A SEGUE INTO
PARTIAL FRACTIONS

JUG FILLING

Here's a classic puzzle.

You are handed a 3-liter jug and a 5-liter jug, neither possessing markings of any kind. Using these jugs is it possible to draw exactly one liter of water from a well?

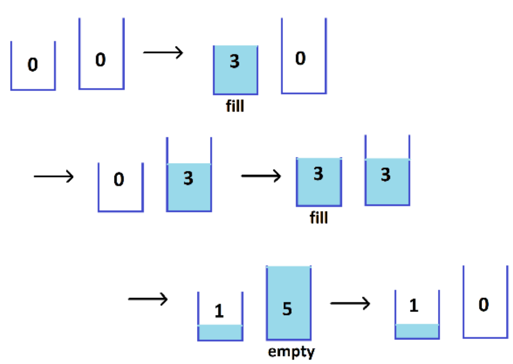
As one is not given the means to measure the exact contents of a partially filled jug, there are essentially only three meaningful maneuvers.

1. Completely fill an empty jug from the well.

2. Completely empty a full jug into the well.

3. Pour water from one jug into another, completely filling or emptying a jug in the process.

Here is one solution to the problem.



This solution involves filling up the 3-liter jug twice and emptying the 5-liter jug once. As

$$2 \times \mathbf{3} - 1 \times \mathbf{5} = 1,$$

this does indeed leave behind precisely one liter of water.

Challenge 1:
a) Find another solution to the problem.
b) Find two more solutions to the problem. (If that feels too hard, read on and come back to this later.)

Challenge 2:
We found two integers a and b so that

$$3a + 5b = 1.$$

(We had $a = 2$, $b = -1$.)

a) Find another pair of integers that satisfy the equation $3a + 5b = 1$.

b) Find two more pairs of integers that satisfy $3a + 5b = 1$.

c) Show that each of your solutions corresponds to a practical means of obtaining 1 liter of water from a 3-liter and a 5-liter jug.

[Now go back and answer 1b)?]

Challenge 3:
a) Suppose, instead, one is given a 4-liter jug and a 9-liter jug (again with no markings). Explain how to obtain exactly one liter of water from a well.

b) Is it possible to obtain one liter of water using a 9-liter jug and a 16-liter jug?

c) Is it possible to obtain one liter of water using a 9-liter jug and a 21-liter jug?

Challenge 4: Explain why each of the following three equations cannot be solved with integer values for a and b.

$$4a + 10b = 1$$

$$9a + 21b = 1$$

$$15a + 35b = 1$$

THE SAME PUZZLE IN DISGUISE

Challenge 5: The number 15 has proper factors 3 and 5. Is it possible to write $\frac{1}{15}$ as a combination of the fractions $\frac{1}{3}$ and $\frac{1}{5}$? That is, are there integers a and b such that

$$\frac{1}{15} = \frac{a}{3} + \frac{b}{5}?$$

If so, find some integers that work. (Is there more than one possibility?) If no integers can work, explain why not.

Challenge 6:

a) The number 36 can be written as $36 = 4 \times 9$. Find a pair of integers a and b that expresses $\frac{1}{36}$ as a combination of $\frac{1}{4}$ and $\frac{1}{9}$. That is, find an integer solution to

$$\frac{1}{36} = \frac{a}{4} + \frac{b}{9}.$$

b) Find an integer solution to

$$\frac{1}{144} = \frac{1}{9 \times 16} = \frac{a}{9} + \frac{b}{16}.$$

Challenge 7: We have $189 = 9 \times 21$. Is there an integer solution to

$$\frac{1}{189} = \frac{1}{9 \times 21} = \frac{a}{9} + \frac{b}{21}?$$

Explain your thinking.

WHY STAY IN BASE TEN? LET'S GO TO BASE *x*!

Challenge 8:

We have

$$x^2 - 4 = (x+2) \times (x-2).$$

Is there an integer solution to

$$\frac{1}{x^2 - 4} = \frac{a}{x+2} + \frac{b}{x-2}?$$

Comment: The notion of an "integer" might be too closely tied to base-ten thinking. If there are no integers a and b that work for $\frac{1}{x^2 - 4}$, might there be real number solutions instead?

OPTIONAL APPENDIX: THE EUCLIDEAN ALGORITHM

The jug-filling problem is tied to a deep number-theory question: *When can a specific integer be written as a combination of two given integers?*

For example, we saw that the number 1 can be written as a combination of 3 and 5,

$$3(2)+5(-1)=1,$$

but it is impossible to write 1 as a combination of 9 and 21. (If $9a+21b=1$ has integer solutions, then the 1 would be a multiple of three.)

The great mathematician Euclid of the third century BCE established that the greatest common factor of any two numbers a and b can be written as an integer combination of a and b. (And by multiplying through, so too can any multiple of the greatest common factor be so written.)

Here are some notes on how to "lean in" to the Euclid's work via more related play.

THE 9-16 GAME

The numbers 9 and 16 are written on a white board.

9 16

Two people will take turns writing a positive number on the board, not already there, that represents the difference of two numbers currently scribed. They will keep doing this until one player cannot make a move. The last player who is able to take a turn is the winner of this game.

For example, there is no choice but for the first player to write 7, the difference of 9 and 16.

9 16

7

There is no choice for the second player either. She must now write 2, the difference of 7 and 9.

9 16

7 2

From this point on players have choices.

a) Play the 9-16 game several times. Is there a player that always seems to win? What do you notice about the entire set of numbers that appear on the board at the end of any game?

b) Play a 7-18 game. What do notice about the set of numbers that appear on the board by the end of a game? In playing this game, which player always wins?

c) Play the 11-25 game. Do you want to play first or second to be the winner?

d) Play the 15-21 game. What happened this time?

e) Find a pair of (whole) numbers, if you can, that satisfy the equation

$$\square \times 15 + \square \times 21 = 1$$

f) Using a 15-liter jug and a 21-liter jug, is it possible to obtain precisely one liter of water from a well?

SOME THOUGHTS

In playing the 9-16 you will notice that all the numbers 1 through 16 appear on the board. (If you knew this in advance, you would thus opt to play second to secure a win.)

Also, the 7-18 game produced all the numbers 1 through 18 and the 11-25 game all the numbers 1 through 25. (You would want to play second in the first of these games and first in the second.)

Alas, the 15-21 game did not produce all the numbers 1 through 21, only the multiples of 3 within this range. And this makes sense: Since 15 and 21 are already multiples of 3 their difference will again be a multiple of 3, as will be all the differences of the differences.

What it curious about this game, however, is that *all* the multiples of 3 from 3 to 21 do appear. (And in the previous games, *all* the multiples of 1 appear.) It is not immediately clear why.

Challenge 9:
a) We predict that a 14-22 game will produce all the multiples of ___ up to 22. Does it?
b) What is your prediction for a 28-84 game? Play it and see if you are correct.

THE EUCLIDEAN ALGORITHM

Let's conduct a simplified version of the 15-21 game. We'll just record two numbers at a time (starting with the initial two numbers 15 and 21) and simply alter the larger number by subtracting the smaller from it. (For ease of typesetting, we'll use notation that looks like fraction notation, but we don't mean to imply fractions.)

$$\frac{15}{21} \rightarrow \frac{15}{6}$$

Let's repeat this process, always subtracting the smaller number from the larger, until we can go no further without heading into zero or negative values.

$$\frac{15}{21} \rightarrow \frac{15}{6} \rightarrow \frac{9}{6} \rightarrow \frac{3}{6} \rightarrow \frac{3}{3}$$

This leads to a common value of 3. And what does 3 have to do with the original two numbers? It's certainly a common factor of 15 and 21. In fact, it is the larger of the two common factors these numbers possess.

As another example, let's conduct this process for the pair 42 and 60.

$$\frac{42}{60} \rightarrow \frac{42}{18} \rightarrow \frac{24}{18} \rightarrow \frac{6}{18} \rightarrow \frac{6}{12} \rightarrow \frac{6}{6}$$

Notice that 6 is the largest common factor 42 and 60 possess.

We have the following claim:

Repeatedly subtracting the smaller from the larger for a pair of numbers eventually produces two identical (positive) numbers. That common result is the greatest common factor of the original pair of numbers.

Our goal is to prove this is true.

THE PROOF

First note that if two numbers a and b are each multiples of 3, say, then a and $a-b$ would still both remain multiples of three. And conversely, if we were told that a and $a-b$ both are multiples of three, then we know that the original numbers a and b were too. (The two numbers a and $a-b$ differ by b.)

That is, if 3 is a common factor of a and b, then it is a common factor of a and $a-b$, and conversely.

There is nothing special about the number 3 in this argument. In general, we can say

The common factors of a and b are the same as the common factors of a and $a-b$.

So in playing our subtraction game of repeatedly subtracting the smaller number from the larger, we are not changing the common factors the numbers possess. So, if we play the game until we reach a repeated value

$$\frac{a}{b} \rightarrow \quad \cdots \quad \rightarrow \frac{d}{d}$$

we know that the common factors of a and b are precisely the same as the common factors of d and d. Thus, whatever the factors of *d* are, they are the common factors of a and b. In particular, *d* is the largest common factor of d and d and so *d* must be the largest common factor of a and b!

This establishes the result.

Comment 1: Technically we should verify that the process of repeatedly subtracting the smaller number from the larger does eventually stop to produce a repeated value. This follows from the fact that as we play the game, we keep producing smaller pairs of numbers without ever allowing ourselves to enter the realm of negative numbers. There is a lower bound on the game (namely that of zero) that stops us from going on forever.

Comment 2: We have actually established more than we set out to do. We proved that the common factors of a and b are the same as the factors of d, with d being the largest common factor.

All common factors of a pair of numbers are factors of the greatest common factor.

For example, the pair 36 and 48 have common factors 1, 2, 3, 4, 6, and 12, and each of these common factors is indeed a factor of the greatest common factor 12.

GOING EVEN FURTHER

There is more to be said.

Consider again the 15-21 example.

$$\frac{15}{21} \rightarrow \frac{15}{6} \rightarrow \frac{9}{6} \rightarrow \frac{3}{6} \rightarrow \frac{3}{3}$$

Let's write this out again but keep track of which number was subtracted from which as we go along, expressing every quantity in terms of the original two numbers.

$$\begin{aligned}
\frac{15}{21} &\rightarrow \frac{15}{21-15} \\
&\rightarrow \frac{15-(21-15)}{21-15} = \frac{2\times 15-21}{21-15} \\
&\rightarrow \frac{2\times 15-21-(21-15)}{21-15} = \frac{3\times 15-2\times 21}{21-15} \\
&\rightarrow \frac{3\times 15-2\times 21}{21-15-(3\times 15-2\times 21)} = \frac{3\times 15-2\times 21}{3\times 21-3\times 15}
\end{aligned}$$

The final answer, of course, is just $\frac{3}{3}$ in disguise.

But notice, this procedure has shown for us precisely how to write the number 3, the greatest common factor of 15 and 21, as a combination of the original two numbers 15 and 21. We have

$$3 = 3 \times 15 + (-2) \times 21$$

and also

$$3 = (-4) \times 15 + 3 \times 21 .$$

This method of finding the greatest common factor of two numbers provides a means to solve jug-filling problems!

Challenge 10: Use this method to find the greatest common factor of 3 and 5. Use it to find two solutions to the equation $3a + 5b = 1$.

Challenge 11: Write the greatest common factor of 42 and 60 as a combination of these two numbers.

In the 9-16 game we were surprised to see all the numbers 1 though 16 appear.

It is clear that the greatest common factor of 9 and 16 is 1. As we obtain the number 1 by performing repeated differences, this means the number 1 will eventually appear as two players play the 9-16 game. As soon as the number 1 appears, players can then choose 16 minus 1, then 15 minus 1, and so on. Thus all the numbers 1 through 16 appear.

Challenge 12: Explain why all the multiples of three from 3 to 27 will appear in a 15-27 game.

We're now ready to state probably the most important theorem of all of number theory. This result was first written and proved by Greek mathematician Euclid (ca. 300 B.C.E.)

THE EUCLIDEAN ALGORITHM

Given two positive integers a and b, it is possible to obtain their greatest common factor d by repeatedly subtracting the smaller number from the larger.

It is also always possible to find integers x and y so that

$$d = ax + by .$$

The word *algorithm* means "method" or "procedure."

Notation

Mathematicians tend to use the word *divisor* rather than *factor*. And the greatest common divisor (that is, factor) *d* of two numbers *a* and *b* is denoted

$$d = \gcd(a,b) .$$

For example

$$\gcd(42,60) = 6$$
$$\gcd(9,16) = 1$$
$$\gcd(35,50) = 5$$

THE EUCLIDEAN ALGORITHM FOR POLYNOMIALS?

Challenge 13:
Consider

$$p(x) = x^3 + x^2 + x - 3$$

and

$$q(x) = x^2 - x.$$

a) Show that $x-1$ is a "common factor" of $p(x)$ and $q(x)$.

b) Do you think $x-1$ is worthy of being dubbed the "greatest common factor" of $p(x)$ and $q(x)$?

c) Are there numbers A and B such that the following holds?

$$x - 1 = A(x^3 + x^2 + x - 3) + B(x^2 - x)$$

d) If the answer to c) is no, might there instead be polynomials $A(x)$ and $B(x)$ such that the following holds?

$$x - 1 = A(x)\cdot(x^3 + x^2 + x - 3) + B(x)\cdot(x^2 - x)$$

SOME ADDITIONAL PRACTICE

Question 1: Find the greatest common factors of

a) 420 and 330
b) 62 and 80
c) 91 and 73
d) 618 and 336

Question 2:
a) Find values for *a* and *b* so that

$$3 = 45a + 33b.$$

b) Suppose you are given unmarked 45-liter and 33-liter jugs and you wish to obtain exactly three liters of a water from a well. Describe a method that allows you to accomplish this feat.

c) Is it possible to obtain exactly 10 liters of water using 33- and 45-liter jugs? Explain.

Question 3 a) Suppose *n* is a counting number. Prove or give a counter example to the claim: $\gcd(n, n+1) = 1$.

b) Describe all the counting numbers for which $\gcd(n, n+2) = 2$. Explain your answer.

Question 4: Suppose *m* is a positive integer. What is $\gcd(0, m)$ and why?

Question 5: True or False (and how do you know?): *The greatest common divisor of two consecutive square numbers is sure to be 1.*

Question 6: a) How many counting numbers n between zero and 1024 satisfy $\gcd(n, 1024) = 1$?

b) How many counting numbers n between zero and 1200 satisfy $\gcd(n, 1200) = 1$?