph.*

Answer: Notice here that all the terms on the right have even coefficients so it feels natural to divide everything throughout by 2. (We could follow the quadrus method right away and multiply through by 2 take make a perfect square up front.)

$$\frac{1}{2}y = x^2 + 4x + 3$$

The quadrus method suggests we rewrite this as

$$\frac{1}{2}y + 1 = x^2 + 4x + 4$$



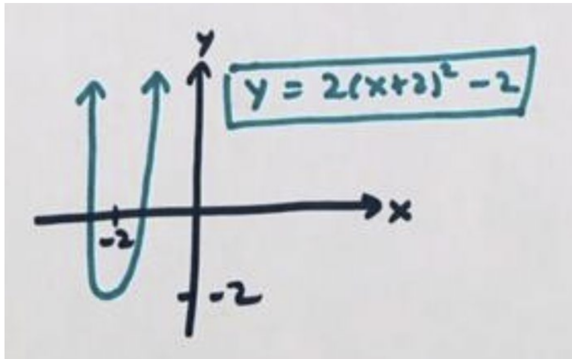
So we have

$$\frac{1}{2}y + 1 = (x+2)^2$$

$$\frac{1}{2}y = (x+2)^2 - 1$$

$$y = 2(x+2)^2 - 2$$

And the graph of this is indeed a symmetrical U-shaped curve. It has $x = -2$ behaving as zero, all heights are shifted down 2 places, and there is a steepness of 2.

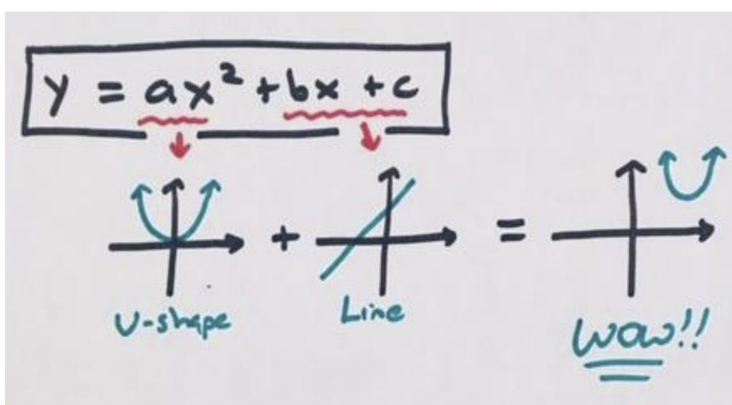


Even if the algebra is a little more complicated, the quadrus method will show each and every time that the graph of a quadratic equation is sure to be a symmetrical U-shaped graph. It will likely be shifted in the plane. There will likely be a steepness factor. But it is guaranteed to be symmetrical!

THE BIG KEY POINT OF THIS ENTIRE SECTION

Every quadratic equation $y = ax^2 + bx + c$ has a symmetrical U-shaped graph.

This is just astounding!



PRACTICE 61 (OPTIONAL): *Show that* $y = 3x^2 + 5x + 1$ *can be rewritten as* $y = 3\left(x + \frac{5}{6}\right)^2 - \frac{13}{12}$ *, and so is sure to have a symmetrical U-shaped graph.*

8.2(B) THE FULL POWER OF SYMMETRY: Graphing with Absolute Ease

We ended the last section on a big note:

> The graph of <u>any</u> quadratic equation $y = ax^2 + bx + c$ is sure to be a symmetric U-shaped curve, situated somewhere in the plane.

And the important word here is *symmetry*. As we full well know, once we have symmetry in a scenario, we have a good friend at our aid!

In this section, we'll show how the simple power of symmetry makes the graphing of quadratic equations beautifully natural and stunningly straightforward!

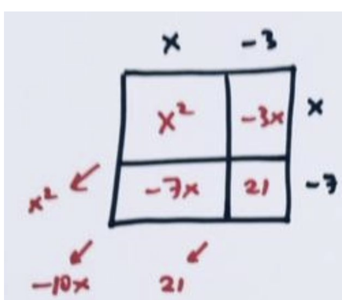
PUTTING SYMMETRY TO USE

We know from the last section that every quadratic equation produces a <u>symmetric</u> U-shaped graph, somewhere in the plane, with some steepness. So let's see now how we can put the power of that symmetry to use.

PROBLEM: *Sketch a graph of* $y = (x-3)(x-7)+10$.

First issue: Is this equation a quadratic equation? It looks a little strange.

Well, if you were to expand out the product in the right side we would see it is a quadratic equation. (It's actually $y = x^2 - 10x + 31$.)



So we can be sure now that its graph will be a symmetric U-shaped graph.

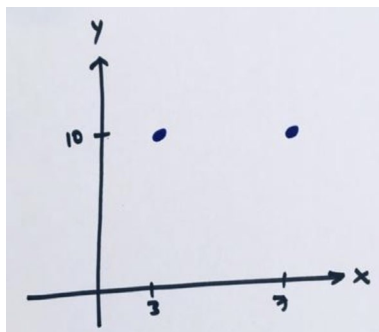
Now stare at the equation

$$y = (x-3)(x-7)+10.$$

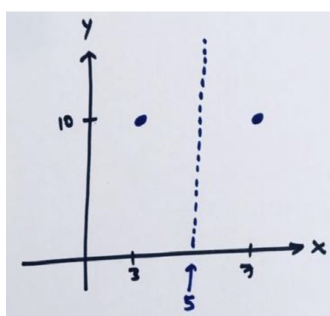
It seems irresistible to put in the x values $x=3$ and $x=7$. And they each give the same y value of 10.

$$x=3 \quad \to \quad y=0+10=10$$
$$x=7 \quad \to \quad y=0+10=10$$

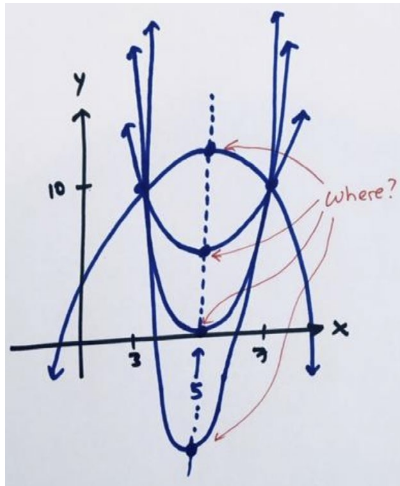
We can plot these two points at least!



Ooh! We have two symmetrical points on a graph that we know is going to be symmetrical.
Common sense tells us that that the line of symmetry of the graph is going to have to be right between $x=3$ and $x=7$, namely, at $x=5$.



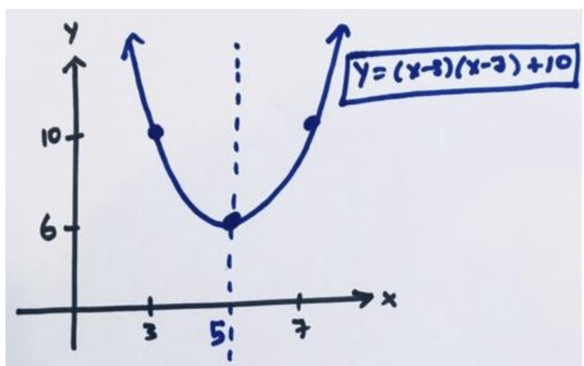
So we have a U-shaped graph with vertex somewhere along this line of symmetry. Where on that line? This will affect the shape of the curve.



Well, we can find out by putting $x = 5$ into the equation!

$$x = 5 \quad \rightarrow \quad y = (2)(-2) + 10 = 6$$

Now it is clear what the graph must be. Wow!



So here's the key idea:

> To sketch the graph of a quadratic equation just find two symmetrical points to go on that symmetrical graph. Then common sense will tell you the line of symmetry, the location of the vertex, and thus the shape of the graph!

Just look for interesting x values!

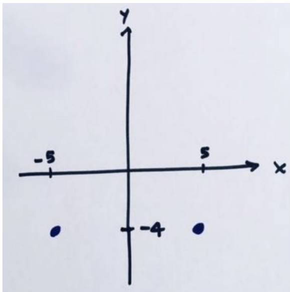
PROBLEM: *Sketch a graph of* $y = -2(x+5)(x-5) - 4$.

If we were to expand out the product on the right side we would see that this is indeed a quadratic equation. It thus will have a symmetrical U-shaped graph.

Are there any obvious and interesting x values? Yes!

$$x = -5 \quad \rightarrow \quad y = 0 - 4 = -4$$
$$x = 5 \quad \rightarrow \quad y = 0 - 4 = -4$$

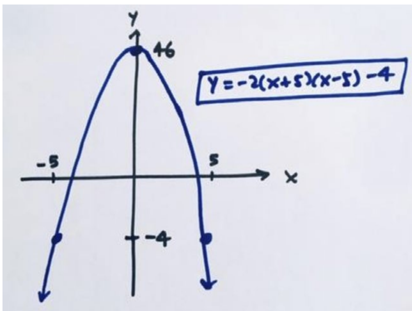
We have two symmetrical points on a symmetrical graph.



The line of symmetry must be the vertical axis and so the vertex must be in this line: $x = 0$.

$$x = 0 \quad \rightarrow \quad y = -2(5)(-5) - 4 = 46$$

We now have a good sketch of the graph.



So swift! So clear! So cool!

Let's do another example.

PROBLEM: *Sketch a graph of* $y = x(x-4) + 7$.

Do we see two interesting x values staring us in the face?

This equation is a little trickier. But, if you like, you can think of it as

$$y=(x-0)(x-4)+7$$

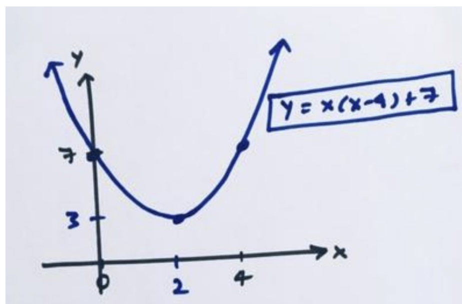
and see directly that $x=0$ and $x=4$ are interesting.

$$x=0 \quad \rightarrow \quad y=7$$
$$x=4 \quad \rightarrow \quad y=7$$

We have two symmetrical points on a symmetrical graph. The line of symmetry is at $x=2$ and the vertex must be on this line.

$$x=2 \quad \rightarrow \quad y=2(-2)+7=3$$

The graph of the equation is now clear.



PRACTICE 62: *The graph of a quadratic equation passes through the points* $(3,81)$, $(4,9)$, *and* $(-10,9)$. *What is the* x*-coordinate of its vertex?*

PRACTICE 63: *Sketch a graph of* $y=2(x-3)(x-23)+200$.

PRACTICE 64: *Sketch a graph of* $y=\frac{1}{25}(x-1)(x-11)-1$.

PRACTICE 65: *Sketch a graph of* $y=-(x-3)(x+5)+6$.

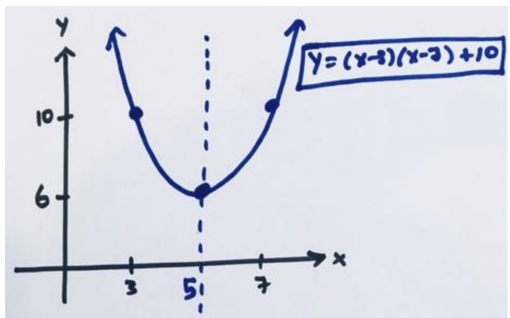
PRACTICE 66: *Sketch a graph of* $y=-2x(x-80)+3$.

THE FULL POWER OF SYMMETRY

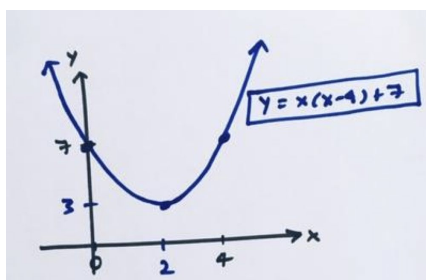
By the power of symmetry we've started graphing quadratic equations with ease. We noticed, for instance, that for

$$y = (x-3)(x-7)+10$$

the x values 3 and 7 are interesting and thus led us to two symmetric points on the symmetric curve. Then common sense allowed us to swiftly sketch the equation's graph.



And for $y = x(x-4)+7$ the x values 0 and 4 are interesting.



But most quadratic equations are presented in a different, namely, in the form

$$y = ax^2 + bx + c\,.$$

Is there a way to identify interesting x values in such equations?

Consider this example.

PROBLEM: *Sketch a graph of* $y = x^2 - 4x + 7$.

There are no interesting x values immediately staring us in the face here. What can we do?

Since we are focused on x values, is there anything we can do with the "$x^2 - 4x$" part of the equation?

The only thing I can think to try is to factor out a common factor of x from those two terms. Write

$$y = x(x-4)+7.$$

Oh! And now we see this is the previous problem we just discussed with interesting x values 0 and 4.

So that's the key!

> If presented with a quadratic expression for which you don't see any obvious interesting x values that lead to a pair of symmetric data points, try doing some basic algebra on the "x part" of the equation.

PROBLEM: *Sketch a graph of* $y = 3x^2 + 9x + 4$.

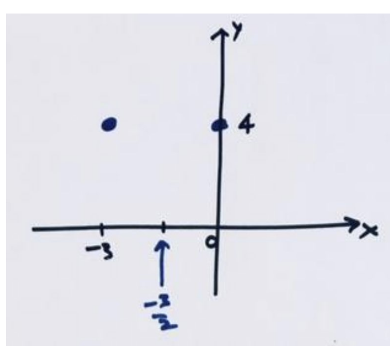
To find some interesting x values, let's play with the "$3x^2 + 9x$" part of the equation. Shall we factor out a common x or $3x$? Well, let's just try $3x$ first (and if it is not useful we can go back and then just try x).

$$y = 3x(x+3)+4$$

Ah! This is good. We see that $x = 0$ and $x = -3$ are interesting.

$$x = 0 \quad \rightarrow \quad y = 4$$
$$x = -3 \quad \rightarrow \quad y = 4$$

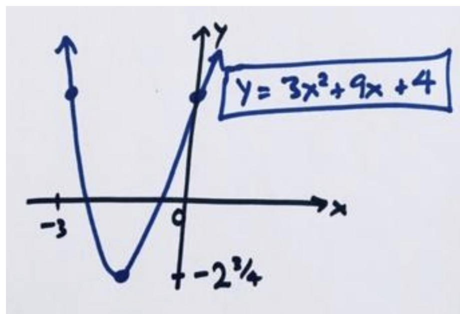
We now have two symmetrical points on a symmetrical graph. The line of symmetry is at $x = -\frac{3}{2}$.



The vertex of the graph will be on this line of symmetry. Let's put $x=-\frac{3}{2}$ into the second form of the equation. (It looks a bit easier.)

$$x=-\frac{3}{2} \quad \rightarrow \quad y=3\left(-\frac{3}{2}\right)\left(\frac{3}{2}\right)+4=-\frac{11}{4}$$

The vertex is below the x axis.



PRACTICE 67: *Sketch a graph of* $y=7x^2+7x-100$.

Here's a curious challenge.

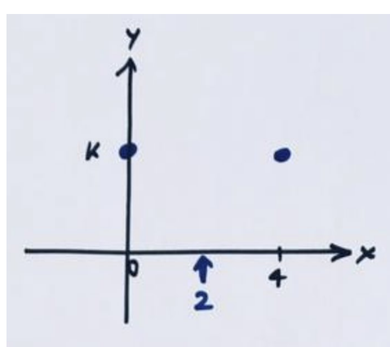
PROBLEM: *Find* k *so that* $y=-2x^2+8x+k$ *gives* 43 *as the largest possible value for* y.

Let's just follow our nose on this one and see how far we can get.

Let's factorise the first two terms, perhaps as

$$y=2x(-x+4)+k.$$

We see that 0 and 4 are interesting x values: they both yield $y=k$.

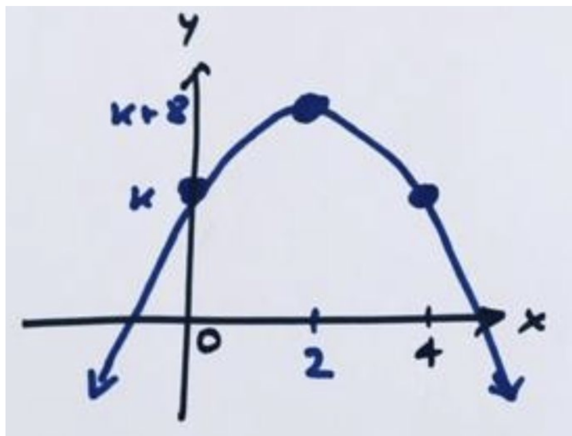


Here I've drawn a picture with k positive. This might not be right—the value could be negative—but let's just go with this for now.

The line of symmetry must be at $x = 2$. For this value we get

$$y = 2(2)(-2+4) + k = k + 8.$$

Hmm. This is a larger than k. Oh, we must have a picture as follows.



The graph is meant to have largest value 43 so we must have $k + 8 = 43$ giving $k = 35$. Done!

PRACTICE 68: *Which value of r forces the graph of $y = 3x^2 + 6x + r$ have 5 as the smallest possible y value?*

PRACTICE 69: *Find a negative value for a so that $y = x^2 + ax + a$ has smallest possible value -3.*

PRACTICE 70: *Find a formula for the location of the line of symmetry of a general quadratic equation* $y = ax^2 + bx + c$.

PRACTICE 71: *Sketch the graph of* $y = -x^2 + 8x + 21$. *What is the largest possible* y *value this equation can produce? What is the vertex of this graph? Make a guess as to what it means to rewrite* $y = -x^2 + 8x + 21$ *in "vertex form."*

MORE PRACTICE

We now have two ways to work with quadratic equations and their graphs.

1. If a quadratic graph has vertex $x = 2$, $y = 3$ say, then we know the equation is of the form

$$y = a(x-2)^2 + 3$$

for some steepness factor a.

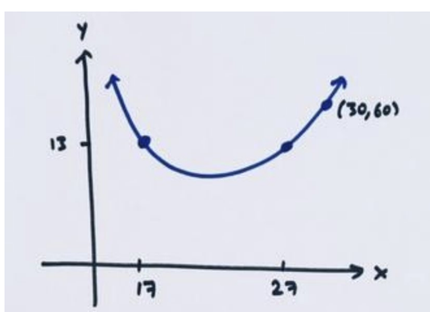
2. Identifying two interesting x values that represent symmetric points in an equation of the form

$$y = a(x-4)(x-12) + 7$$

say, allows us to identify the line of symmetry of the graph and readily sketch its graph.

Let's practice these ideas some more.

PROBLEM: *Find a quadratic equation whose graph is as shown.*



Answer: The graph shows two interesting symmetrical points and suggests then an equation of the form

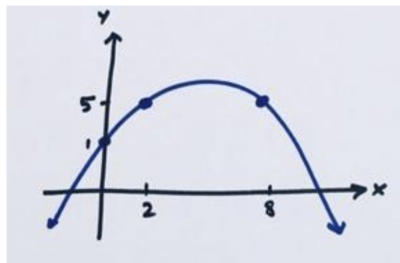
$$y = a(x-17)(x-27) + 13$$

for some steepness a. Since the graph is also meant to pass through the point $x = 30$, $y = 60$ we must have that

$$60 = a(13)(3) + 13$$

is a true number sentence. This forces $a = \frac{47}{39}$, a positive steepness larger than 1, as expected. The equation we seek is $y = \frac{47}{39}(x-17)(x-27)+13$.

PRACTICE 72: *Find the quadratic equation whose graph appears as shown.*



PRACTICE 73: *Write down a quadratic equation whose graph has x intercepts $x=-3$ and $x=11$ and y intercept 10.*

PRACTICE 74: *Sketch the graph of $y=-3x^2-18x+5$ and then use the graph to rewrite the equation in "vertex form."*

Here's a typical textbook question.

PROBLEM: *Consider the equation $y = 2x^2 - 8x + 6$.*

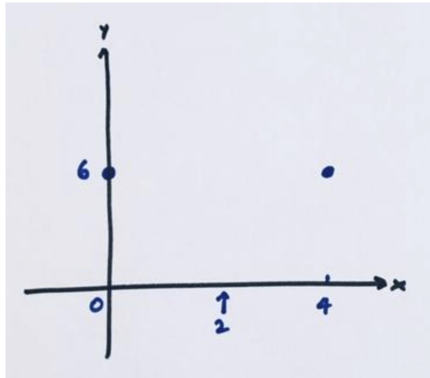
a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x intercepts of the graph.
f) Find the y intercept of the graph.

Answer: Firstly, no one says that you must answer questions in the order presented to you. Since a picture tends to reveal all, let's just go straight to part d) and sketch a graph of the quadratic.

We have

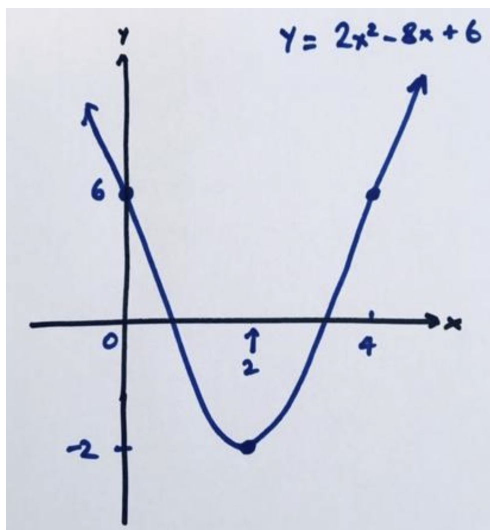
$$y = 2x(x-4)+6$$

giving us $x=0$ and $x=4$ as interesting.



The line of symmetry is at $x = 2$, answering part b). (Some people call this the *axis of symmetry*.)

The vertex is on this line too. When $x = 2$ we have $y = 2(2)(-2) + 6 = -2$. So the vertex is $(2, -2)$, answering part a). And we have a good sketch of the graph for part d).



We also see that the answer to part f) is clear. The y intercept is the point $(0, 6)$.

To answer a), we can see the equation of this graph has the form $y = a(x-2)^2 - 2$. Since the original form of the equation contains "$2x^2$" it must be that $a = 2$. So the vertex form of the equation is

$$y = 2(x-2)^2 - 2.$$

This leaves now only part e).

An x intercept is a point with y-coordinate zero. So we are seeking the x-values which satisfy the equation

$$0 = 2x^2 - 8x + 6.$$

This is back to the algebra of quadratic equations. We can solve this any number of ways—with the square method, with the quadratic formula, perhaps with unsymmetrical factoring. We could also choose to work with an alternative form of the equation, say, with the vertex form of the equation from part a).
For fun, let's do that here. Let's solve

$$0 = 2(x-2)^2 - 2.$$

This gives

$2(x-2)^2 = 2$

$(x-2)^2 = 1$

$x - 2 = 1$ or -1

$x = 3$ or 1.

The x intercepts are $(1,0)$ and $(3,0)$.

The point of the previous question was to show that YOU are in control of your own fabulous thinking and doing. Just use your wits and common sense and all will start to fall into place.

PRACTICE 75: *Consider the equation* $y = 3x^2 - 6x + 20$.

a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x intercepts of the graph.
f) Find the y intercept of the graph.

PRACTICE 76: *Consider the equation* $y = 5x^2 - 10x$.

a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x intercepts of the graph.
f) Find the y intercept of the graph.

PRACTICE 77: *Solve the following quadratic equations.*

a) $(x-3)(x+5)=1$

b) $x^2=(2x-1)(2x+1)-5$

c) $(x-10)(x+1)+5=12$

PRACTICE 78: *Find, in terms of* c*, the value* k *so that* $y=(x+c)(x-c)+k$ *gives* -2 *as the smallest possible* y *value.*

Here are a few more practice problems. This next one illustrates the interplay between geometry and algebra.

PRACTICE 79:

a) Solve $x^2+10x+30=0$ *and see what happens when you try.*

b) Sketch the graph of $y=x^2+10x+30$.

c) Use the graph to explain what happened in part a)

d) Will $x^2+10x+30=11$ *have a solution? If so, how many solutions?*

e) For which value(s) k *does* $x^2+10x+30=k$ *have only one solution?*

PRACTICE 80: *How many solutions does* $-x^2+4x-5=0$ *have? Answer this question not by algebra, but by graphing.*

PRACTICE 81:

a) *Find a value* k *so that the graph of*

$$y=5x^2-10x+k$$

just touches the x *axis.*

b) Find a value m *so that* $y=-2x^2-18x+m$ *gives the highest output value of* 100.

c) Find a value p *so that* $y=(x-p)(x-3p)$ *has smallest output value of* -10.

Some more questions!

PRACTICE 82: *Find, in terms of* r*, a value* k *so that the graph of* $y = 3x(2r - x) - k$ *just touches the* x *axis.*

PRACTICE 83: *Consider the equation* $y = 2(x-1)(x+1) + (x-3)(x+4) + x(x-2)$

a) Is this an equation in the form $y = ax^2 + bx + c$ *in disguise for some numbers* a*,* b*, and* c*? (This is just a YES/NO question!)*

b) Explain how one readily sees $a = 2 + 1 + 1 = 4$*.*

c) Explain why putting $x = 0$ *into the right side of the equation allows us to conclude that* $c = -14$*.*

d) What is the value of b*?*

PRACTICE 84: *A rectangle has side lengths* $7 - r$ *and* $3 + r$ *for some value* r*. What value for* r *gives a rectangle of maximal area?*

PRACTICE 85: *Here are three quadratic equations:*

(A) $y = 3(x-3)(x+5)$

(B) $y = 2x^2 + 6x + 8$

(C) $y = 2(x-4)^2 + 7$

i) For which of these three expressions is it very easy to answer the question: "What is the smallest y*-value the expression can produce?"*

ii) For which of these three expressions is it very easy to answer the question: "Where does the graph of the equation cross the x*-axis?"*

iii) For which of these three expressions is it very easy to answer the question: "Where does the graph of the quadratic cross the y *axis?"*

iv) For which of these three expressions is it very easy to answer the question: "What are the coordinates of the vertex in this equation's graph?"

PRACTICE 86: Here is a silly question.

I make and sell BIPS. It costs me $80,000 + 200n$ *rupees to make* n *BIPS.*

If I sell n *BIPS I bring in* $2n\left(600 - n\right)$ *rupees.*

a) Ignoring costs, what number of BIPS sold brings in the largest number of rupees?

b) What number of BIPS should I sell to make the biggest profit?

8.3(A) HOW TO SPELL YOUR NAME IN MATH: Finding Formulas to Fit Data

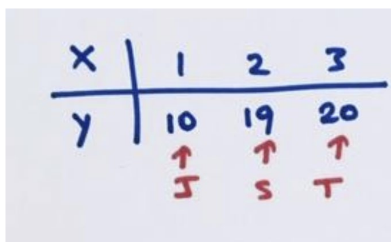
My full names is *James Stuart Tanton* and my initials thus are JST. So consider this quadratic equation:

$$y = -4x^2 + 21x - 7.$$

Put in the values 1, 2, and 3 in turn and out come the y values:

$$-4 \cdot 1^2 + 21 \cdot 1 - 7 = 10$$
$$-4 \cdot 2^2 + 21 \cdot 2 - 7 = 19$$
$$-4 \cdot 3^2 + 21 \cdot 3 - 7 = 20$$

And notice: The 10th letter of the alphabet is J, the 19th letter of the alphabet is S; the 20th letter of the alphabet is T. This quadratic spells out my initials!



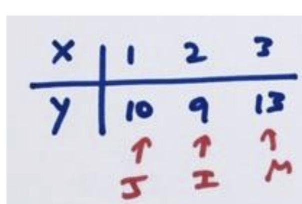
Many people like to call me JIM. So consider $y = \frac{5}{2}x^2 - \frac{17}{2}x + 16$.

Put in $x = 1$ and one gets $y = \frac{5}{2} - \frac{17}{2} + 16 = 10$.

Put in $x = 2$ and one gets $y = \frac{5}{2} \cdot 4 - \frac{17}{2} \cdot 2 + 16 = 9$.

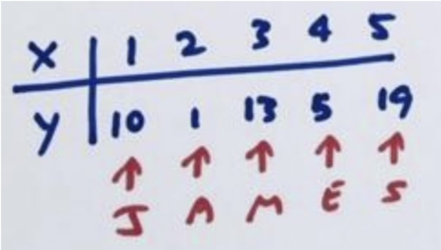
Put in $x = 3$ and one gets $y = \frac{5}{2} \cdot 9 - \frac{17}{2} \cdot 3 + 16 = 13$.

This quadratic spells my nickname!



I actually prefer to be called JAMES. Going beyond quadratics I can write down

$$y = \frac{83}{24}x^4 - \frac{497}{12}x^3 + \frac{4141}{24}x^2 - \frac{3463}{12}x + 164$$



WHOA!

How am I coming up with these crazy equations that spell initials and names?

So my puzzle to you is:

> Choose three letters (a three-letter word, three initials) and try to find a quadratic equation that "spells" those three letters. Can you do it?

THE TECHNIQUE

Let me reveal the technique I used to find these formulas with the specific example of spelling JAMES. The technique is due to French mathematician Joseph-Louis Lagrange (1736-1813) and is today called *Lagrange's Interpolation Formula*. It looks shockingly scary at first.

A formula that gives

value 10 for $x=1$,
value 1 for $x=2$,
value 13 for $x=3$,
value 5 for $x=4$,
value 19 for $x=5$

is

$$y=10\times\frac{(x-2)(x-3)(x-4)(x-5)}{(-1)(-2)(-3)(-4)}+1\times\frac{(x-1)(x-3)(x-4)(x-5)}{(1)(-1)(-2)(-3)}$$

$$+13\times\frac{(x-1)(x-2)(x-4)(x-5)}{(2)(1)(-1)(-2)}+5\times\frac{(x-1)(x-2)(x-3)(x-5)}{(3)(2)(1)(-1)}$$

$$+19\times\frac{(x-1)(x-2)(x-3)(x-4)}{(4)(3)(2)(1)}$$

And this looks horrific!

But it's actually easy to understand after one has taken a deep breath.

DEEP BREATH!

Let's process the scary-looking formula in stages.

1. There are five terms added together.
2. There is one term for each appearance of the values 10, 1, 13, 5, and 19.
3. Each term is designed to vanish for all but one of the values $x=1$, 2, 3, 4, and 5.

To make sense of this third statement, put $x=1$, say, into the formula and see what it does to each term of in the expression. For example, the second term is $1\times\frac{(x-1)(x-3)(x-4)(x-5)}{(1)(-1)(-2)(-3)}$ with the factor $x-1$ in its numerator. When we put $x=1$ into this expression we thus get zero in the numerator of the expression and thus the whole expression equals zero.

The third, fourth, and fifth terms in the sum also have a factor of $x-1$ in their numerators, and so each equal zero when x equals 1.

Only the first term will be non-zero for $x=1$.

Check: *Put $x=4$ into the formula. Which terms vanish for $x=4$? Which one term "survives" for $x=4$?*

4. Each term in the sum is a number times a fraction. The fraction is designed to equal 1 at a specific value of x.

To make sense of this, consider putting in $x=1$ again. The final four terms in the sum are zero and only the first term "survives." But look what happens when we put in $x=1$ into the first term. We get

$$10\times\frac{(1-2)(1-3)(1-4)(1-5)}{(-1)(-2)(-3)(-4)}=10\times1=10.$$

The denominator was designed to match the numerator for this instance and the term has value 10. The sum of all five terms equals $10+0+0+0+0=10$ for $n=1$, just as wanted.

Check: *Put $x=4$ into the entire formula and see it gives the value $0+0+0+5\times1+0=5$, just as hoped.*

Practice 87: *Convince yourself that the formula*

$$8\frac{(x-10)(x-20)}{(-4)(-14)}+122\frac{(x-6)(x-20)}{(4)(-10)}+4600\frac{(x-6)(x-10)}{(14)(10)}$$

gives the value 8 for $x=6$, the value 122 for $x=10$, and the value 4600 for $x=20$.

Practice 88: *Find a formula that gives the value* 9000 *for* $x=3$*, the value* -45 *for* $x=5$*, and the value* $\frac{2}{3}$ *for* $x=8$*. (Don't bother simplifying your formula.)*

Practice 89: *Write down a formula that fits this data set.*

x	1	2	7	15
y	5	10	8	-3

The formula that spells JAMES

$$y=10\times\frac{(x-2)(x-3)(x-4)(x-5)}{(-1)(-2)(-3)(-4)}+1\times\frac{(x-1)(x-3)(x-4)(x-5)}{(1)(-1)(-2)(-3)}$$

$$+13\times\frac{(x-1)(x-2)(x-4)(x-5)}{(2)(1)(-1)(-2)}+5\times\frac{(x-1)(x-2)(x-3)(x-5)}{(3)(2)(1)(-1)}$$

$$+19\times\frac{(x-1)(x-2)(x-3)(x-4)}{(4)(3)(2)(1)}$$

can be made to look friendlier. Can see that we can at least write

$$y=\frac{10}{24}(x-2)(x-3)(x-4)(x-5)\quad-\frac{1}{6}(x-1)(x-3)(x-4)(x-5)$$

$$+\frac{13}{4}(x-1)(x-2)(x-4)(x-5)\quad-\frac{5}{6}(x-1)(x-2)(x-3)(x-5)$$

$$+\frac{19}{24}(x-1)(x-2)(x-3)(x-4)$$

And if you have the patience to keep working on it—and I did—you will eventually come to

$y = \frac{83}{24}x^4 - \frac{497}{12}x^3 + \frac{4141}{24}x^2 - \frac{3463}{12}x + 164$ as I first presented. (Whoa!)

Practice 90: *Use Lagrange Interpolation to find a formula that spells JST. Then do the necessary work to show that it simplifies to* $y = -4x^2 + 21x - 7$.

Practice 91: a) *Can you see that* $y = 7\frac{(x-4)(x-5)}{(-5)(-6)} + \frac{(x+1)(x-5)}{(5)(-1)} + 10\frac{(x-1)(x-4)}{(4)(1)}$ *is a quadratic equation in disguise? Verify that the graph of this equation passes through the data points* $(-1,7)$, $(4,1)$, *and* $(5,10)$.

b) Find a quadratic formula that fits the data $(2,5)$, $(-1,6)$, *and* $(5,46)$ *and make your answer look as friendly as possible.*

x	**2**	**-1**	**5**
y	**5**	**6**	**46**

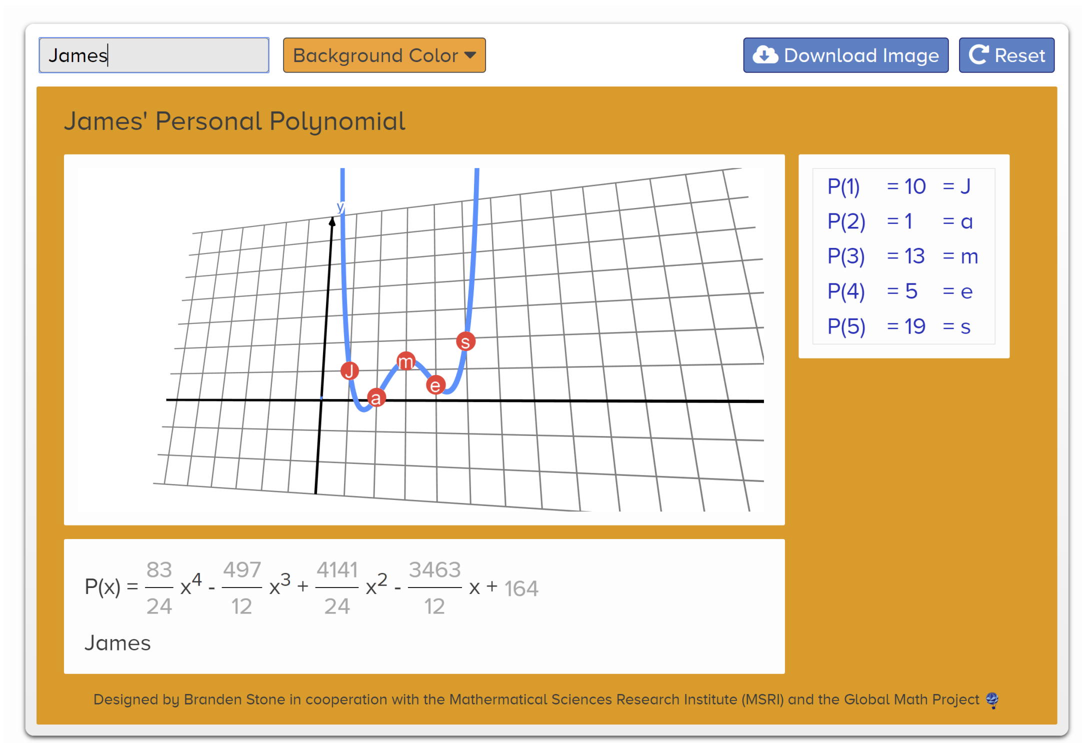
c) *Find a quadratic formula that fits the data* $(2,5)$, $(3,8)$, *and* $(5,14)$ *and make your answer look as friendly as possible. Explain what happens!*

d) *There is no quadratic formula* $y = ax^2 + bx + c$ *that fits the data* $(2,5)$, $(10,6)$, *and* $(10,100)$. *(The same input value of* $x = 10$ *cannot give two different output values.) So then, how does Lagrange's Interpolation method fail when you try to use it?*

Practice 92: *Use Lagrange's Interpolation Formula to find the equation of the line between two points* (p,m) *and* (q,n) *with* $p \neq q$. *Is your equation equivalent to "*$y = mx + b$*"where* m *is the slope of the line segment between the two points?*

A COOL WEB APP

Go to www.globalmathproject.org/personal-polynomial/ for a really cool web app that finds—and graphs—your personal polynomial for you! (Also see videos explaining the mathematics.)

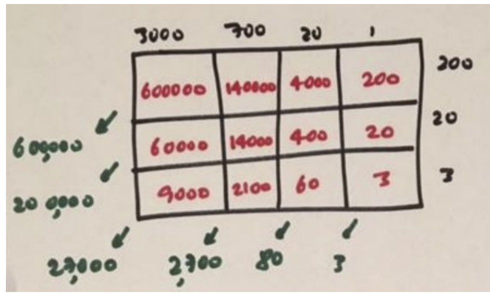


FINAL EXERCISE: Find the formula that spells **YOUR** name!

MODULE SOLUTIONS

Practice 1: *Use the area model to compute* 3721×223*. (Into how many pieces might you divide your rectangle?)*

Answer: One might divide a rectangle into twelve pieces by thinking of 3721 as 3000 + 700 + 20 + 1 and 223 as 200 + 20 + 3. The areas of the individual pieces then add to 829783.

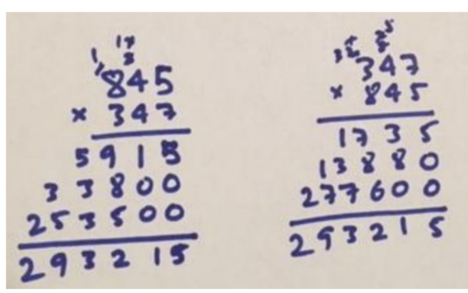


Of course, one need not divide the rectangle this way. One could think of 3721 as 2000 + 1500 + 10 + 10 + 1 and 223 as 100+100+10+10+1+1+1, for instance.

All ways of subdividing the rectangle should, of course, yield the same final answer of 829783.

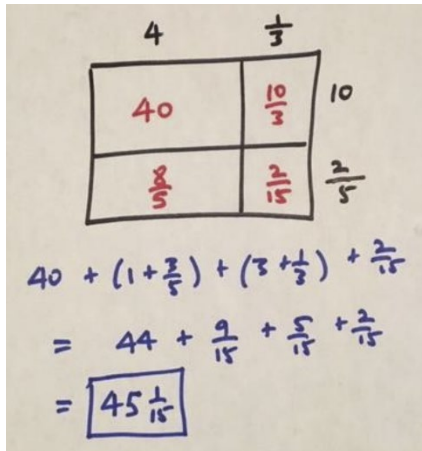
Practice 2: *Use the traditional long multiplication algorithm to compute* 845×387*. And use it again to compute* 387×845*, but as you do so this second time ask yourself: Is it obvious the algorithm will give the same final answer?*

Answer: It is not at all obvious to me why at face value the traditional long multiplication algorithm naturally gives the same answers whether you compute $a \times b$ or $b \times a$. What you write on the page looks different. It is actually somewhat surprising that the final lines after each summation are the same.



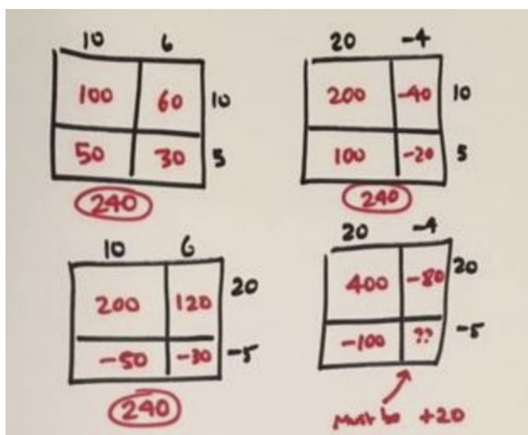
Practice 3: *Use the area model to compute* $4\frac{1}{3}\times 10\frac{2}{5}$.

Answer:



Practice 4: *Compute* 16×15 *four different ways to conclude that* $(-4)\times(-5)$ *is positive* 20.

Answer:



Practice 5:

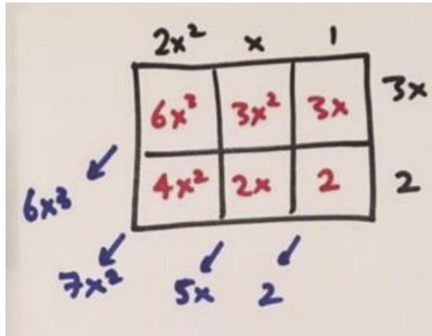
a) Use the area model to compute $(2x^2+x+1)(3x+2)$.

b) Use your answer to quickly see the value of 211×32.

c) Put $x=-10$ *into your answer from a). What multiplication problem is this the answer to?*

Answer:

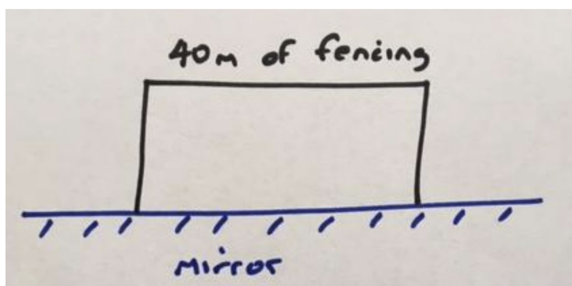
a) $(2x^2+x+1)(3x+2)=6x^3+7x^2+5x+2$



b) Put $x = 10$ to see that this reads: $(200+10+1)(30+2)$ equals $6000+700+50+2=6752$.

c) We have $(200-10+1)(-30+2)$, that is, $191\times(-28)$, equals $-6000+700-50+2$, which is -5348.

Practice 6: (**CHALLENGE**) *A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. But she has huge mirrored wall in her field and wants to use the mirror as one side of her rectangular pen.*



What should the dimensions of the rectangle be in order to obtain a pen of maximal area?

Answer:
This problem is not-symmetrical as the pen has two full vertical sides and only one horizontal side. The problem we discussed in the essay just before this was symmetrical with two sides of each type.

But if one looks into the mirrored wall, it will look as though we have a rectangular pen, double the area and double the perimeter (80 meters), with FOUR sides of fencing.

We know the answer to the symmetrical problem is a symmetrical square, so the mirrored reflection must be a 20 meter-by-20 meter square pen to get maximal area. But the actual pen without the reflection is half of this, a 10 meter-by-20 meter pen!

She should build a 10-by-20 pen.

SNEAKY!

PRACTICE 7: *Solve*

a) $2x^2 = 50$
b) $y^2 + 5 = 14$
c) $100x^2 = 1$
d) $4p^2 = 0.25$
e) $a^2 + 7 = 7$
f) $x^2 = 20$
g) $x^2 = -20$

Answer:

a) $x^2 = 25$ so $x = 5$ or -5.

b) $y^2 = 9$, so $y = 3$ or -3.

c) $x^2 = \frac{1}{100}$, so $x = \frac{1}{10}$ or $-\frac{1}{10}$.

d) $p^2 = \frac{1}{16}$, so $p = \frac{1}{4}$ or $-\frac{1}{4}$.

e) $a^2 = 0$, so $a = 0$.

f) $x = \sqrt{20}$ or $-\sqrt{20}$.

g) Has no solutions.

PRACTICE 8: *Solve*

a) $(4x-6)^2 = -7$
b) $(4x-6)^2 = 0$
c) $(4x-6)^2 = 4$
d) $(4x-6)^2 = 5$

Answer:

a) No solutions. (No quantity squared can be negative.)

b) $4x - 6 = 0$, so $x = \frac{3}{2}$.

c) $4x - 6 = 2$ or -2 , so $x = 2$ or 1.

d) $4x-6=\sqrt{5}$ or $-\sqrt{5}$,

so $x=\dfrac{6+\sqrt{5}}{4}$ or $\dfrac{6-\sqrt{5}}{4}$.

PRACTICE 9: *Solve*

a) $(y+1)^2-2=23$

b) $4(p-2)^2-16=0$

c) $9+\left(34x-77\dfrac{1}{2}\right)^2=0$

d) $\left(x-\sqrt{2}\right)^2=5$

Answer:

a) $(y+1)^2=25$

$y+1=5$ or -5

$y=4$ or -6

b) $(p-2)^2=4$

$p-2=2$ or -2

$p=4$ or 0

c) $\left(34x-77\dfrac{1}{2}\right)^2=-9$ has no solutions.

d) $x-\sqrt{2}=\sqrt{5}$ or $-\sqrt{5}$

$x=\sqrt{2}+\sqrt{5}$ or $\sqrt{2}-\sqrt{5}$.

PRACTICE 10: *Solve*

a) $p^2-6p+9=9$

b) $x^2-4x+4=1$

c) $x^2-20x+100=7$

d) $r^2-16r+64=-2$

e) $x^2+2\sqrt{5}x+5=36$

f) $x^2-2\sqrt{2}x+2=19$

Answer:

a) $\left(p-3\right)^2=9$
$p-3=3$ or -3
$p=6$ or 0

b) $\left(x-2\right)^2=1$
$x-2=1$ or -1
$x=3$ or 1

c) $\left(x-10\right)^2=7$
$x=10+\sqrt{7}$ or $10-\sqrt{7}$

d) $\left(r-8\right)^2=-2$ has no solutions.

e) $\left(x+\sqrt{5}\right)^2=36$
$x=\sqrt{5}+6$ or $\sqrt{5}-6$

f) $\left(x-\sqrt{2}\right)^2=19$
$x=\sqrt{2}+\sqrt{19}$ or $\sqrt{2}-\sqrt{19}$.

PRACTICE 11: *Solve for* x *giving your answer in terms of* A and B.

$$x^2+2Ax+A^2=B^2$$

Answer:
$\left(x+A\right)^2=B^2$
$x+A=B$ or $-B$
$x=B-A$ or $-\left(A+B\right)$

PRACTICE 12: *Solve*

a) $f^2+8f+15=80$
b) $w^2+90=22w-31$
c) $x^2-6x=3$.

Brief Answers:

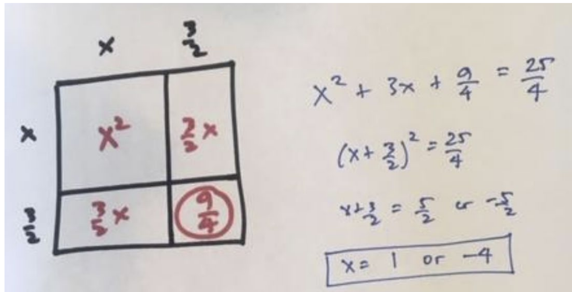
a) $f^2+8f+16=81$
$\left(f+4\right)^2=81$
$f=5$ or -13

b)
$w^2-22w+90=-31$
$w^2-22w+121=0$
$\left(w-11\right)^2=0$
$w=11$

c) $x^2-6x+9=12$
$\left(x-3\right)^2=12$
$x=3+\sqrt{12}$ or $3-\sqrt{12}$

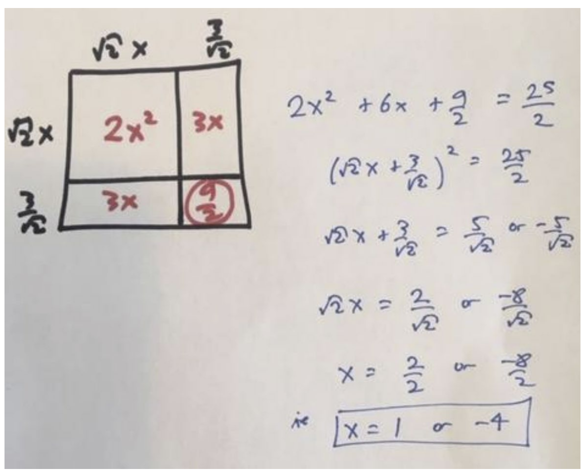
PRACTICE 13 (OPTIONAL): *Push on with this problem and do work with fractions. Show that the square method eventually gives the solutions* $x=1$ *or* $x=-4$. (The square method will <u>never</u> let you down!)

Answer:



PRACTICE 14 (OPTIONAL): *Push on with this problem and do work with the square roots. Show that the square method again eventually gives the solutions* $x=1$ *or* $x=-4$.

Answer:



PRACTICE 15: *Solve as many of these as you feel like doing.*

a) $w^2-5w+6=2$

b) $x^2+9x+1=11$

c) $p^2+p+1=0.75$

d) $x^2=10-3x$

e) $x^2-x-1=2\frac{3}{4}$

f) $x^2+3=9$

Brief Answers:

a) $w = 1$ or 4
b) $x = 1$ or -10
c) $p = -0.5$
d) $x = 2$ or -5
e) $x = \frac{3}{2}$ or $-\frac{5}{2}$
f) $x = \sqrt{6}$ or $-\sqrt{6}$

PRACTICE 16: *Solve as many of these you feel like doing.*

a) $2x^2 = 9$
b) $4 - 3x^2 = 2 - x$
c) $\alpha^2 - \alpha + 1 = \frac{7}{4}$
d) $3x^2 + 3x + 1 = 19$
e) $-3x^2 + 3x + 1 = 19$
f) $10k^2 = 1 + 10k$

Brief Answers:

a) $x = \frac{3}{\sqrt{2}}$ or $-\frac{3}{\sqrt{2}}$.

(Does your curriculum prefer to write this as $\frac{3\sqrt{2}}{2}$ and $-\frac{3\sqrt{2}}{2}$?)

b) $x = 1$ or $-\frac{2}{3}$.
c) $\alpha = \frac{3}{2}$ or $-\frac{1}{2}$.
d) $x = 2$ or -3.
e) $x = \frac{3+\sqrt{249}}{6}$ or $\frac{3-\sqrt{249}}{6}$.
f) $k = \frac{5+\sqrt{35}}{10}$ or $\frac{5-\sqrt{35}}{10}$

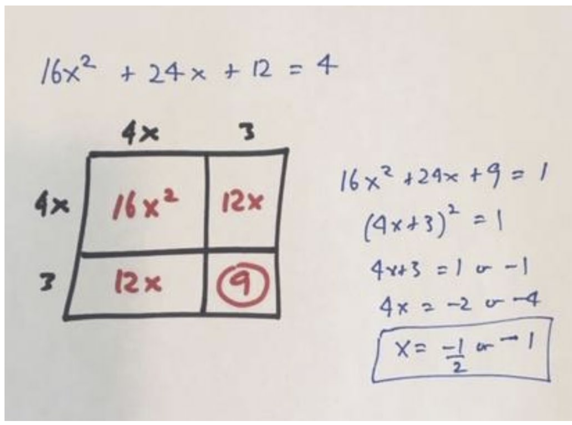
PRACTICE 17: *Consider* $4x^2 + 6x + 3 = 1$*. Does it look like this quadratic equation will have problems when solving it? Does it have problems as you try to solve it? What can you do to obviate the difficulties you encounter?*

Answer: We have a perfect square up front and an even middle term. So it looks good!

But if you try to solve it with the square method you find yourself dealing with fractions!

To avoid them, let's try multiplying through by 4. Then all is good!

We get $x = -\frac{1}{2}$ or -1.



PRACTICE 18:

a) Design a quadratic equation that has two negative solutions.
b) Design a quadratic equation with just one solution, namely, $x = 4$.
c) Design a quadratic equation with $x = 2$ and $x = 10$ as solutions.

Answer:

a) Lots of different answer are possible for a). One strategy is to work with $(x+500)^2 = 1$, for instance. This is the quadratic equation

$$x^2 + 1000x + 250000 = 1.$$

b) Work with $(x-4)^2 = 0$. This is the quadratic equation

$$x^2 - 8x + 16 = 0.$$

c) This one is harder. Maybe think

$$x = 6 - 4 \text{ or } 6 + 4$$

to then think

$$\left(x-6\right)^2 = 16$$

which is the quadratic equations

$$x^2 - 12x + 36 = 16.$$

PRACTICE 19:

a) A rectangle is twice as long as it is wide. Its area is 30 square meters. What are the dimensions of the rectangle?
b) A rectangle has one side 4 meters longer than the other. Its area is 30 square meters. What are the dimensions of the rectangle?

Answer:

a) If we have a $2x$ by x rectangle, then we need $2x^2 = 30$. This means x must be $\sqrt{15}$. It is a $2\sqrt{15}$ by $\sqrt{15}$ rectangle.

b) If we have an x by $x+4$ rectangle, then we need $x\left(x+4\right) = 30$, that is, we need

$$x^2 + 4x = 30.$$

This is $\left(x+2\right)^2 = 34$ and so $x = \sqrt{34} - 2$. (We must choose the positive length.) We thus have a $\sqrt{34} - 2$ by $\sqrt{34} + 2$ rectangle.

PRACTICE 20:

a) Solve $\tau^2 - 5\tau + 7 = 1$.

The symbol $\sqrt{}$ means the <u>positive</u> root of a number. (For instance, $\sqrt{9} = 3$, and not -3, even though there are two numbers whose squares are 9.) This is a mathematics convention, and it can be confusing as it is ignoring symmetry.

But we can say that if x is a positive number, then we have $x = \left(\sqrt{x}\right)^2$.

b) Solve $x - 5\sqrt{x} + 7 = 1$.
HINT: Look at part a).

c) Solve $x - 2\sqrt{x} = -1$

d) Solve $x + 2\sqrt{x} - 5 = 10$ and be clear why this equation has only one solution!

e) Solve $2u^4 + 8u^2 + 7.5 = 0$.

Answers:

a) $\tau = 2$ or 3.

b) $\sqrt{x} = 2$ or 3 , suggesting $x = 4$ or 9. (One checks that these are both valid solutions.)

c) $\left(\sqrt{x} - 1\right)^2 = 0$, so $\sqrt{x} = 1$, suggesting $x = 1$ is a solution (which it is).

d) $\left(\sqrt{x} + 1\right) = 16$

$\sqrt{x} + 1 = 4$ or -4

$\sqrt{x} = 5$ or -3.

But $\sqrt{x}$ can't be a negative value, so we can only consider $\sqrt{x} = 5$, suggesting $x = 25$. (And this is a valid solution.)

e) This is a quadratic equation in the variable u^2. Multiply though by 2 to make a perfect square up front.

$$4u^4 + 8u^2 + 15 = 0$$

$$4\left(u^2\right)^2 + 8\left(u^2\right) + 15 = 0$$

$$4\left(u^2\right)^2 + 8\left(u^2\right) + 16 = 1$$

$$\left(2u^2 + 4\right)^2 = 1$$

So we have

$2u^2 + 4 = 1$ or -1

$2u^2 = -3$ or -5

This can never be the case as $2u^2$ cannot be negative. This equation has no solutions.

PRACTICE 21: *Consider* $y = 2\left(x - 4\right)^2 + 6$. *What value for* x *produces the smallest possible value for* y *? Why?*

Answer: The quantity $\left(x - 4\right)^2$ is always postive, or zero if $x = 4$. Thus y has smallest possible value $2 \times 0 + 6 = 6$, occuring when $x = 4$.

PRACTICE 22: *Find one solution to* $\left(x + 1\right)^3 = 27$.

Answer: Something cubed is 27. That something could be 3.

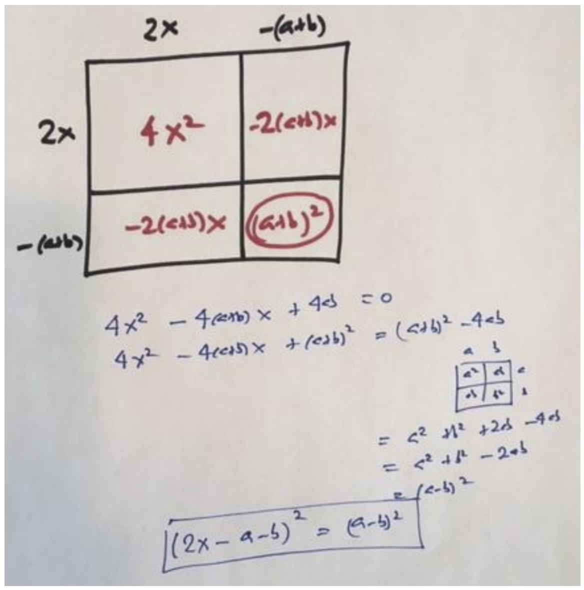
So $x + 1 = 3$, that is, $x = 2$, is one solution.

PRACTICE 23 (TOUGH!): *In the following equation, solve for x in terms of a and b.*

$$x^2 - (a+b)x + ab = 0.$$

<u>HINT 1</u>: Multiply through by 4 just in case $a+b$ is odd.
<u>HINT 2</u>: $(a+b)^2 - 4ab$ equals $a^2 - 2ab + b^2$, which happens to equal $(a-b)^2$. (Check these claims.)

Answer: $4x^2 - 4(a+b)x + 4ab = 0$



So

$$2x - a - b = a - b \text{ or } b - a$$
$$2x = 2a \text{ or } 2b$$
$$x = a \text{ or } b$$

PRACTICE 24 (OPTIONAL): This problem will require you to multiplying through by 4 many times!

a) Solve $x^2 + x = 2$.
b) Solve $2x^2 + x = 3$.
c) Solve $4x^2 + x = 5$.
d) Solve $8x^2 + x = 9$.
e) Solve $16x^2 + x = 17$.

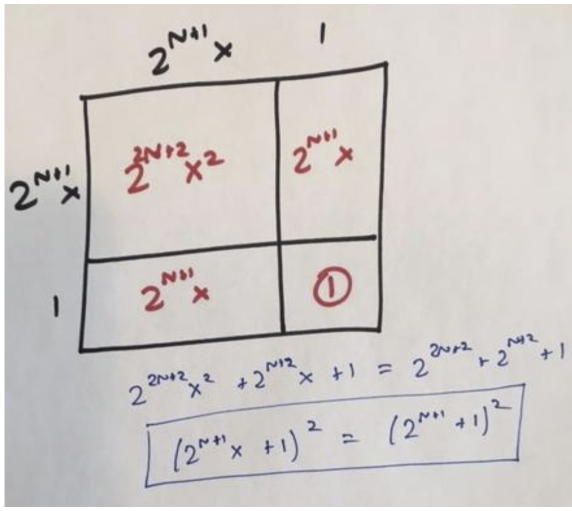
If you are game …

f) Find the solutions to $2^N x + x = 2^N + 1$.

Brief Answer: We'll answer f).

Multiply through by 2^{N+2} to examine

$$2^{2N+2}x^2 + 2^{N+2}x = 2^{2N+2} + 2^{N+2} .$$



So

$$2^{N+1}x + 1 = 2^{N+1} + 1 \text{ or } -2^{N+1} - 1$$
$$2^{N+1}x = 2^{N+1} \text{ or } -2^{N+1} - 2$$
$$x = 1 \text{ or } -1 - \frac{1}{2^N} = -\frac{2^N + 1}{2^N}$$

PRACTICE 25: *Solve* $3x^2+5x+1=9$ *by using the quadratic formula and then again by using the square method. (Of course, you should get the same answers each time!)*

Answer: Rewrite the equation as $3x^2+5x-8=0$. The quadratic formula then gives

$x=\dfrac{-5\pm\sqrt{25+96}}{6}=\dfrac{-5\pm 11}{6}$ and so

$x=1$ or $x=-\dfrac{8}{3}$.

The square method suggests working with

$$36x^2+60x+25=121$$

that is,

$$(6x-5)^2=121$$

which gives the same solutions.

PRACTICE 26 (OPTIONAL): *Solve*

$$1.3x^2+\frac{\pi}{3}x-\frac{17}{\sqrt{2\frac{1}{2}}}=0\,.$$

Answer:

$x=\dfrac{-\dfrac{\pi}{3}\pm\sqrt{\dfrac{\pi^2}{9}-\dfrac{88.4}{\sqrt{2.5}}}}{2.6}$ which I do not feel like simplifying.

PRACTICE 27: *Solve whichever of these quadratic equations you feel like doing, using whatever method you like. (Or solve some twice with two different methods!)*

a) $6x^2-x+10=11$
b) $30x^2-17x=2$
c) $x^2-4x+4=0$
d) $2x^2+5=11x$
e) $93x^2-117=0$
f) $x^2+x+1=2$
g) $x^2+x+1=1$
h) $x^2+x+1=0$

Brief Answers:

a) $x = \frac{1}{2}$ or $-\frac{1}{3}$.

b) $x = \frac{2}{3}$ or $-\frac{1}{10}$.

c) $x = 2$.

d) $x = 5$ or $\frac{1}{2}$.

e) $x = \pm\sqrt{\frac{117}{93}}$.

f) $x = \frac{-1+\sqrt{5}}{2}$ or $\frac{-1-\sqrt{5}}{2}$.

g) $x = 0$ or -1.

h) No solutions.

PRACTICE 28: *Solve for x in terms of a and b.*

$$x^2 - (a+b)x + ab = 0.$$

Answer: We did this question using the square method in a previous essay. Let's use the quadratic formula this time.

$$\begin{aligned} x &= \frac{a+b \pm \sqrt{(a+b)^2 - 4ab}}{2} \\ &= \frac{a+b \pm \sqrt{a^2 + b^2 - 2ab}}{2} \\ &= \frac{a+b \pm \sqrt{(a-b)^2}}{2} \end{aligned}$$

So

$$x = \frac{a+b+a-b}{2} \text{ or } \frac{a+b-(a-b)}{2}$$

giving

$$x = a \text{ or } b.$$

PRACTICE 29: *Find a number p so that $1 + \frac{1}{p}$ equals p.*

Answer: Notice that this question implicitly assumes that p is not zero. (We'll see if this is an issue or not later on, I guess.)

Multiplying the equation through p gives

$$p+1=p^2$$

that is,

$$p^2-p-1=0.$$

This has solutions

$p=\frac{1+\sqrt{5}}{2}$ and $p=\frac{1-\sqrt{5}}{2}$.

Neither of these is zero. They are both valid solutions.

PRACTICE 30: *A quadratic equation* $ax^2+bx+c=0$ *has precisely one solution. What is the value of* b^2-4ac *?*

Answer: According to the quadratic formula, the solutions are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

For there to be precisely one solution, we need b^2-4ac to be zero. This gives the solutions

$$x=\frac{-b+\sqrt{0}}{2a} \text{ and } \frac{-b-\sqrt{0}}{2a}.$$

And since $\sqrt{0}=0$, these both equal $\frac{-b}{2a}$ and we have one solution.

PRACTICE 31: *The list of* oblong numbers *begins* $2, 6, 12, 20, 30, 42, 56, \ldots$. *Here each number is the product of two consecutive integers.*

$$2 = 1 \times 2$$
$$6 = 2 \times 3$$
$$12 = 3 \times 4$$
etc.

a) What is the one-hundredth number in the list?
b) 5402 *is an oblong number. At which position in the list does it sit?*

Answer: The nth number in the list is given by $n(n+1)$.

a) The 100^{th} number in the list is 10100.

b) We need to solve $n(n+1) = 5402$, that is,

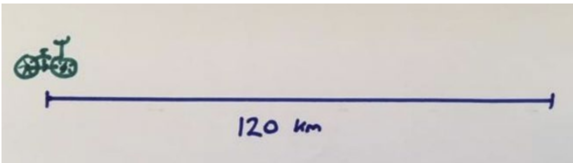
$$n^2 + n - 5402.$$

We get $n = \dfrac{-1 \pm \sqrt{21609}}{2} = \dfrac{-1 \pm 147}{2}$.
So $n = 73$. (The negative option is not relevant to this question.)

PRACTICE 32: *I ride my bike along a straight stretch of road. The road is* 120 *km long and I ride at a constant speed.*

I then ride my bike back along the same stretch of road, again at a constant speed, but this time 10 *km/hr faster than I did before.*

I was one hour quicker on my return journey.
What were my two riding speeds?



Answer: Following on from the start of the solution in the essay, we have

$$\frac{120}{v+10} = \frac{120}{v} - 1.$$

Let's avoid fractions! Let's multiply through by v,

$$\frac{120v}{v+10} = 120 - v \ ,$$

and then through by $v+10$,

$$120v = 120(v+10) - v(v+10).$$

This simplifies to

$$0 = 1200 - v^2 - 10v,$$

that is, to

$$v^2 + 10v - 1200 = 0.$$

Solving gives

$$v^2 + 10v + 25 = 1225$$
$$(v+5)^2 = 1225$$
$$v+5 = 35 \text{ or } -35$$
$$v = 30 \text{ or } -40$$

It seems only the positive number for a speed is relevant for this problem, so my two speeds were

$v = 30$ km/hr

and

$v + 10 = 40$ m/hr.

PRACTICE 33: *Xavier, in a speed banana-eating contest, ate his first set of six bananas in t seconds, but took 5 seconds longer eating his second set of six bananas. His banana-eating rate was $\frac{6}{t}$ bananas-per-second for his first set. His rate was 0.1 bananas-per-second less for his second set.*

How long did Xavier spend eating his first six bananas?

Answer: We are being told that Xavier's second banana-eating rate, $\frac{6}{t+5}$, is 0.1 less than $\frac{6}{t}$. So

$$\frac{6}{t+5} = \frac{6}{t} - 0.1.$$

Multiplying through by t and then $t+5$ and then 10 eventually gives the equation

$$t^2 + 5t - 300 = 0.$$

This has solutions $t = 15$ or -20. Only the first solution is relevant to the context.

Xavier ate his first six bananas in 15 seconds.

PRACTICE 34: *Solve* $w + 3\sqrt{w} + 2 = 0$.

Answer: The quadratic formula gives

$$\sqrt{w} = \frac{-3 \pm 1}{2} = -1 \text{ or } -2.$$

But $\sqrt{w}$ is a positive quantity, and so no proposed solution is valid. This equation has no solutions.

PRACTICE 35: *Find at least two solutions to* $x^4 - 3x^2 - 4 = 0$.

Answer: The quadratic formula gives

$$x^2 = \frac{3 \pm 5}{2} = 4 \text{ or } -1.$$

As x^2 cannot be negative, we can only entertain $x^2 = 4$, that is, $x = 2$ or $x = -2$. One checks that these are indeed valid solutions to the original equation.

PRACTICE 36: *Find all possible solutions to* $\frac{4}{x^2} + \frac{4}{x} + 1 = 0$.

Answer: This question tacitly assumes that x is not zero. We'll see if this might be an issue later on.

Let's multiply through by x^2 to avoid fractions.

$$4 + 4x + x^2 = 0$$

The quadratic formula gives

$$x = \frac{-4 \pm 0}{2} = -2.$$

There is only one potential solution (and it is non-zero) and one checks that $x = -2$ really is a solution!

PRACTICE 37: *Find all values* k *for which there is a pair of numbers that sum to* 10 *and have product* k *.*

$$sum = 10$$
$$product = k$$

(For instance, we saw that $k = 24$ *and* $k = 25$ *are both possible values for* k *, and* $k = 26$ *is not.)*

Answer: Following the ideas in this essay, we need two numbers p and q with

$$p + q = 10$$
$$pq = k.$$

The first equation gives $q = 10 - p$ so we can read the second equation as

$$p(10 - p) = k$$

that is, as

$$p^2 - 10p + k = 0.$$

The square method has as look at

$$(p-5)^2 = 25 - k,$$

which has solution(s) only if $25 - k \geq 0$, that is, if k is less than or equal to 25 .

PRACTICE 38: *Let* S *be a fixed number. Find, in terms of* S *, the largest possible number* k *for which there is a pair of numbers that sum to* S *and have product* k *.*

$$sum = S$$
$$product = k$$

Answer: Call the two numbers p and q . Then we want

$$p + q = S$$
$$pq = k$$

to have a solution with k as large as possible.

Writing $q = S - p$, the second equation gives

$$p^2 - Sp + k = 0.$$

We want this to have a solution with k as large as possible.

One can use the square method here. But let's use this quadratic formula this time.

The discriminant of this equation is $S^2 - 4k$, and this must be ≥ 0 for this equation to have solutions. That is, we need $k \leq \frac{S^2}{4}$.

The largest possible value of k possible then is $\frac{S^2}{4}$.

PRACTICE 39: *Find the smallest possible value k for which the equation $x + \frac{1}{x} = k$ has a solution for a positive value x.*

Answer: The question wants us to look at the equation $x + \frac{1}{x} = k$ only for positive values of x. It follows then that the value of k we seek will be positive.

Let's multiply the equation through by x to avoid fractions. We need to solve

$$x^2 - kx + 1 = 0.$$

This has a solution if the discriminant is ≥ 0, that is, if

$$k^2 - 4 \geq 0.$$

Since we are considering only positive values of k, this means we need $k \geq 2$.

The smallest value of k that yields a solution to the equation for a positive x is $k = 2$. (And $x = 1$ works in this case.)

PRACTICE 40: *a) Find two numbers that differ by 100 and have product 5069.*

b) When Anu thought about this question she first said to herself: "Symmetry is my friend. A symmetrical solution would have the two numbers the same. So let me represent the numbers by how different they are from being the same." She decided to write the two numbers as $n - 50$ and $n + 50$. How do you think she then proceeded with the problem?

c) *Find two numbers whose sum is* 100 *and whose product is 2491. (What is a "symmetrical" way to set up this problem?)*

Answer:
For a) and b):
Two numbers of the form $n-50$ and $n+50$ differ by 100 and have product n^2-2500.
We thus need $n^2=5069+2500=7569$, giving $n=87$ or -87. Our two numbers could thus be 37 and 137, or -137 and -37.

c) The "symmetrical" answer would have the two numbers be 50 and 50. Let's represent two general numbers then as $50-n$ and $50+n$. (These sum to 100.) Then their product is $2500-n^2$ and we want this to equal 2491. So we need $n^2=9$ giving $n=3$ or -3. Either way, the two numbers are 47 and 53.

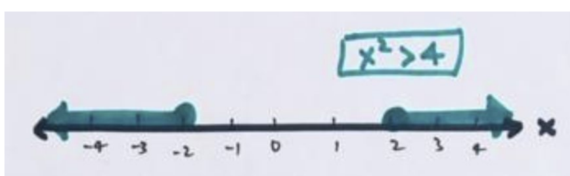
PRACTICE 41:

a) Sketch a graph of the one-variable equation $x^2=4$. *(So it's graph will require only one number line, one for* x *values.)*

b) Sketch a graph of the one-variable inequality $x^2\geq 4$.

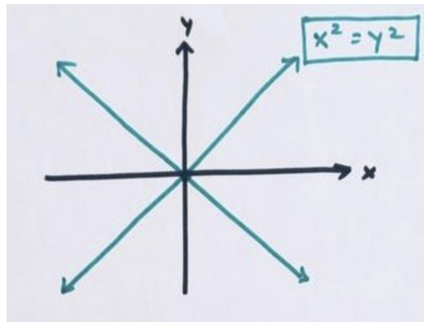
Answer: a) Only $x=2$ and $x=-2$ make $x^2=4$ a true number sentence. A natural way to represent this visually might be as follows.

b) Any value greater than or equal to 2, or less than or equal to -2, makes $x^2\geq 4$ a true number sentence. One might represent this visually as shown.



PRACTICE 42: *Sketch a graph of the two-variable equation* $x^2 = y^2$.

Answer: One gets and X-shaped graph.



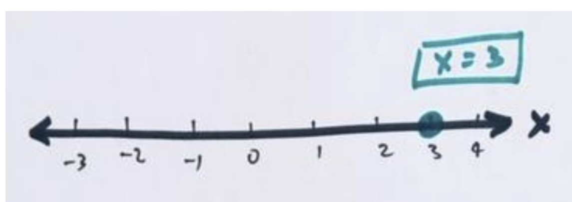
PRACTICE 43:

a) *Sketch a graph of the one-variable equation* $x = 3$.

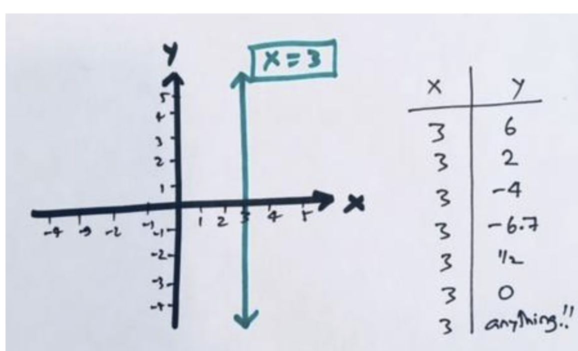
b) *Sketch a graph of* $x = 3$ *thinking of it as a two-variable equation. (Imagine it as* $x + 0 \cdot y = 3$ *if you like.)*

c) *Sketch a graph of* $x = 3$ *thinking of it as a three-variable equation. (Imagine it as* $x + 0 \cdot y + 0 \cdot z = 3$ *if you like.) How will you draw your three number lines?*

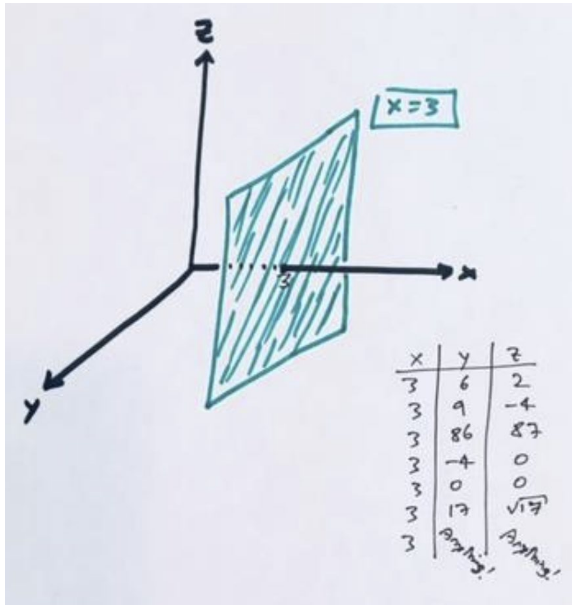
Answer:
a) There is only one value for x that makes "$x = 3$" a true sentence, namely, x being 3.



b)



c) Draw three mutually perpendicular number lines. The graph is an entire plane of points.



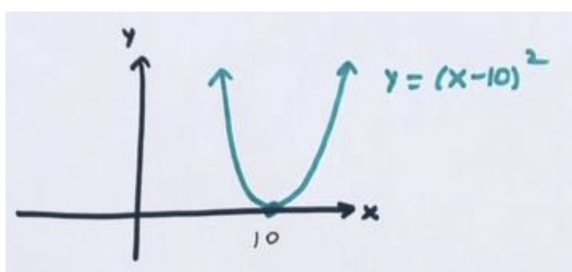
PRACTICE 44:

a) Sketch the graph of $y=(x-10)^2$.

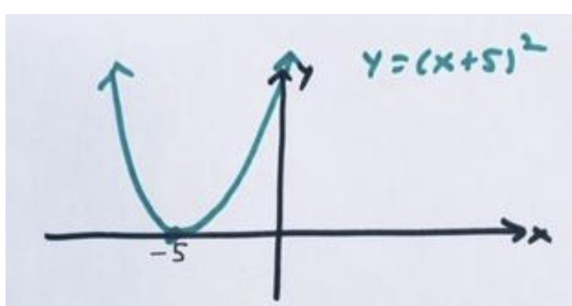
b) Sketch the graph of $y=(x+5)^2$.

Answer:

a) We see $x=10$ is behaving like zero.



b) We see $x=-5$ is behaving like zero.

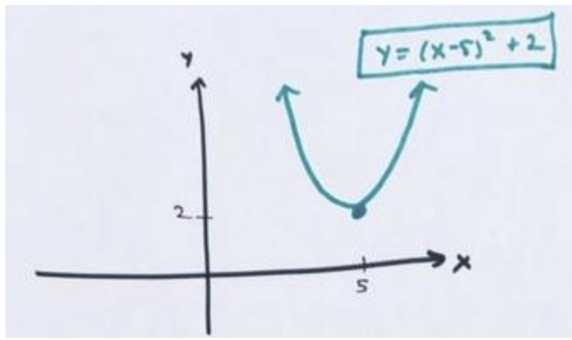


PRACTICE 45:

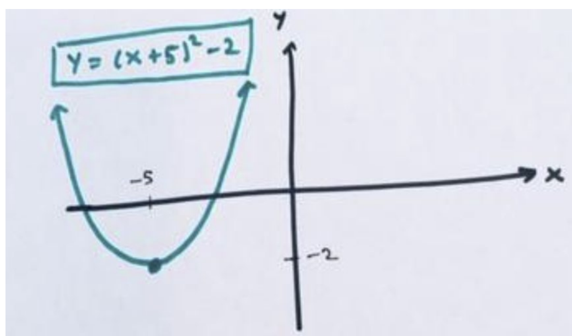
a) *Sketch the graph of* $y = (x-5)^2 + 2$.

b) *Sketch the graph of* $y = (x+5)^2 - 2$.

Answer:
a) We see $x = 5$ is behaving like zero and everything is shifted upwards 2 units.



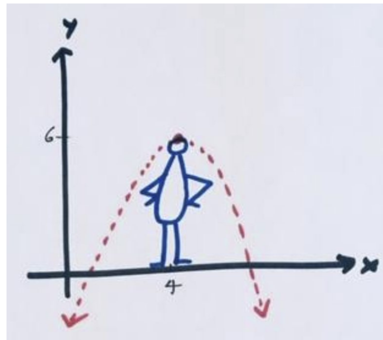
b) We see $x = -5$ is behaving like zero and everything is shifted upwards -2 units, that is, down two units.



PRACTICE 46: *When we say that the graph of* $y = x^2$ *is a U-shaped graph, is that a correct analogy? The two sides of the letter "U" are vertical. Does the graph of* $y = x^2$ *possess vertical lines?*

Answer: The graph does not possess vertical lines. For instance, if the graph became vertical at say $x = 100$, then the graph never extends to x-values beyond 100, which is absurd, because we know for example that $x = 101$, $y = 10201$ is a data point to be plotted.

PRACTICE 47: *Find three different equations that give U-shaped graphs that balance on my head this way.*



Answer: We need to take the equation $y=-x^2$ and adjust it so that $x=4$ behaves like zero and all data points are shifted 6 units higher. $y=-(x-4)^2+6$ works.

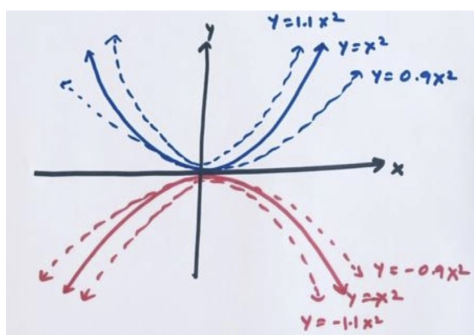
Actually, $y=-2(x-4)^2+6$, $y=-\frac{1}{3}(x-4)^2+6$, and $y=-7(x-4)^2+6$ work too.

PRACTICE 48: *Draw, on the same sets of axes, rough sketches of each the following equations.*

$y=x^2$ $\quad y=1.1x^2$ $\quad y=0.9x^2$

$y=-x^2$ $\quad y=-1.1x^2$ $\quad y=-0.9x^2$

Answer: Roughly, we get:



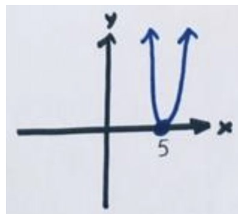
PRACTICE 49: *Sketch graphs of*

a) $y = 3(x-5)^2$

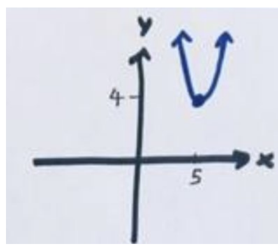
b) $y = 3(x-5)^2 + 4$

c) $y = -2(x+4)^2 + 40$.

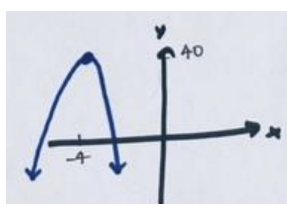
Answer: a) A steep U-shaped graph shifted so that $x = 5$ behaves like zero.



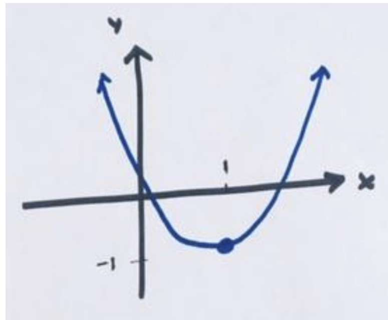
b) The same as the previous graph, except all data points are 4 units higher.



c) This is a steep upside-down U graph, with $x = -4$ behaving like zero, and all data points 40 units up.



PRACTICE 50: *Which of the following equations could have the graph shown?*



a) $y=\frac{4}{3}(x+1)^2+1$

b) $y=-\frac{4}{3}(x+1)^2+1$

c) $y=\frac{4}{3}(x-1)^2+1$

d) $y=-\frac{4}{3}(x-1)^2+1$

e) $y=\frac{4}{3}(x+1)^2-1$

f) $y=-\frac{4}{3}(x+1)^2-1$

g) $y=\frac{4}{3}(x-1)^2-1$

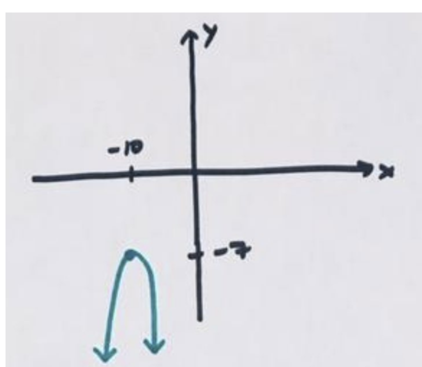
h) $y=-\frac{4}{3}(x-1)^2-1$

Answer: This is an upward-facing U-shaped graph with $x=1$ behaving like zero and all data points shifted down 1. Only option g) can work.

PRACTICE 51: *Sketch the graph of*

$$y=-2(x+10)^2-7\,.$$

Answer: We see $x=-10$ is behaving like zero, with a steepness factor of -2 at play (so the graph will be a steep downward-facing U-shape), with all data values shifted down 7 units.



PRACTICE 52: *The graph of a quadratic equation has a vertical line of symmetry at* $x = 3$*, and has highest value* $y = 17$*. Which of the following could be an equation for that quadratic?*

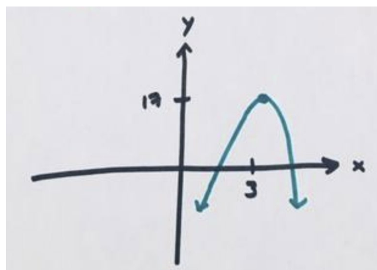
a) $y = 200(x-3)^2 + 17$

b) $y = -200(x-3)^2 + 17$

c) $y = 200(x-3)^2 - 17$

d) $y = -200(x-3)^2 - 17$

Answer: We must have a graph like this:



Only option b) can work.

PRACTICE 53: *Sketch a graph for each of the following equations.*

a) $y = 2 - x^2$

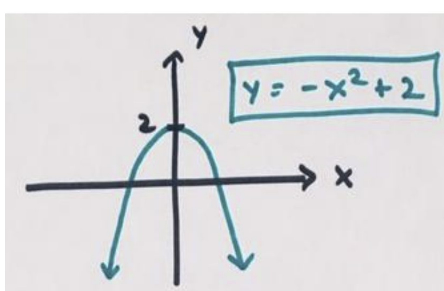
b) $y = \frac{1}{3}\left(x - \frac{1}{2}\right)^2 - 4$

c) $y = 0.0034(x + 0.276)^2 + 0.778$

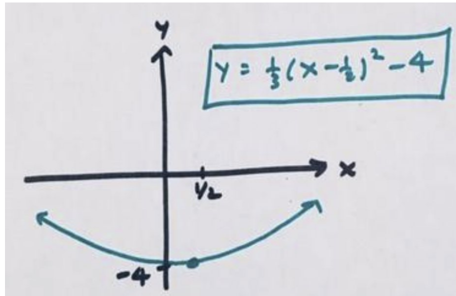
d) $y = 200000(x - 200000)^2 - 200000$

Answer:

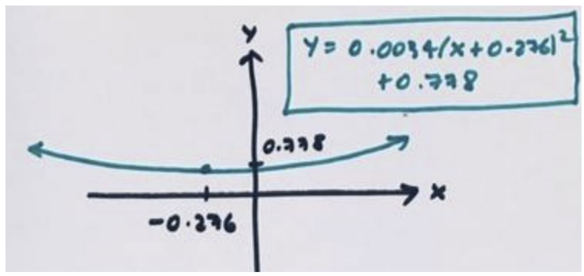
a)



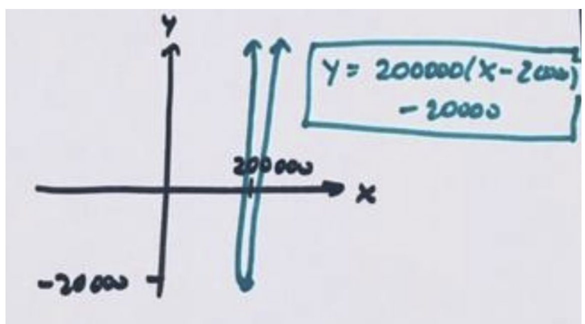
b)



c)



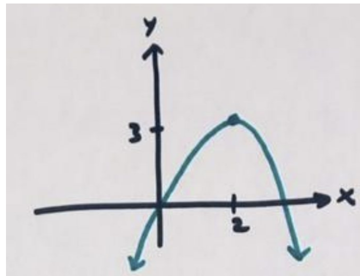
d)



PRACTICE 54: *If* $y = a(x+b)^2 + c$ *has a graph passing through the origin and with* $(2,3)$ *as the vertex, then what is the value of* $a+b+c$*?*

a) $\frac{1}{4}$ b) $1\frac{3}{4}$ c) $4\frac{1}{4}$ d) $5\frac{1}{4}$

Answer: The graph must look like this:



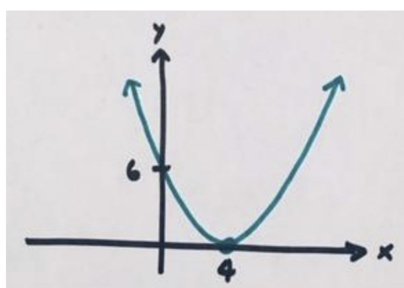
So we see the equation must be of the form $y = a(x-2)^2 + 3$, and so $b = -2$ and $c = 3$.

The graph passes through $x = 0$, $y = 0$ and so

$$0 = a(-2)^2 + 3$$

must be a true sentence about numbers. This forces $a = -\frac{3}{4}$ and so $a + b + c = \frac{1}{4}$, option a).

PRACTICE 55: *Write a quadratic equation that fits this graph.*



Answer: We see that it is a graph basically coming from $y = x^2$ but with $x = 4$ behaving like zero. So we can try

$$y = (x-4)^2.$$

But we see this is not right: when $x = 0$ we get $y = 16$, not 6. We are missing a steepness factor!

Try $y = a(x-4)^2$.

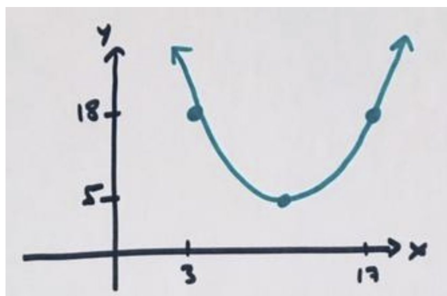
Now $x = 0$, $y = 6$ should yield a true number sentence, so $6 = 16a$ forcing $a = \frac{3}{8}$.

We have

$$y = \frac{3}{8}(x-4)^2 .$$

PRACTICE 56: *Write down a quadratic equation whose graph passes through the points* $(3,18)$ *and* $(17,18)$ *and has lowest value* 5*.*

Answer: We must have a graph that looks like this:



So let's just use common sense to figure things out.

We want a symmetrical graph and so the line of symmetry must be at $x=10$, midway between 3 and 17. So the quadratic equation producing this graph must have the form

$$y = a(x-10)^2 + 5$$

for some steepness a.

When $x=3$ we should have $y=18$, showing that

$$18 = a(-7)^2 + 5$$

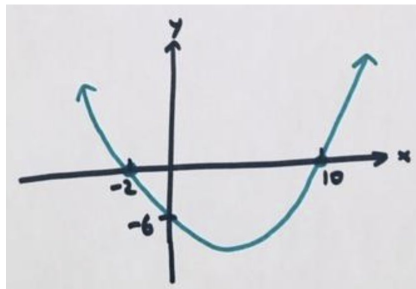
should be a true number sentence. This forces

$$a = \frac{13}{49}.$$

So $y = \frac{13}{49}(x-10)^2 + 5$ works.

PRACTICE 57: *Write down a quadratic equation whose graph passes through the* x *axis at* $x=-2$ *and at* $x=10$ *and passes through the* y *axis at* $y=-6$.

Answer: We have this picture.



The equation must be of the form

$$y=a(x-4)^2+b$$

When $x=10$, $y=0$ and so

$$0=36a+b$$

must be a true number sentence. We have $b=-36a$.

When $x=0$, $y=-6$ and so

$$-6=16a+b$$

must also be a true number sentence. We see that

$$-6=16a-36a$$

showing that $a=\dfrac{3}{10}$. Consequently $b=-\dfrac{54}{5}$.

The equation we need is

$$y=\frac{3}{10}(x-4)^2-\frac{54}{5}.$$

PRACTICE 58: *Write down quadratic equations with symmetrical U-shaped graphs possessing the following properties:*

a) *Crosses the* x*-axis at* 3 *and* 5 *and the* y*-axis at* 1000.

b) *Passes through* $(4,10)$, $(6,10)$ *and* $(8,13)$.

c) *Has vertex* $(5,5)$ *and passes through* $(4,4)$.

d) *Has vertex the origin and passes through the point* $\left(\sqrt{2},\pi\right)$.

Brief Answers: Sketch a picture of each scenario and use logic to determine which x-value is behaving like zero in each case. Then go from there!

a) $y=\frac{200}{3}(x-4)^2-\frac{200}{3}$

b) $y=\frac{3}{8}(x-5)^2+\frac{77}{8}$

c) $y=-(x-5)^2+5$

d) $y=\frac{\pi}{2}x^2$

PRACTICE 59: *Use algebra to prove that the graph of* $y=x^2-6x+10$ *is sure to be a symmetrical U-shaped graph.*

Answer: Following the quadrus method we have

$$y-1=x^2-6x+9$$

$$y-1=(x-3)^2$$

and so

$$y=(x-3)^2+1.$$

The graph of $y=x^2-6x+10$ is thus the graph of $y=x^2$ with $x=3$ behaving like zero and with all heights shifted 1 higher. It is thus the same symmetrical U-shape!

PRACTICE 60: *Use algebra to prove that the graph of* $y=x^2+8x-7$ *is sure to be a symmetrical U-shaped graph.*

Brief Answer: We get $y+23=(x+4)^2$ and so

$$y=(x+4)^2-23.$$

We get the same symmetrical U-shaped graph just shifted in the plane.

PRACTICE 61 (OPTIONAL): *Show that* $y = 3x^2 + 5x + 1$ *can be rewritten as* $y = 3\left(x + \frac{5}{6}\right)^2 - \frac{13}{12}$*, and so is sure to have a symmetrical U-shaped graph.*

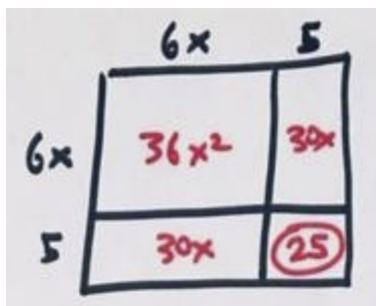
Answer: Let's follow the quadrus method. First multiply three by 3 to create a perfect square up front.

$$3y = 9x^2 + 15x + 3$$

Now multiply though by 4 to make an even middle term.

$$12y = 36x^2 + 60x + 12$$

Draw the square.



$$12y + 13 = (6x + 5)^2$$

Pause here as we have $6x$ in the parentheses, and not just x. Since $(6x+5) = 6\left(x + \frac{5}{6}\right)$ we can rewrite matters as

$$12y + 13 = 36\left(x + \frac{5}{6}\right)^2 .$$

So

$$12y = 36\left(x + \frac{5}{6}\right)^2 - 13$$

giving

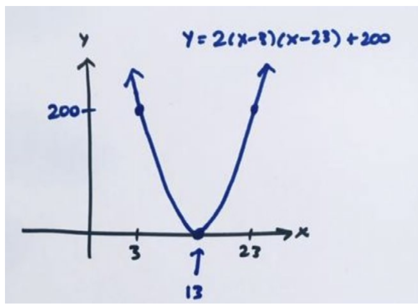
$$y = 3\left(x + \frac{5}{6}\right)^2 - \frac{13}{12} .$$

PRACTICE 62: *The graph of a quadratic equation passes through the points* $(3,81)$*,* $(4,9)$*, and* $(-10,9)$*. What is the* x*-coordinate of its vertex?*

Answer: We see that $(4,9)$ and $(-10,9)$ are two symmetrical points on a symmetrical graph, and so its line of symmetry must be halfway between $x=4$ and $x=-10$, namely, at $x=-3$. And $x=-3$ must be the x-coordinate of the vertex.

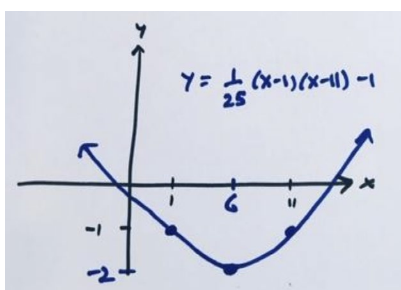
PRACTICE 63: *Sketch a graph of* $y=2(x-3)(x-23)+200$.

Answer:



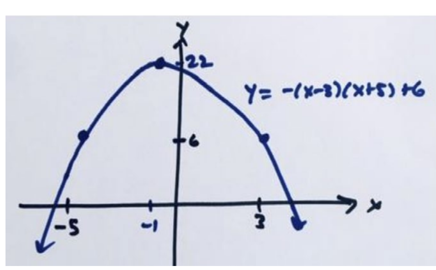
PRACTICE 64: *Sketch a graph of* $y=\frac{1}{25}(x-1)(x-11)-1$.

Answer:



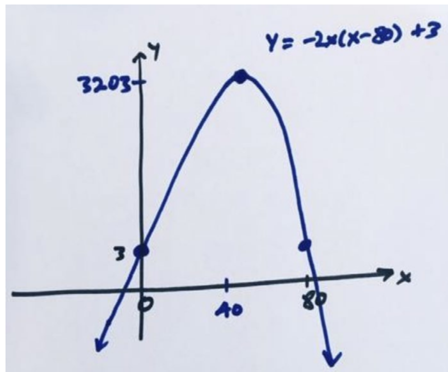
PRACTICE 65: *Sketch a graph of* $y=-(x-3)(x+5)+6$.

Answer:



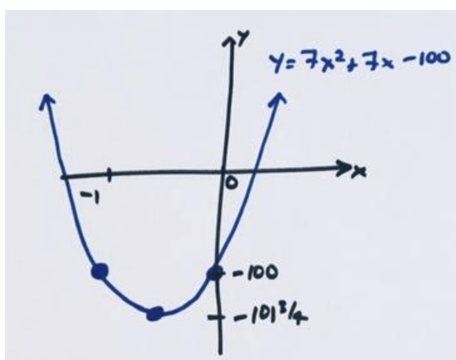
PRACTICE 66: *Sketch a graph of* $y = -2x(x-80)+3$.

Answer:



PRACTICE 67: *Sketch a graph of* $y = 7x^2 + 7x - 100$.

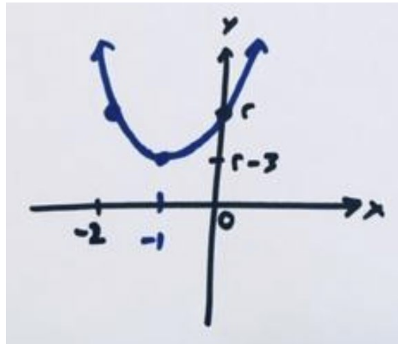
Answer: We have $y = 7x(x+1) - 100$ showing that both $x = 0$ and $x = -1$ are interesting values.



PRACTICE 68: *Which value of* r *forces the graph of* $y = 3x^2 + 6x + r$ *have* 5 *as the smallest possible* y *value?*

Answer: We have $y = 3x(x+2) + r$ showing that the line of symmetry is halfway between $x = 0$ and $x = -2$, namely, at $x = -1$. The vertex is on this line.

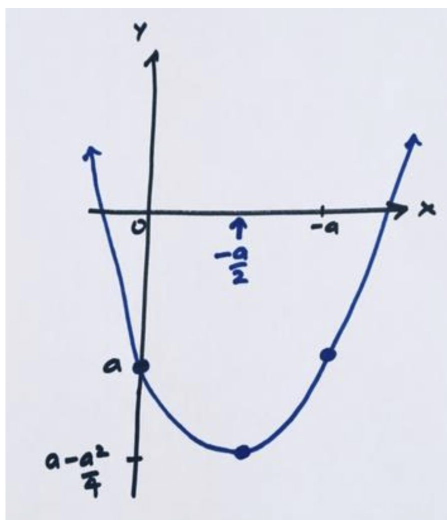
At $x = -1$ we have $y = 3 - 6 + r = r - 3$.



We must have $r-3=5$ giving $r=8$.

PRACTICE 69: *Find a negative value for* a *so that* $y=x^2+ax+a$ *has smallest possible value* -3.

Answer: We have $y=x(x+a)+a$.



(The picture assumes a is negative.)

We see that the line of symmetry is at $-\dfrac{a}{2}$ and for $x=-a/2$, $y=\left(-\dfrac{a}{2}\right)\left(\dfrac{a}{2}\right)+a=a-\dfrac{a^2}{4}$

We see we need

$$a-\frac{a^2}{4}=-3.$$

That is, we need $a^2-4a-12=0$.

Solving

$$(a-2)^2=16$$

$$a-2=4 \text{ or } -4$$

$a = 6$ or -2.

Choose $a = -2$.

PRACTICE 70: *Find a formula for the location of the line of symmetry of a general quadratic equation* $y = ax^2 + bx + c$.

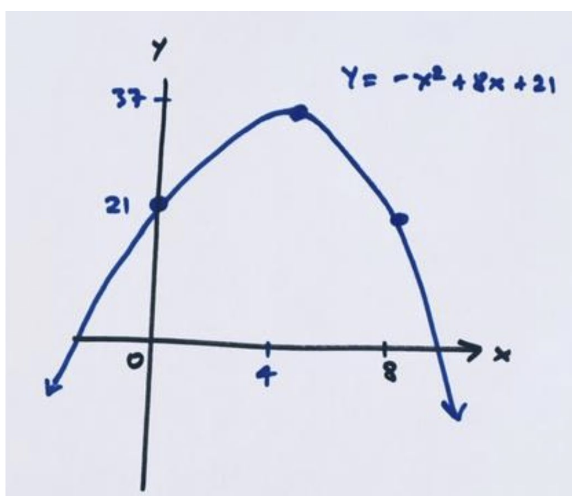
Answer: We have $y = x(ax + b) + c$ which shows

$$x = 0 \quad \to \quad y = c$$
$$x = -\frac{b}{a} \quad \to \quad y = c$$

The line of symmetry is halfway between $x = 0$ and $x = -\dfrac{b}{a}$, which is at $x = -\dfrac{b}{2a}$.

PRACTICE 71: *Sketch the graph of* $y = -x^2 + 8x + 21$. *What is the largest possible* y *value this equation can produce? What is the vertex of this graph? Make a guess as to what it means to rewrite* $y = -x^2 + 8x + 21$ *in "vertex form."*

Answer: We have $y = x(-x + 8) + 21$, with line of symmetry at $x = 4$.



The largest value y value occurs at $x = 4$ with $y = 4(4) + 21 = 37$. The vertex is $(4, 37)$.

"Vertex form" of the equation probably means a form of the equation that makes the vertex clear in the equation. Going back to last lecture, we see this graph as the $y = x^2$ shifted in the plane with negative steepness. We have

$$y = a(x-4)^2 + 37 .$$

When $x = 0$ we should have $y = 21$. This gives

$$21 = 16a + 37$$

Showing that $a = -1$.

Thus

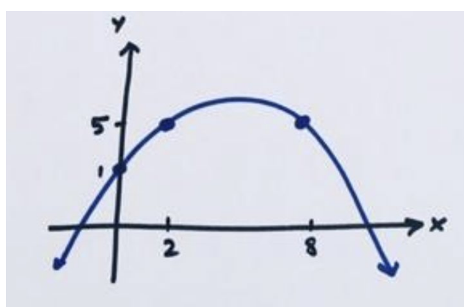
$$\begin{aligned} y &= -x^2 + 8x + 21 \\ &= -(x-4)^2 + 37 \end{aligned}$$

Note: We could have deduced the steepness was going to be $a = -1$ by looking at the coefficient of the x^2 term in the original equation. The x^2 terms here

$$-x^2 + 8x + 21 = a(x-4)^2 + 37$$

will match only for $a = -1$.

PRACTICE 72: *Find the quadratic equation whose graph appears as shown.*



Answer: The picture suggests an equation of the form $y = a(x-2)(x-8) + 5$. When $x = 0$ we should have $y = 1$, so we need

$$1 = a(-2)(-8) + 5$$

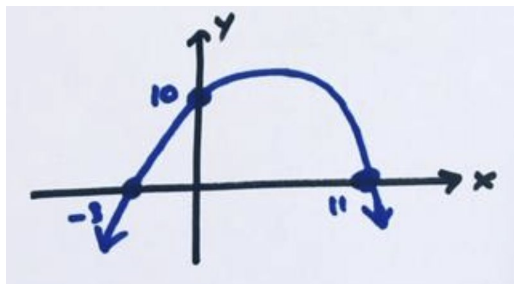
giving $a = -\frac{1}{4}$. So the equation is

$$y = -\frac{1}{4}(x-2)(x-8) + 5 .$$

PRACTICE 73: *Write down a quadratic equation whose graph has* x *intercepts* $x = -3$ *and* $x = 11$ *and* y *intercept* 10.

Answer: The sketch suggests the equation

$y = a(x+3)(x-11) + 0$.

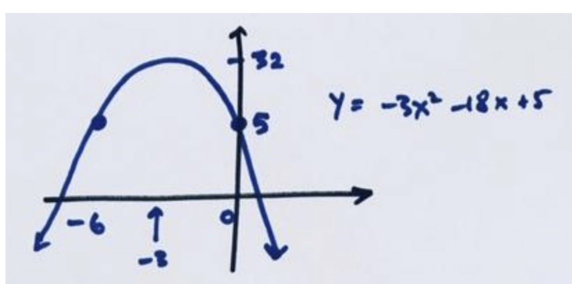


When $x = 0$ we need $y = 10$ giving $a = -\frac{10}{33}$.
The equation is

$$y = -\frac{10}{33}(x+3)(x-11).$$

PRACTICE 74: *Sketch the graph of* $y = -3x^2 - 18x + 5$ *and then use the graph to rewrite the equation in "vertex form."*

Answer: We have $y = -3x(x+6) + 5$ showing the line of symmetry is at $x = -3$. For this x value, we have $y = -3(-3)(3) + 5 = 32$. The graph thus appears



In vertex form, we must have

$$y = a(x+3)^2 + 32$$

with $a = -3$ to yield a "$-3x^2$" term when expanded. So the vertex form of the equation is

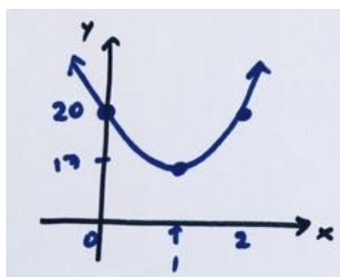
$$y = -3(x+3)^2 + 32.$$

PRACTICE 75: *Consider the equation* $y = 3x^2 - 6x + 20$.

a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x *intercepts of the graph.*
f) Find the y *intercept of the graph.*

Brief Answer:

a) $(1,17)$
b) At $x = 1$
c) $y = 3(x-1)^2 + 17$
d)



e) There are none
f) $(0,20)$.

PRACTICE 76: *Consider the equation* $y = 5x^2 - 10x$.

a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x *intercepts of the graph.*
f) Find the y *intercept of the graph.*

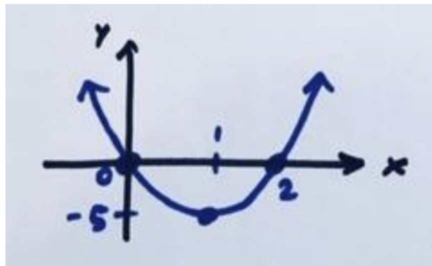
Brief Answer:

a) $(1,-5)$

b) At $x=1$

c) $y-=5(x-1)^2-5$

d)



e) $(0,0)$ and $(2,0)$

f) $(0,0)$

PRACTICE 77: *Solve the following quadratic equations.*

a) $(x-3)(x+5)=1$

b) $x^2=(2x-1)(2x+1)-5$

c) $(x-10)(x+1)+5=12$

Brief Answers:

a) Rewrite as $x^2+2x-15=1$. Solving gives $x=-1+\sqrt{17}$ or $x=-1=\sqrt{17}$.

b) Rewrite as $x^2=4x^2-1-5$, that is, as $x^2=2$ with solutions $x=\sqrt{2}$ or $x=-\sqrt{2}$.

c) Rewrite as $x^2-9x-5=12$ which has solutions $x=\dfrac{9\pm\sqrt{13}}{2}$.

PRACTICE 78: *Find, in terms of c, the value k so that*

$$y=(x+c)(x-c)+k$$

gives -2 as the smallest possible y value.

Brief Answer: We have that $x=c$ and $x=-c$ give symmetrical points on a symmetrical graph. The line of symmetry is thus at $x=0$, and this is where the vertex lies.

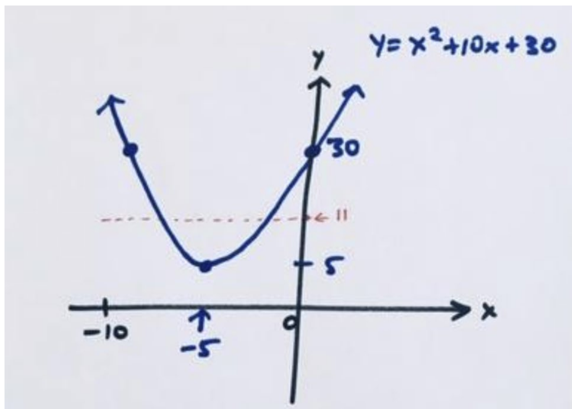
At $x=0$, $y=k-c^2$. We want this to equal -2 and so we must have

$$k=c^2-2.$$

PRACTICE 79:

a) Solve $x^2 + 10x + 30 = 0$ *and see what happens when you try.*
b) Sketch the graph of $y = x^2 + 10x + 30$.
c) Use the graph to explain what happened in part a)
d) Will $x^2 + 10x + 30 = 11$ *have a solution? If so, how many solutions?*
e) For which value(s) k *does* $x^2 + 10x + 30 = k$ *have only one solution.*

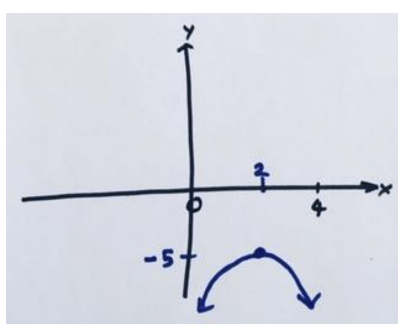
Brief Answers:
a) We have $(x+5)^2 = -5$. There are no solutions.
b)



c) We see from the graph that there are no x values that give a y value of zero.
d) We see from the graph the there are two x values that give $y = 11$.
e) We see from the graph that the answer is $k = 5$.

PRACTICE 80: *How many solutions does* $-x^2 + 4x - 5 = 0$ *have? Answer this question not by algebra, but by graphing.*

Brief Answer: We see from the graph of $y = -x^2 + 4x - 5$ that there are no solutions to $0 = -x^2 + 4x - 5$.



PRACTICE 81:

a) *Find a value* k *so that the graph of*

$$y = 5x^2 - 10x + k$$

just touches the x *axis.*

b) Find a value m *so that* $y = -2x^2 - 18x + m$ *gives the highest output value of* 100*.*

c) Find a value p *so that* $y = (x-p)(x-3p)$ *has smallest output value of* -10*.*

Brief Answers:

a) Writing $y = 5x(x-2)+k$ shows that $x = 1$ is the line of symmetry. We need the vertex of the graph of this equation to have height zero, so we need $0 = 5(1)(-1)+k$ meaning we need $k = 5$.

b) Write $y = -2x(x+9)+m$. We need $\left(-\frac{9}{2}, 100\right)$ to be the vertex, so we need

$-2\left(-\frac{9}{2}\right)\left(\frac{9}{2}\right)+m = 100$ giving $m = \frac{119}{2}$.

c) The line of symmetry of the equation's graph is $x = 2p$. We need $(p)(-p) = -10$ so we need $p = \sqrt{10}$ or $-\sqrt{10}$.

PRACTICE 82: *Find, in terms of* r*, a value* k *so that the graph of* $y = 3x(2r-x)-k$ *just touches the* x *axis.*

Brief Answer: The line of symmetry is at $x = r$. We need $3(r)(r)-k = 0$ giving $k = 3r^2$.

PRACTICE 83: *Consider the equation* $y = 2(x-1)(x+1)+(x-3)(x+4)+x(x-2)$

a) Is this an equation in the form $y = ax^2+bx+c$ *in disguise for some numbers* a*,* b*, and* c*? (This is just a YES/NO question!)*

b) Explain how one readily sees $a = 2+1+1 = 4$*.*

c) Explain why putting $x = 0$ *into the right side of the equation allows us to conclude that* $c = -14$*.*

d) What is the value of b*?*

Brief Answer: a) YES. If we expand each product and collect like terms, we'd have an expression of the form given.

b) Expanding the products shows that

$$\begin{aligned} y &= 2x^2 + stuff + x^2 + stuff + x^2 + stuff \\ &= 4x^2 + stuff \end{aligned}$$

We see $a = 4$.

c) Putting $x = 0$ into $y = ax^2 + bx + c$ gives $y = c$. Putting $x = 0$ into the original expression gives $y = 2(-1)(1) + (-3)(4) + 0 = -14$.
Thus $c = -14$.

d) We have so far

$$2(x-1)(x+1) + (x-3)(x+4) + x(x-2)$$
$$= 4x^2 + bx - 14$$

Let's put in another value for x. And let's have the first product vanish by choosing $x = 1$.

$$0 + (-2)(5) + 1(-1) = 4 + b - 14$$

We see $b = -1$.

PRACTICE 84: *A rectangle has side lengths $7 - r$ and $3 + r$ for some value r. What value for r gives a rectangle of maximal area?*

Brief Answer: The area is $A = (7 - r)(3 + r)$. This is a quadratic equation with $r = 7$ and $r = -3$ both interesting. The line of symmetry is at $r = 2$ and this is where the vertex of its graph lies. As this is a downward-facing quadratic, $r = 2$ gives the maximal value.

PRACTICE 85: *Here are three quadratic equations:*

(A) $y = 3(x-3)(x+5)$

(B) $y = 2x^2 + 6x + 8$

(C) $y = 2(x-4)^2 + 7$

i) For which of these three expressions is it very easy to answer the question: "What is the smallest y-value the expression can produce?"

ii) For which of these three expressions is it very easy to answer the question: "Where does the graph of the equation cross the x-axis?"

iii) For which of these three expressions is it very easy to answer the question: "Where does the graph of the quadratic cross the y axis?"

iv) For which of these three expressions is it very easy to answer the question: "What are the coordinates of the vertex in this equation's graph?"

Answers: These answers are subjective.

i) (C) might be the easiest. We can see the smallest value of 7 occurs for $x = 4$.

ii) (A) It crosses at $x = 3$ and $x = -5$.

iii) (B) Putting in $x = 0$ gives $y = 8$.

iv) (C) The vertex is $(4,7)$.

PRACTICE 86: Here is a silly question.

I make and sell BIPS. It costs me $80,000 + 200n$ rupees to make n BIPS.
If I sell n BIPS I bring in $2n(600-n)$ rupees.

a) Ignoring costs, what number of BIPS sold brings in the largest number of rupees?

b) What number of BIPS should I sell to make the biggest profit?

Brief Answers:

a) $2n(600-n)$ is a quadratic expression with $n = 0$ and $n = 600$ both giving a value of zero. The maximum value thus occurs for $n = 300$.

b) My profit is

$$\begin{aligned}&2n(600-n)-(80000+200n)\\&=-2n^2+1000n-80000\\&=-2n(n-500)-80000.\end{aligned}$$

We see $n=0$ and $n=500$ give symmetrical outputs and so the maximal profit occurs for $n=250$.

Practice 87: *Convince yourself that the formula*

$$8\frac{(x-10)(x-20)}{(-4)(-14)}+122\frac{(x-6)(x-20)}{(4)(-10)}+4600\frac{(x-6)(x-10)}{(14)(10)}$$

gives the value 8 *for* $x=6$, *the value* 122 *for* $x=10$, *and the value* 4600 *for* $x=20$.

Brief Answer: Did you verify this?

Practice 88: *Find a formula that gives the value* 9000 *for* $x=3$, *the value* -45 *for* $x=5$, *and the value* $\frac{2}{3}$ *for* $x=8$. *(Don't bother simplifying your formula.)*

Answer:

$$y=9000\frac{(x-5)(x-8)}{(-2)(-6)}-45\frac{(x-3)(x-8)}{(2)(-3)}+\frac{2}{3}\cdot\frac{(x-3)(x-5)}{(5)(3)}.$$

Practice 89: *Write down a formula that fits this data set.*

x	1	2	7	15
y	5	10	8	-3

Answer:

$$y=5\frac{(x-2)(x-7)(x-15)}{(-1)(-6)(-14)}+10\frac{(x-1)(x-7)(x-15)}{(1)(-5)(-13)}$$
$$+8\frac{(x-1)(x-2)(x-15)}{(6)(5)(-8)}-3\frac{(x-1)(x-2)(x-7)}{(14)(13)(8)}$$

Practice 90: *Use Lagrange Interpolation to find a formula that spells JST. Then do the necessary work to show that it simplifies to* $y=-4x^2+21x-7$.

Answer: It's $y = 10\frac{(x-2)(x-3)}{(-1)(-2)} + 19\frac{(x-1)(x-3)}{(1)(-1)} + 20\frac{(x-1)(x-2)}{(2)(1)}$ which does simplify as shown.

Practice 91: a) *Can you see that* $y = 7\frac{(x-4)(x-5)}{(-5)(-6)} + \frac{(x+1)(x-5)}{(5)(-1)} + 10\frac{(x-1)(x-4)}{(4)(1)}$ *is a quadratic equation in disguise? Verify that the graph of this equation passes through the data points* $(-1,7)$, $(4,1)$, *and* $(5,10)$.

b) *Find a quadratic formula that fits the data* $(2,5)$, $(-1,6)$, *and* $(5,46)$ *and make your answer look as friendly as possible.*

x	**2**	**-1**	**5**
y	**5**	**6**	**46**

c) *Find a quadratic formula that fits the data* $(2,5)$, $(3,8)$, *and* $(5,14)$ *and make your answer look as friendly as possible. Explain what happens!*

d) *There is no quadratic formula* $y = ax^2 + bx + c$ *that fits the data* $(2,5)$, $(10,6)$, *and* $(10,100)$. *(The same input value of* $x = 10$ *cannot give two different output values.) So then, how does Lagrange's Interpolation method fail when you try to use it?*

Answer. a) We have $y = \frac{7}{30}(x-4)(x-5) - \frac{1}{5}(x+1)(x-5) + \frac{10}{4}(x-1)(x-4)$. If we were to expand the product in each term, we see that we would obtain and expression of the form $y = Ax^2 + Bx + C$.

One can see that substituting $x = -1$ into the original equation really does give $y = 7$, and so on.

b) We get $y = 5\frac{(x+1)(x-5)}{(3)(-3)} + 6\frac{(x-2)(x-5)}{(-3)(-6)} + 46\frac{(x-2)(x+1)}{(3)(6)}$, which simplifies to

$$y = -\frac{5}{9}\left(x^2 - 4x - 5\right) + \frac{1}{3}\left(x^2 - 7x + 10\right) + \frac{23}{9}\left(x^2 - x - 2\right)$$
$$= \left(-\frac{5}{9} + \frac{1}{3} + \frac{23}{9}\right)x^2 + \left(\frac{20}{9} - \frac{7}{3} - \frac{23}{9}\right)x + \left(\frac{25}{9} + \frac{10}{3} - \frac{46}{9}\right) \quad = \frac{7}{3}x^2 - \frac{8}{3}x + 1$$

c) We get an expression that simplifies to $y = 3x - 1$. The data turns out to be linear.

d) You find that the Lagrange's Interpolation formula has a denominator equal to zero in one of its terms. It breaks down.

Practice 92: *Use Lagrange's Interpolation Formula to find the equation of the line between two points* (p,m) *and* (q,n) *with* $p \neq q$. *Is your equation equivalent to "* $y = mx + b$ *"where* m *is the slope of the line segment between the two points?*

Answer: We have $y = m\dfrac{(x-q)}{(p-q)} + n\dfrac{(x-p)}{(q-p)}$, which simplifies to

$$y = \frac{m(x-q)-n(x-p)}{p-q} = \frac{m-n}{p-q}x + \frac{np-mq}{p-q}$$

And $\dfrac{m-n}{p-q}$, the coefficient of x in this equation, is indeed the slope of the line.