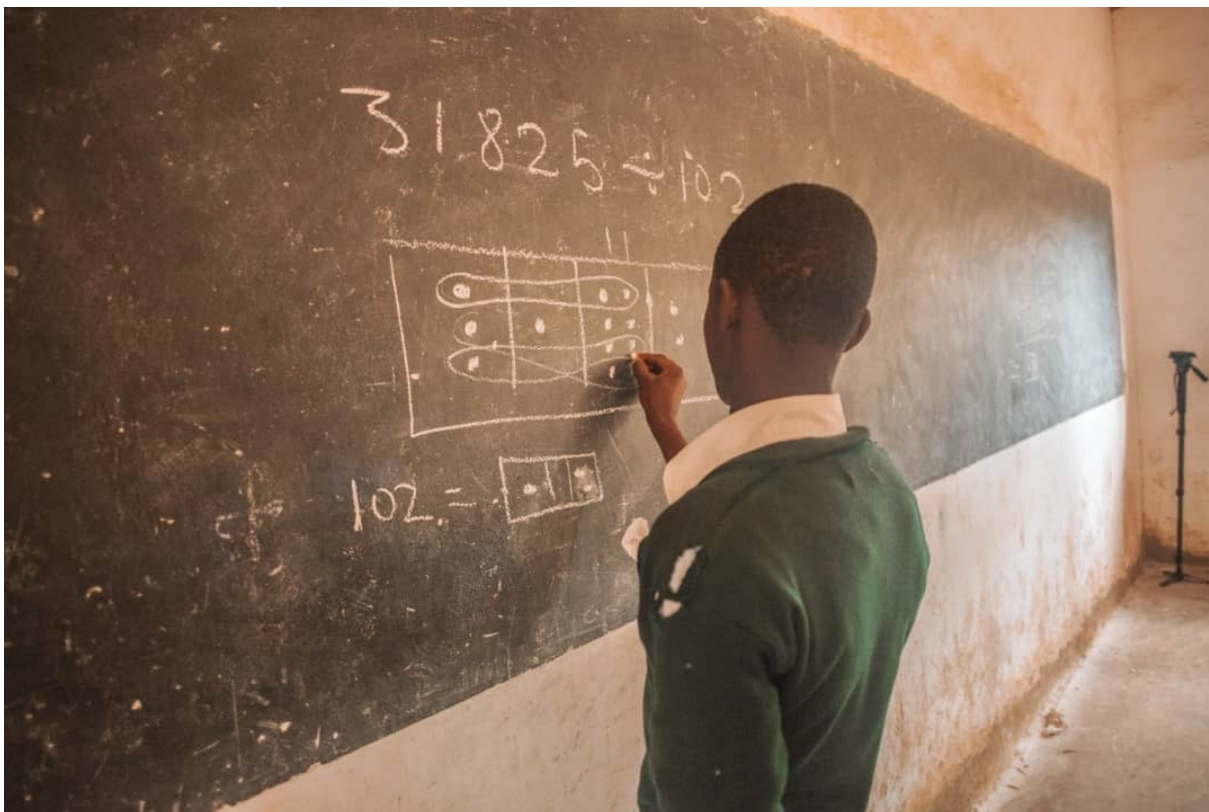


Adventures in Exploding Dots



LESSON 1

Teacher Guides brought to you by
THE GLOBAL MATH PROJECT





The *Global Math Project* was founded in 2016 with the mission of showing that learning mathematics can and should inspire joy. Over 6.25 million students and teachers across the planet have looked at our curriculum materials and used them to ignite and sustain their love of mathematics.

More than 95% of teachers agree that our lessons make mathematics more approachable, enjoyable, and understandable.

Our mathematics all starts with the story of *Exploding Dots*, which is the story of place-value: how we write and do arithmetic with numbers, and how the joy of algebra and much more opens from there.

Here we present six lessons for ages 10 and up that get the young—and the young at heart—started on this wonderful journey.

The full *Exploding Dots* experience, leading into high-school mathematics, college mathematics, and unsolved research challenges, can be found at www.globalmathproject.org. (All so naturally and readily follows from what we present here!)

Enjoy!



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Lesson 1 OVERVIEW

The Five-Card Mind-Reading Trick

Setting the scene for deep understanding of place value.

OBJECTIVES

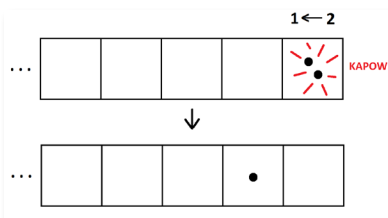
A classic mind-reading trick leads us to the introduction of a $1 \leftarrow 2$ machine and the discovery of binary codes. This is our first foray into a deep understanding of place value.

The Experience in a Nutshell

This introduction lays the groundwork for the full mathematics to come. Here we explore our first *Exploding Dots* machine and become adept at converting numbers into a curious code.

To illustrate:

In a $1 \leftarrow 2$ machine, pairs of dots in any one box “explode” – that is disappear – to produce one dot, one box to their left. (This explains the machine name “ $1 \leftarrow 2$ ” written this backwards way.) We see that two dots turn into one dot and zero dots in a $1 \leftarrow 2$ machine. We say that two has $1 \leftarrow 2$ machine code “10.”



These codes and their meanings reveal the secret of a mind-reading trick.



Some Thoughts on Presenting Mathematics Online

James Tanton

Mathematics is best presented in real time, organically, as conversation unfolds. This is why mathematics classrooms are full of chalkboards and white boards.

But in teaching online, it is not easy to write mathematics and present mathematics in a real-time way. What are some possible approaches?

Use Online Whiteboard Tools

Many people use Zoom to conduct online lessons and Zoom, like most conference platforms, comes with its own whiteboard option.

[Ziteboard.com](https://ziteboard.com) offers a more flexible whiteboard and is worth looking at. It has a free option.

But to write mathematics effectively and with ease, one really needs a writing pad and stylus to plug into one's computer. Wacom and other manufacturer's produce inexpensive tablets.

Use a Plug-In Document Camera

This is my preferred option. I share my screen on Zoom and write on paper as per usual.



Prices for plug-in document cameras seem to vary considerably. My IPEVO "Ziggi" wasn't expensive when I bought it several years ago. Some people like the INSWAN INS-1 Tiny 8MP USB document camera.



Use your SmartPhone as a Document Camera

Turn your phone into a document camera for screen-sharing on Zoom!

Here are some videos on this:

<https://youtu.be/v8IT9c3Ju2E>

<https://youtu.be/6Il008KwcUE>

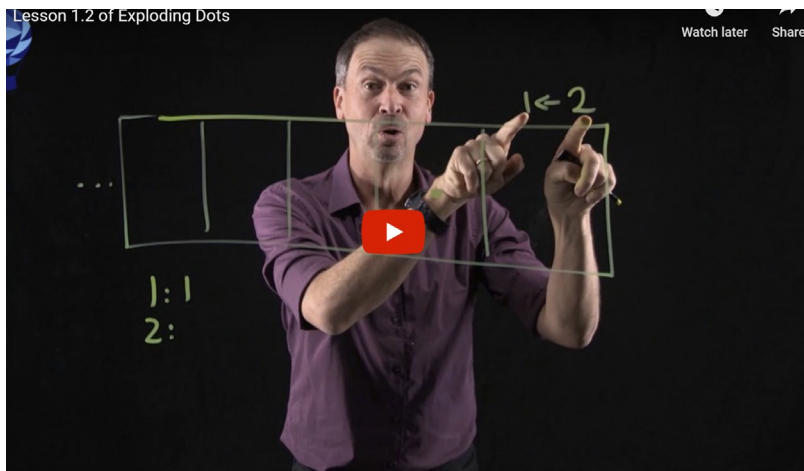
<https://youtu.be/NAOALweAyc>

Just type “Make your smartphone into a document camera,” or some such, in YouTube to find more advice.

If you have a flexible tripod for your phone, then no need for soup cans or clamps on light stands!

Make a Light Board

Those who know my work know I use a light board too.



I made an investment and purchased a pre-made one at www.learning.glass/product-line/, but these are far from cheap.

But there are videos on the internet showing how to build your own (inexpensive) ones if you are game. Here’s one, for example. <https://youtu.be/L1au1JxMSaA>



Lesson 1 Teacher Guide

The Five-Card Mind-Reading Trick

Setting the scene for deep understanding of place value.

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Further Resources for this Lesson

Videos: Explorations 1 and 2 of *Exploding Dots* at www.globalmathproject.org.

Web App: Interactive online play at www.explodingdots.org. Collect KAPOWS!



Part 1: THE FIVE-CARD MIND-READING TRICK

Here's a classic trick. On a board or a screen, write out the following five groups of numbers. Or write out the numbers on five cards.

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

Ask your students to each silently think of the day of the month they were born, or any other number between 1 and 31 they like.

Now perform the mindreading trick by having the following conversation with a student.

"Suzzy. Is the number you are thinking of in group A?"	"Yes."
"Is the number you are thinking of in group B?"	"Yes."
"Is it in group C?"	"No."
"Is it in group D?"	"No."
"Group E?"	"Yes!"

"Ahh ... your number is 25."	"Wow! Yes it is!"
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GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31
YES	YES	NO	NO	YES

Suzzy's responses for the number 25.

Have the same conversation with a few more students, noting each time which groups elicit a "yes" answer from a given student. The secret number they have in mind is simply the sum of the top-left corner numbers in each group with a yes answer. For example, Suzzy answered YES YES NO NO YES. Groups A, B, and E have top left numbers 16, 8, and 1, respectively, and indeed $16 + 8 + 1 = 25$.



As practice, check that if Sameer is thinking the number 13, he will answer NO YES YES NO YES and indeed $8 + 4 + 1 = 13$.

Do this as many times as your students desire. Invite them to figure out what you are doing, and then, why what you are doing works! (Can you figure out why the top left numbers are key?) Perhaps facilitate thinking by writing a student's YES/NO answers under each group each time as done for Suzzy above.

Some Things Students Might Notice or Question

1. Students might notice that group E contains all the odd numbers.
2. Students might notice that group A contains all the numbers 16 and above.
3. Students might question the number 31. "Why must we choose between 1 and 31. What's special about the number 31?"
4. Students might notice the power of two (a doubling number: 1, 2, 4, 8, 16) in the top left corner of each group.

(If you like, you could jumble the numbers in each group when you write them out so as to make this trick more mysterious!)

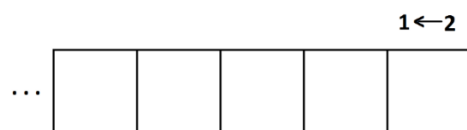


Part 2: A MYSTERIOUS MACHINE

Let's change subject for a moment and also change tone. I like to tell the following as though I am telling a personal story, a story that I am just going to say up front is not true.

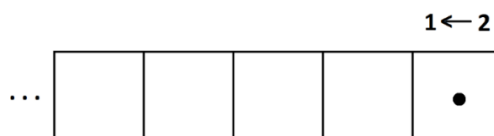
When I was a child, I invented a machine – not true – and this machine is nothing more than a row of boxes that extends as far to the left as I could ever desire.

I gave this machine of mine a name. I called it a “two-one machine” both written and read in a funny backwards way. (I knew no different as a child.)



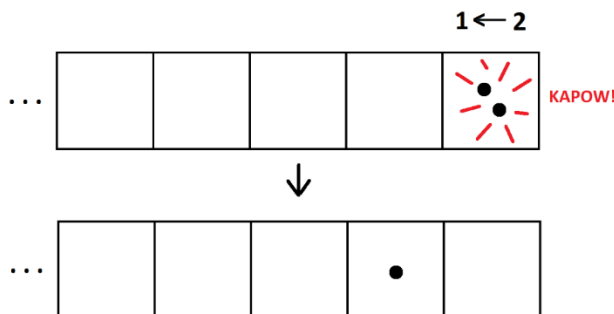
And what do you do with this machine? You put in dots. Dots always go into the rightmost box.

Put in one dot, and, well, nothing happens: it stays there as one dot. Ho hum!



But put in a second dot – always in the rightmost box – and then something exciting happens.

Whenever there are two dots in a box they explode and disappear – KAPOW! – to be replaced by one dot, one box to the left.



(Do you see now why I called this a “1 ← 2 machine” written in this funny way?)

We see that two dots placed into the machine yields one dot followed by zero dots.



Putting in a third dot – always the rightmost box – gives the picture one dot followed by one dot.

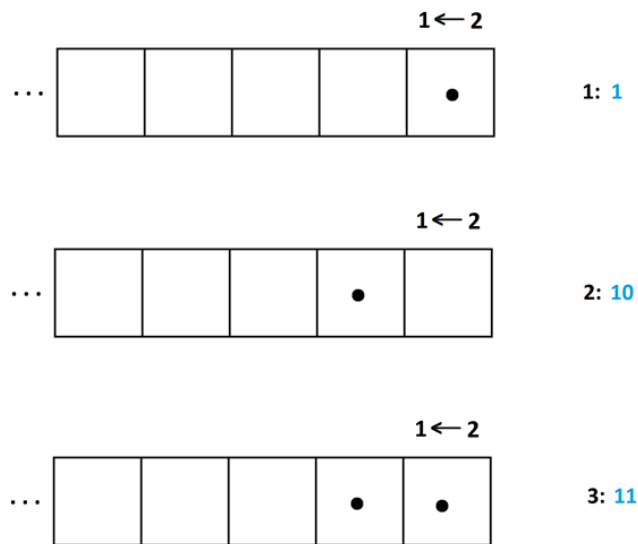


I realized that this machine, in my untrue story, was giving codes for numbers.

Just one dot placed in the machine, stayed as one dot. Let's say that that the $1 \leftarrow 2$ machine code for the number one is 1.

Two dots placed into the machine, one after the other, yielded one dot in a box followed by zero dots. Let's say that the $1 \leftarrow 2$ machine code for the number two is 10.

Putting a third dot in the machine gives the code 11 for three.

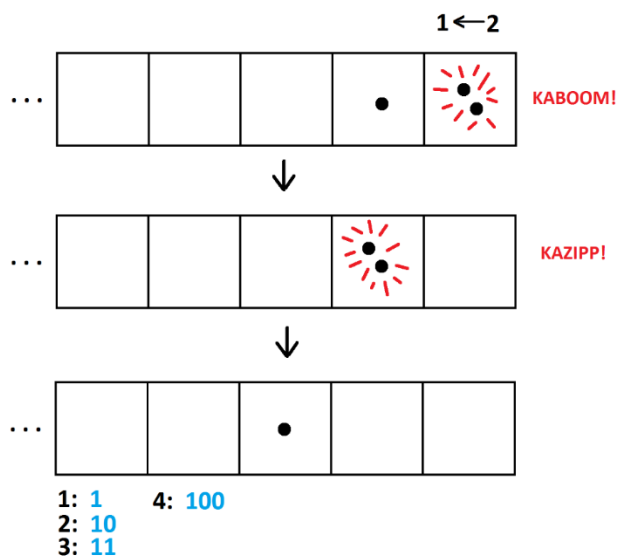


What's the $1 \leftarrow 2$ machine code for four?

(We'll see the answer in a moment.)



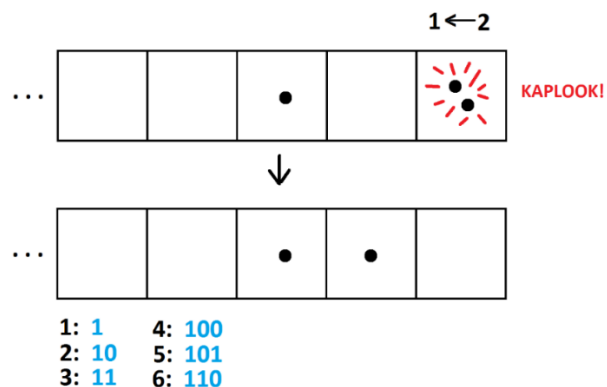
Putting a fourth dot into the machine is particularly exciting: we are in for multiple explosions!
 The $1 \leftarrow 2$ code for four is 100.



What will be the code for five?

Can you see it's 101?

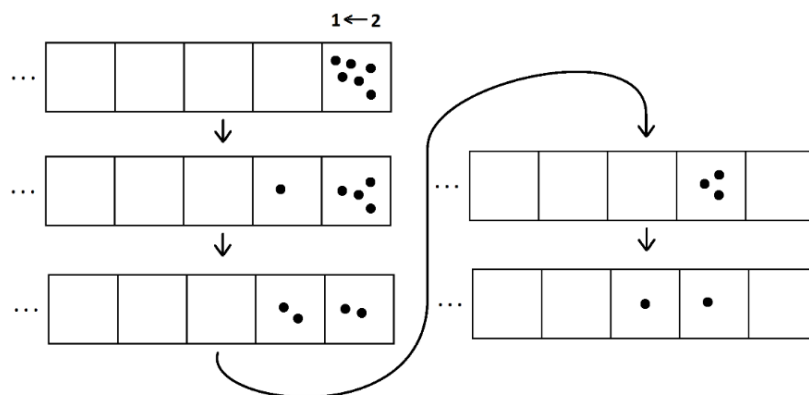
And the code for six? Adding one more dot to the code for five gives 110 for six.





Actually, we can also get this code for six by clearing the machine and then putting in six dots all at once. Pairs of dots will explode in turn to each produce one dot, one box to their left.

Here is one possible series of explosions. Sound effects omitted!



Do you get the same final code of 110 if you perform explosions in a different order? (Try it!)



What is the $1 \leftarrow 2$ machine code for the number thirteen?

(It turns out to be 1101. Can you get that answer?)

There are hours of fun to be had playing with codes in a $1 \leftarrow 2$ machine.

In fact, explaining the codes of these $1 \leftarrow 2$ machine codes explains the five-card mind-reading trick!

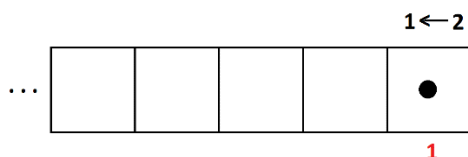


Part 3: EXPLAINING THE $1 \leftarrow 2$ MACHINE

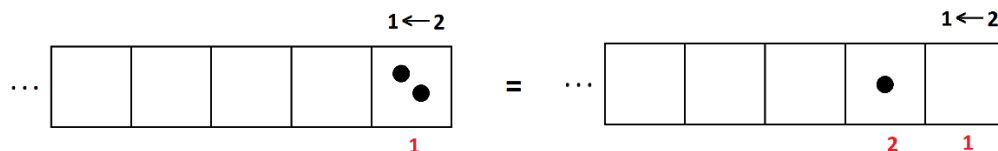
Recall that a $1 \leftarrow 2$ machine follows the rule

Whenever there are two dots in any one box they “explode,” that is, disappear, and are replaced by one dot, one place to their left.

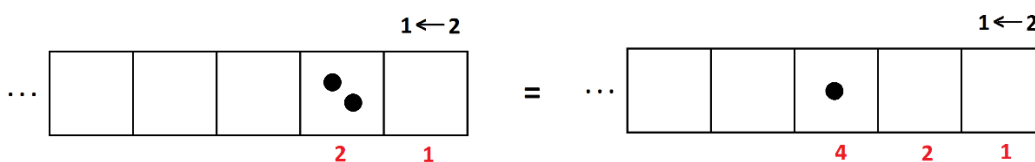
And the machine is set up so that dots in the rightmost box are always worth one.



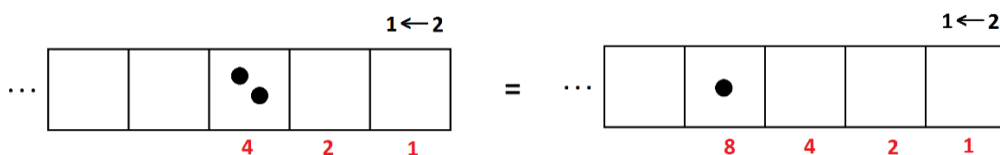
With an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1's, that is, 2.



And two dots in this second box is equivalent to one dot, one place to the left. Such a dot must be worth two 2's, that is, worth 4.



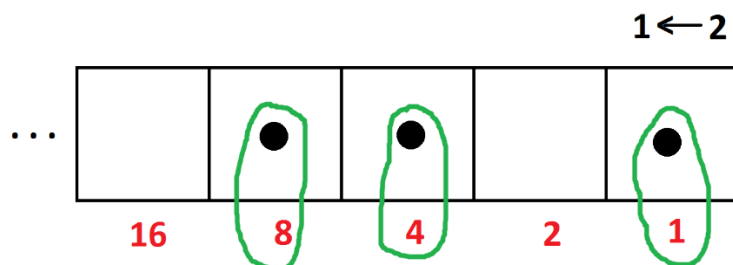
And two 4's makes 8 for the value of a dot the next box over.



And two 8's make 16, and two 16's make 32, and two 32's make 64, and so on.



We saw earlier that the code for thirteen in a $1 \leftarrow 2$ machine is 1101. Now we can see that this is absolutely correct: one 8 and one 4 and no 2's and one 1 does indeed make thirteen.



In fact, the $1 \leftarrow 2$ machine shows how to write any number as a sum of the doubling numbers 1, 2, 4, 8, 16, and so on. Just put in that many dots into the box, let them explode, and the code of 0s and 1s tells you which doubling numbers to use to make that number.



Which number has $1 \leftarrow 2$ machine code 11001?

(Can you see it's 25, Suzy's chosen number?)



What is the $1 \leftarrow 2$ machine code for the number fifty?

(Does it help to know that 50 equals 32 + 16 + 2?)



What is the largest number with a five-digit $1 \leftarrow 2$ machine code?

(That number would have code "11111." What number is that?)



What is the smallest number with a five-digit $1 \leftarrow 2$ machine code?

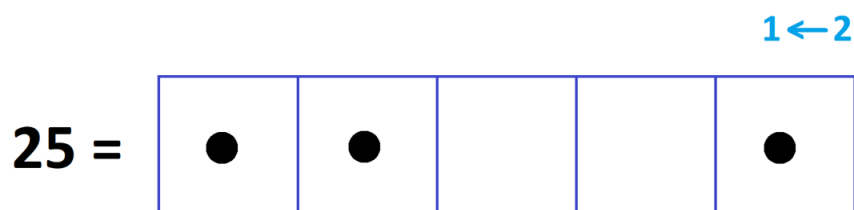
People call the $1 \leftarrow 2$ codes for numbers the *binary* representations of numbers (with the prefix *bi-* meaning "two"). They are also called *base two* representations. One only ever uses the two symbols 0 and 1 when writing numbers in binary.

Computers are built on electrical switches that are either on, or off. So, it is very natural in computer science to encode all arithmetic in a code that uses only two symbols: say 1 for "on" and 0 for "off." Thus, base two, binary, is the right base to use in computer science.



Part 4: EXPLAINING FIVE-CARD MIND-READING TRICK

Look at the $1 \leftarrow 2$ machine codes (that is, the binary codes) of the numbers from 1 up to 31. (We put some extra zeros at front to make each code five digits long.)



1: 00001	9: 01001	17: 10001	25: 11001
2: 00010	10: 01010	18: 10010	26: 11010
3: 00011	11: 01011	19: 10011	27: 11011
4: 00100	12: 01100	20: 10100	28: 11100
5: 00101	13: 01101	21: 10101	29: 11101
6: 00110	14: 01110	22: 10110	30: 11110
7: 00111	15: 01111	23: 10111	31: 11111
8: 01000	16: 10000	24: 11000	

These are the codes that can be written with at most five digits. (The codes for the numbers 32 and above are six or more digits long.)

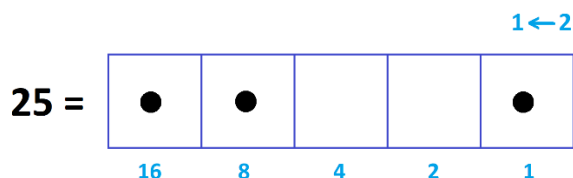
Now look at the numbers in the groups of the mind-reading trick.

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31



Notice that $25 = 16 + 8 + 1$ and that the number 25 appears in the group with the number 16 at its top left, the number 8 at its top left, and the number 1 at its top left. Answering YES YES NO NO YES reveals precisely these groups!

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31
YES	YES	NO	NO	YES



The trick is that if a particular number “uses” 16 or 8 or 4 or 2 or 1 it appears in each of the group with 16 or 8 or 4 or 2 or 1.

For example, $12 = 8 + 4$ and the number 12 appears in the groups containing 8 and 4, and no other groups. And $30 = 16 + 8 + 4 + 2$, and the number 30 appears in every group except the final group that contains 1.

Show that every number that has a $1 \leftarrow 2$ machine code that ends with a 1 appears in group E.

What is common about the $1 \leftarrow 2$ machine codes of each of the numbers that appear in group C?



In fact, one can check:

The numbers listed in group E are precisely the numbers with a 1 in the rightmost box for their $1 \leftarrow 2$ machine codes.

The numbers listed in group D are precisely the numbers with a 1 in the second-to-right box for their $1 \leftarrow 2$ machine codes.

The numbers listed in group C are precisely the numbers with a 1 in the third-to-right box for their $1 \leftarrow 2$ machine codes.

The numbers listed in group B are precisely the numbers with a 1 in the fourth-to-right box for their $1 \leftarrow 2$ machine codes.

The numbers listed in group A are precisely the numbers with a 1 in the fifth-to-right box for their $1 \leftarrow 2$ machine codes.

So, when a student answers “YES” to a particular group, she is informing you that there is a 1 in that matching box of her secret number’s code. When she answers “NO,” there is no 1. That is, she is informing us which of the numbers 1, 2, 4, 8, and 16 make her number.

Question 1: Does it make sense that only the odd numbers have a 1 in the rightmost position for their $1 \leftarrow 2$ codes?

Question 2: Does it make sense that group E matches the numbers 16 through 31?





Adventures in Exploding Dots

Lesson 1 Optional Student Handout

The Five-Card Mind-Reading Trick

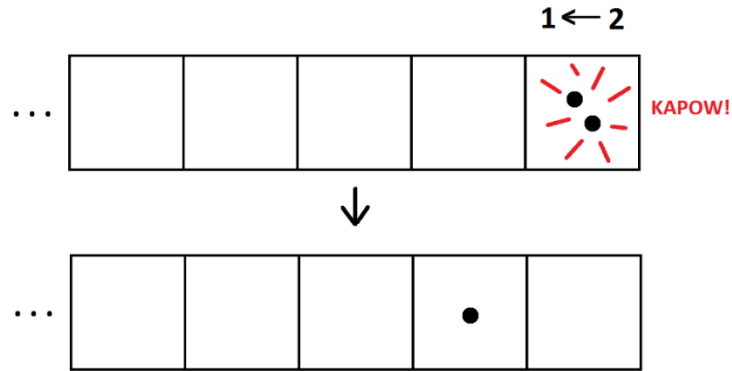
Setting the scene for deep understanding of place-value.



THE $1 \leftarrow 2$ MACHINE

Here's the rule for a $1 \leftarrow 2$ machine.

Whenever there are two dots in a box they explode and disappear – KAPOW! – to be replaced by one dot, one box to the left.



Here are some questions you might, or might not, want to try.

Question 1: a) What is the $1 \leftarrow 2$ machine code for the number thirteen?
(It turns out to be 1101. Can you get that answer?)

b) What is the code for fifty in this machine? (Whoa!)

Question 2: Could a number ever have code 100211 in a $1 \leftarrow 2$ machine, assuming we always choose to explode dots if we can?

Challenge 3: Which number has code 10011 in a $1 \leftarrow 2$ machine?



THE FIVE-CARD MIND-READING TRICK

Write out the following five groups of numbers on cards.

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31

Now ask a friend or family member to silently think of the day of the month they were born, or any other number between 1 and 31 they like.

Perform the mindreading trick by having the following conversation.

“Suzzy. Is the number you are thinking of in group A?”	“Yes.”
“Is the number you are thinking of in group B?”	“Yes.”
“Is it in group C?”	“No.”
“Is it in group D?”	“No.”
“Group E?”	“Yes!”

“Ahh ... your number is 25.”	“Wow! Yes it is!”
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Question 4: Do you remember how this trick works? Do you remember why this trick works?

Exploration 5: What if we were willing to play with six groups of numbers? What would be a six-group version of this mind-reading trick? “Think of a number between 1 and ___”?





Adventures in Exploding Dots

Lesson 1 Student Handout SOLUTIONS

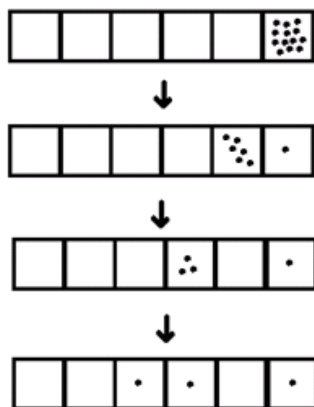
The Five-Card Mind-Reading Trick

Setting the scene for deep understanding of place-value.



Solutions to Optional Practice

Question 1: a) Here's how the code 1101 appears from thirteen dots.



b) The number fifty has code 110010.

Question 2: Assuming we want to make the agreement that we'll always choose to explode dots if we can, then the code 100211 is not complete: the two dots in the third-to-last box can explode to give a final code of 101011.

Question 3: This is the code for the number nineteen.

Question 4: Look at the numbers in the top left corner of each group. Add together these numbers to each YES answer to reveal the secret number. Each of the number 1 through 31 can be written as a sum of some, or all of, the numbers 1, 2, 4, 8, and 16. We see this from playing with the binary codes of numbers.

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 24 28	8 12 24 28	4 12 20 28	2 10 18 26	1 9 17 25
17 21 25 29	9 13 25 29	5 13 21 29	3 11 19 27	3 11 19 27
18 22 26 30	10 14 26 30	6 14 22 30	6 14 22 30	5 13 21 29
19 23 27 31	11 15 27 31	7 15 23 31	7 15 23 31	7 15 23 31
YES	YES	NO	NO	YES

Suzzy's responses for the number 25.

Exploration 5: We'll leave this for you to ponder on. You'll need six cards for all the numbers that can be written with six-digits in base two. (The largest of these numbers has code 111111. What number is that?)