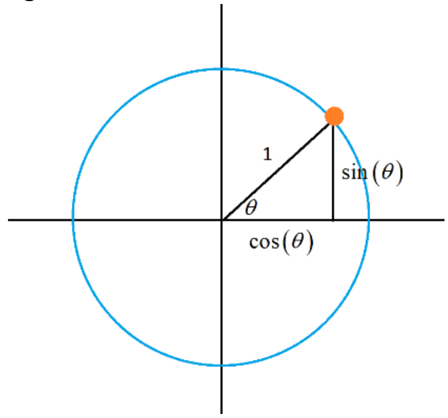
INVERSE TRIGONOMETRIC FUNCTIONS

We can think of the act of computing the sine of an angle as a function operation. "Sine" assigns to each angle θ the height of the Sun at angle of elevation θ on a circle of radius 1.



Similarly, "cosine" is a function that assigns to each real number, an angle measure θ, a real number between -1 and 1, the "overness" of the Sun at that angle of elevation θ.

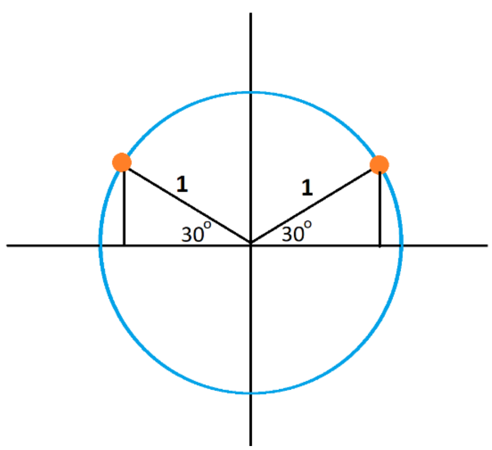
To each input, one and only one output value is assigned.

> **Question 1:** Can "tangent" in trigonometry also be thought of as a function? If so, what are the allowable inputs (the *domain*)? What possible outputs could you see (the *range*)?

It's fun to wonder if we can go backwards.

> Given a height value first, can we determine which angle has that sine value?
> Given an overness value first, can we determine which angle has that cosine value?

The trouble here, of course, that there that are many different angle of elevation values that yield a Sun position of the same height. For example, for a height of $h = 0.5$, we see that $\sin\left(30^\circ\right)$, $\sin\left(150^\circ\right)$, and $\sin\left(-210^\circ\right)$, for example, all have value h. (Make sure you truly understand this.)



Question 2: Find **seven** different angles θ for which $\cos(\theta) = -0.5$.

Question 3: Find **three** different angles θ for which $\tan(\theta) = -1$.

There are an infinitude of angle measures θ for which $\sin(\theta) = 0.5$. They are

30° and 150°, and these two angles adjusted by positive or negative integer multiples of 360° (to give $30 + 360 = 390^\circ$ and $30 - 720 = -690^\circ$ and $150 + 3600 = 3750^\circ$, and such).

People might write

"$30^\circ + k360^\circ$ and $150^\circ + k360^\circ$ for $k \in \mathbb{Z}$,"

for instance.

Question 4: Write down all the angle measures θ for which $\cos(\theta) = -\frac{\sqrt{3}}{2}$.

Thus, in order to "go backwards," we need to be willing to work with many angle measures and decide, in context, which of the infinitely many possibilities apply to the scenario being considered.

Question 5: Which angle measure $180^\circ \le \theta \le 360^\circ$ has $\tan(\theta) = -\sqrt{3}$?

Definition: For x a real number define

$\arcsin(x)$ to be the set of all angles with sine value x.

$\arccos(x)$ to be the set of all angles with cosine value x.

$\arctan(x)$ to be the set of all angles with tangent value x.

We have, for example,

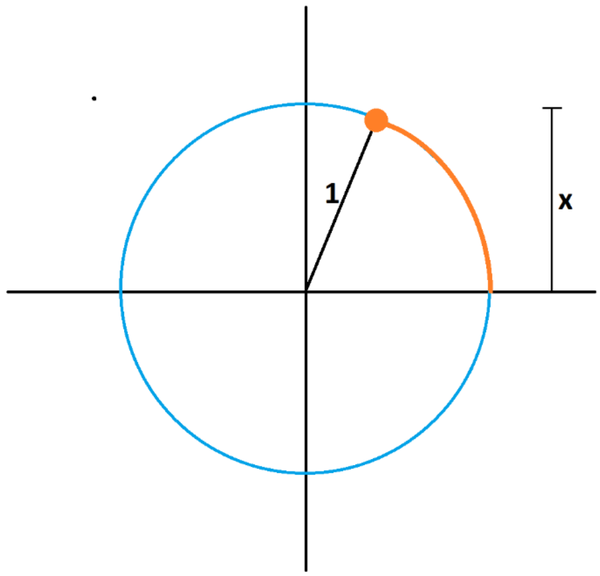
$$\begin{aligned}\arcsin(0.5) = \{&30^\circ \pm 360^\circ, 30^\circ \pm 720^\circ, 30^\circ \pm 1080^\circ, \ldots, \\ &150^\circ \pm 360^\circ, 150^\circ \pm 720^\circ, 150^\circ \pm 1080^\circ, \ldots\}\end{aligned}.$$

Question 6:

a) Describe the set of angles $\arcsin(x)$ if x is a real number greater than 1 or less than -1 in as much or as little detail you care choose.

b) Describe the set of angles $\arctan(\text{seventeen billion})$.

Advanced Comment 1: If one is working in radian measure, then angles ("turning") are measured by the length of arc moved through on a circle of radius 1. (One full turn is 2π radian.) Thus $\arcsin(x)$ is literally the length of the arc of a unit circle for a point on it of a given height x.



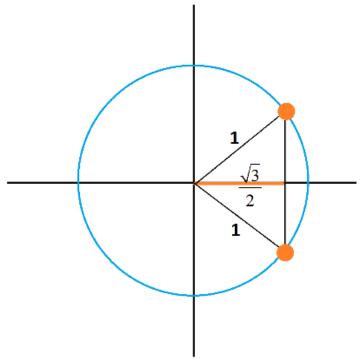
Some countries use *ang sin(x)* for an angle whose sine is *x.*

Advanced Comment 2: One often sees the notation $\sin^{-1}(x)$ for $\arcsin(x)$, which is mimicking inverse function notation: we "undo" the sine function. Of course, $\sin^{-1}(x)$ adopts either infinitely values or no values (depending on whether x is inside or outside of the range $[-1,1]$) and this is considered disturbing to those wish to adhere strictly to the definition of a function: "just *one* output is to be associated with each input." Consequently, some textbooks avoid this notation. (Though, one can, of course, define $\sin^{-1}$ to be a function with inputs real numbers and outputs sets. In this context, each input does now have just one output associated with it!)

Working with multi-values is hard!

Example: *Compute* $\arccos\left(\sin\left(60^\circ\right)\right)$.

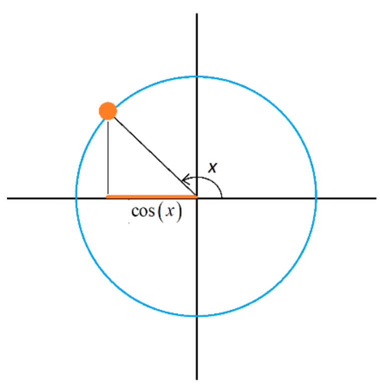
Answer: Now $\sin\left(60^\circ\right)=\frac{\sqrt{3}}{2}$ and $\arccos\left(\frac{\sqrt{3}}{2}\right)=\pm 30^\circ + k360^\circ$ for k an integer.



It is tempting to write $\arccos\left(\sin\left(60^\circ\right)\right)=\arccos\left(\cos\left(30^\circ\right)\right)=30^\circ$ since arc cosine "undoes" cosine. But we have to careful to consider all angles with matching cosine values and this approach, apparently, only focused on one.

Example: *Write a general formula for* $\arccos\left(\cos\left(x\right)\right)$.

Answer: We need to identify all angles with cosine value $\cos\left(x\right)$. We see from the picture (I happened to draw an obtuse angle x this time) that both x and $-x$ are angles with cosine value $\cos\left(x\right)$.



Thus

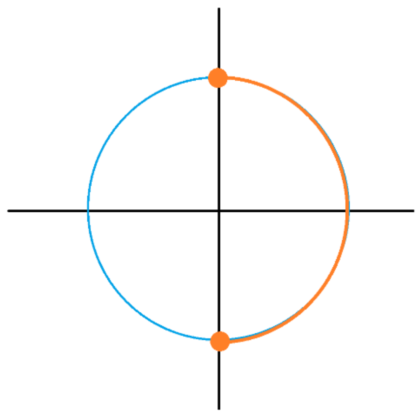
$$\arccos\left(\cos\left(x\right)\right)=\pm x + k360^\circ \text{ for an integer } k.$$

Question 7:
a) Write down a general formula for $\arcsin(\sin(x))$.
b) Write down a general formula for $\arctan(\tan(x))$.

Question 8: Is there a value x between -1 and 1 so that the two sets $\arcsin(x)$ and $\arccos(x)$ are the same?

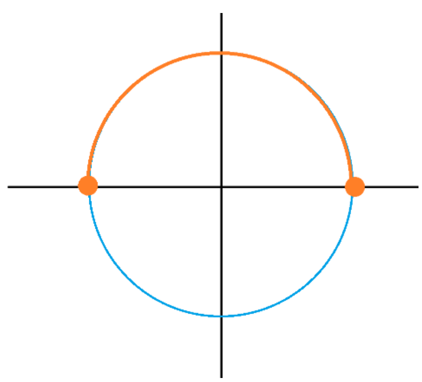
SIMPLIFYING MATTERS: Principal Branches

Sine values, the height of the Sun on a circle of radius 1, range in value from -1 to 1. And a very natural representative set angles of elevation that have those heights are the angles between -90° and 90°.



To settle on just *one* representative value for $\arcsin(x)$ and generally make our work easier, let's define $\text{Arcsin}(x)$, with a capital A, to be the one angle θ between -90° and 90° with $\sin(\theta) = x$. This is called the *principal branch* of arc sine.

In the analogous way, cosine values, the overness of the Sun on a circle of radius 1, range in value from -1 to 1. And a very natural representative set angles of elevation that have those overness values are the angles between 0° and 180°.



Let's define the principal branch of arc cosine as follows: $\text{Arccos}(x)$ is the one angle θ between 0° and 180° with $\cos(\theta) = x$.

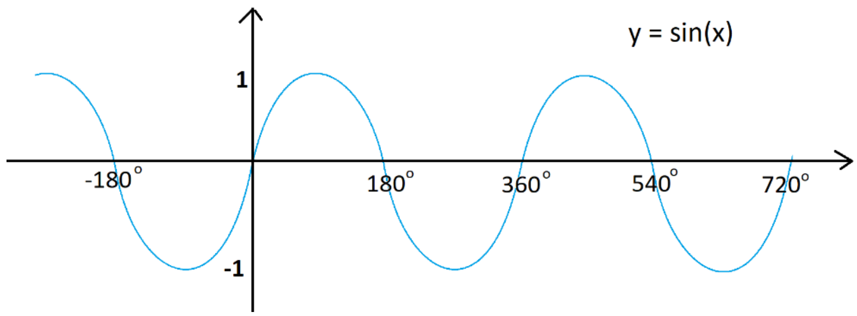
Question 9:

a) Convince yourself that for each real number x that there is just one angle θ strictly between -90° and 90° with $\tan(\theta) = x$.

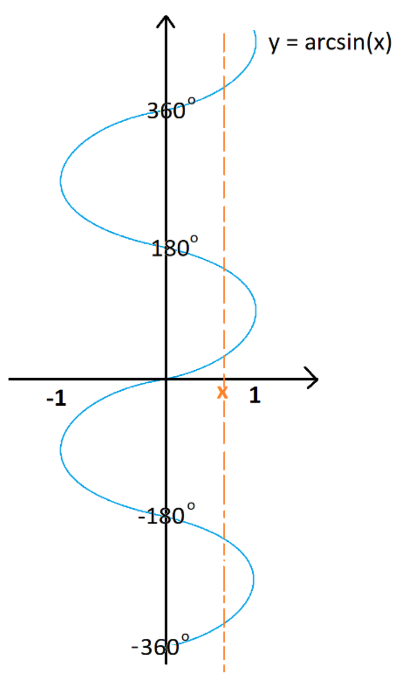
b) Convince yourself that for each real number $x \neq 0$ that there is just one angle θ between 0° and 180° with $\tan(\theta) = x$. Could that angle be 90°? And what if $x = 0$?

c) What do you think is a sensible definition for $\mathrm{Arctan}(x)$? What do textbooks typically say?

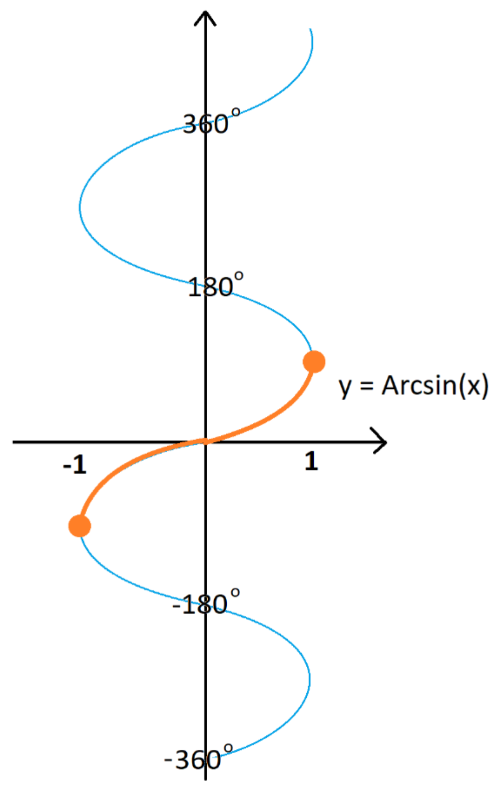
Advanced Comment 3: In light of circle-ometry, the principal branches of the inverse trigonometric functions are very natural. Textbooks, on the other hand, motivate the need for the definitions of principal branches in terms of the graphs of trigonometric functions. For example, here is a graph of $y = \sin(x)$.



And here is the graph of $y = \arcsin(x)$, as a multivalued inverse function. (One can see that for each value $-1 \leq x \leq 1$ on the horizontal axis, there are infinitely many angles θ above and below it on the graph with $\sin(\theta) = x$.)



As working with multi-values is tricky, people highlight a portion of this graph as representative of all angle values on the vertical axis. This is dubbed the “principal branch” of the graph. For Arcsin it is this branch shown.



Optional Question 10: Draw the graphs of $y = \arccos(x)$ and $y = \arctan(x)$ and identify within them the graphs of $y = \text{Arccos}(x)$ and $y = \text{Arctan}(x)$.

WARNING: Many textbooks do not distinguish between $\arcsin(x)$, a set typically with an infinite number of angle values, and $\text{Arcsin}(x)$, a specific representative angle from that set. They use the “small a” notation, $\arcsin(x)$, to mean the one principal value. Similarly for $\text{Arccos}(x)$ and $\text{Arctan}(x)$.

WARNING: Calculators only display principal branch values of inverse trigonometric functions.

Question 11: There is something to think through for each of the six claims presented. Think through them and make sure you are convinced of them!

$\sin\left(\text{Arcsin}(x)\right) = x$ for all numbers $-1 \le x \le 1$.
$\cos\left(\text{Arccos}(x)\right) = x$ for all numbers $-1 \le x \le 1$.
$\tan\left(\text{Arctan}(x)\right) = x$ for all numbers x.

$\text{Arcsin}\left(\sin(\theta)\right) = \theta$ for all angles $-90^\circ \le \theta \le 90^\circ$.
$\text{Arccos}\left(\cos(\theta)\right) = \theta$ for all angles $0^\circ \le \theta \le 180^\circ$.
$\text{Arctan}\left(\tan(\theta)\right) = \theta$ for all angles $-90^\circ \le \theta \le 90^\circ$.

Here's a relationship often used in textbooks. There are subtle issues to think through.

Question 12: Let a be a value with $-1 \le a \le 1$.

i) Show that $\cos\left(\text{Arcsin}(a)\right)$ is sure to be a non-negative value.

ii) Show that $\sin\left(\text{Arccos}(a)\right)$ is sure to be a non-negative value.

iii) Show that $\cos\left(\text{Arcsin}(a)\right) = \sqrt{1-a^2}$, the non-negative root.

iv) Show that $\sin\left(\text{Arccos}(a)\right) = \sqrt{1-a^2}$, the non-negative root.

Here's a typical textbook question using these observations.

Question 13: Let a and b be values with $-1 \le a, b \le 1$. Write $\cos\left(\text{Arcsin}(a) + \text{Arccos}(b)\right)$ solely on terms of a and b.

Question 14:

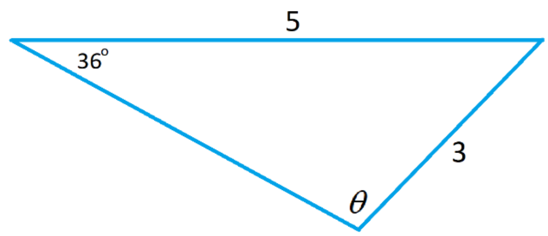
a) Show that $\tan\left(\text{Arcsin}(a)\right) = \dfrac{a}{\sqrt{1-a^2}}$.

b) Find a formula for $\tan\left(\text{Arccos}(a)\right)$.

Advanced Comment 4: ## The Supposed "Ambiguous" Case

That textbooks often de-emphasize the multi-valued nature of $\arcsin$ and $\arccos$, often choosing to only mention the principle branches of these quantities, there come situations in which apparent ambiguities arise. For example, consider a "real-world" problem about a farmer making a triangular pen against river edges that lead to the mathematical task:

Find the measure of the obtuse angle θ in this diagram.

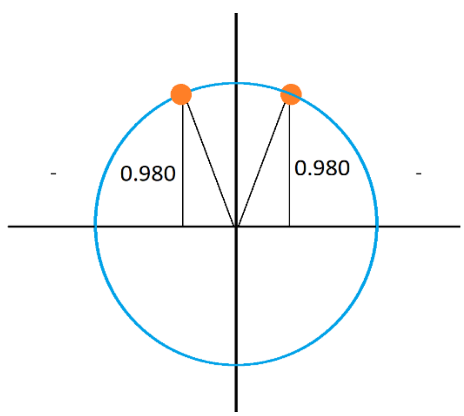


A student might use the Law of Sines and deduce $\frac{3}{\sin(36^\circ)} = \frac{5}{\sin(\theta)}$ giving

$\sin(\theta) = \frac{5}{3}\sin(36^\circ) \approx 0.980$ and so deduce, using a calculator, that $\theta \approx arcin(0.980) \approx 78.5^\circ$.

But this angle is clearly not an obtuse angle, as the question stated.

But a student with circle-ometry imagery in mind realises that there are multiple angles with the same given sine value and that they are free to choose the value that is appropriate for the context at hand.



It is clear for this problem that we need $90^\circ < \theta < 180^\circ$ and that we should thus work with

$$\theta \approx 180^\circ - 78.5^\circ = 101.5^\circ.$$

No special "ambiguous case" discussions to be had!

BRIEF SOLUTIONS

Question 1: Yes. Tangent can be thought of as a function with domain the set of all angle measures different from

$90^\circ + k360^\circ$ for an integer k
and $-90^\circ + k360^\circ$ for an integer k,

and range the set of *all* real numbers.

Question 2: Here are infinitely many examples (and, in fact, all of them).

$\pm 120^\circ + k360^\circ$ for an integer k

Question 3: Here are infinitely many examples (and, in fact, all of them).

$135^\circ + k360^\circ$ for an integer k
$-45^\circ + k360^\circ$ for an integer k

Question 4: $\pm 150^\circ + k360^\circ$ for an integer k.

Question 5: 300°.

Question 6: a) It is an infinite set.
If $-1 < x < 1$ and θ is one angle in the set, then $180^\circ - \theta$ is another, and the set consists of these two angles and multiples of 360° added to them. If $x = 1$, then the set of all angles $90^\circ + k360^\circ$ for an integer k, and it is the set of all angles $-90^\circ + k360^\circ$ for an integer k if $x = -1$.
b) It is an infinite set consisting of an angle θ ever-so-slightly less in measure than 90° and $\theta + 180^\circ$, along with these values adjusted by multiples of 360°.

Question 7: a) $x + k360^\circ$ or $180^\circ - x + k360^\circ$ for an integer k.
b) $x + k360^\circ$ or $x + 180^\circ + k360^\circ$ for an integer k.

Question 8: No.

Question 9:
a) Thinking of tangent as the slope of the line from the origin to the Sun, this is clear.
b) This is clear too, but the line in question is vertical for an angle of elevation of 90°. Also, angles of elevation 0° and 180° both give a slope (tangent value) of zero.
c) $\text{Arctan}(x)$ is that unique angle θ strictly between -90° and 90° with $\tan(\theta) = x$.

Question 10: Consult any trigonometry textbook. This is the standard approach to matters.

Question 11: Do you honestly feel you've properly "taken in" each of the six items?

Question 12:
a) Because $\text{Arcsin}(a)$ is defined to be an angle between -90° and 90°, its cosine is sure to be non-negative.
b) Because $\text{Arccos}(a)$ is defined to be an angle between 0° and 180°, its sine is sure to be non-negative.
c) and d): Draw a diagram of the appropriate right triangle in a circle of radius 1. This right triangle has hypotenuse 1 and one side of length $|a|$. By the Pythagorean Theorem, the remaining side length is $\sqrt{1-|a|^2} = \sqrt{1-a^2}$.

Question 13: Using the standard angle-sum formula and the previous two questions we have

$$\begin{aligned}
&\cos\left(\text{Arcsin}(a) + \text{Arcsin}(b)\right) \\
&= \cos\left(\text{Arcsin}(a)\right)\cos\left(\text{Arcsin}(b)\right) \\
&\quad - \sin\left(\text{Arcsin}(a)\right)\sin\left(\text{Arccos}(b)\right) \\
&= \sqrt{1-a^2} \times b - a \times \sqrt{1-b^2} \\
&= b\sqrt{1-a^2} - a\sqrt{1-b^2}
\end{aligned}$$

Question 14:

a) $\tan\left(\text{Arcsin}(a)\right) = \dfrac{\sin\left(\text{Arcsin}(a)\right)}{\cos\left(\text{Arcsin}(a)\right)} = \dfrac{a}{\sqrt{1-a^2}}$

b) $\tan\left(\text{Arccos}(a)\right) = \dfrac{\sqrt{1-a^2}}{a}$.