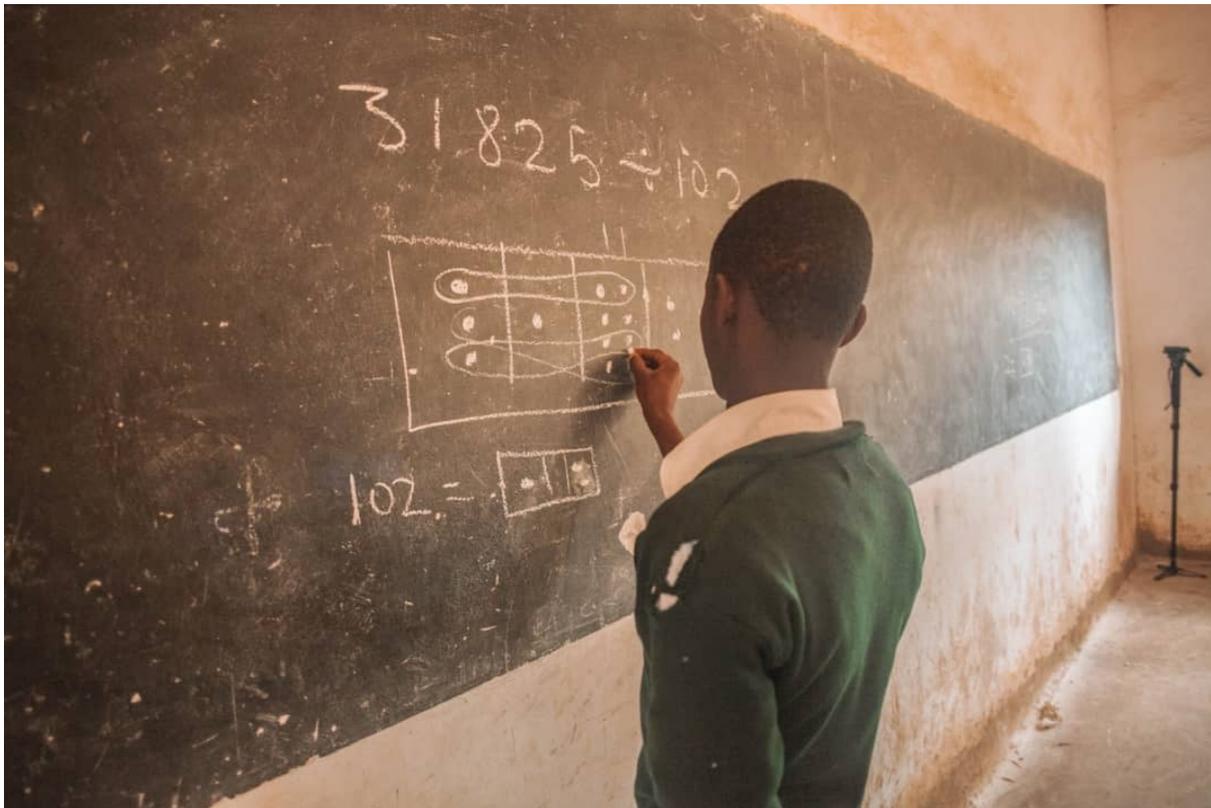


Adventures in Exploding Dots



LESSON 2

Teacher Guides brought to you by
THE GLOBAL MATH PROJECT





The *Global Math Project* was founded in 2016 with the mission of showing that learning mathematics can and should inspire joy. Over 6.25 million students and teachers across the planet have looked at our curriculum materials and used them to ignite and sustain their love of mathematics.

More than 95% of teachers agree that our lessons make mathematics more approachable, enjoyable, and understandable.

Our mathematics all starts with the story of *Exploding Dots*, which is the story of place-value: how we write and do arithmetic with numbers, and how the joy of algebra and much more opens from there.

Here we present six lessons for ages 10 and up that get the young—and the young at heart—started on this wonderful journey.

The full *Exploding Dots* experience, leading into high-school mathematics, college mathematics, and unsolved research challenges, can be found at www.globalmathproject.org. (All so naturally and readily follows from what we present here!)

Enjoy!



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Lesson 2 OVERVIEW

Wild Machines

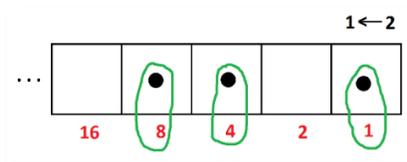
A focus on base ten and why we humans are drawn to it.

OBJECTIVES

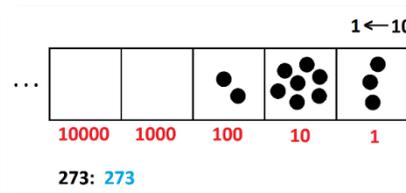
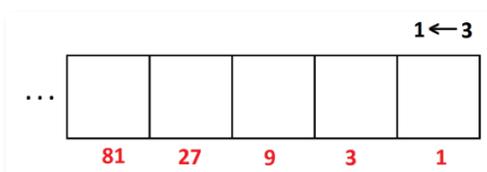
Now we go beyond the $1 \leftarrow 2$ machine and discover more general dots-and-boxes machine codes as representations of numbers in different bases. The $1 \leftarrow 2$ machine codes are the binary (base two) representations of numbers, the $1 \leftarrow 3$ machine codes are the ternary (base three) representations of numbers, and the $1 \leftarrow 10$ machine codes are our familiar base ten representations. We solve another mind-reading trick using the $1 \leftarrow 3$ machine.

The Experience in a Nutshell

In a $1 \leftarrow 2$ machine a pair of dots in any one box is equivalent to a single dot, one box to their left. As dots in the rightmost box are set to be worth 1, the value of each dot in subsequent boxes must arise from a process of doubling. We can now readily see that the $1 \leftarrow 2$ machine code 1101 for the number thirteen, for instance, is correct: thirteen is one 8, one 4, and one 1.



The $1 \leftarrow 3$ machine gives the ternary representations of numbers. The $1 \leftarrow 10$ machine gives our familiar base-ten number representations.



We even speak the language of base ten representations.

273 = two hundred seventy three



Some Thoughts on Presenting Mathematics Online

James Tanton

Mathematics is best presented in real time, organically, as conversation unfolds. This is why mathematics classrooms are full of chalkboards and white boards.

But in teaching online, it is not easy to write mathematics and present mathematics in a real-time way. What are some possible approaches?

Use Online Whiteboard Tools

Many people use Zoom to conduct online lessons and Zoom, like most conference platforms, comes with its own whiteboard option.

[Ziteboard.com](https://ziteboard.com) offers a more flexible whiteboard and is worth looking at. It has a free option.

But to write mathematics effectively and with ease, one really needs a writing pad and stylus to plug into one's computer. Wacom and other manufacturer's produce inexpensive tablets.

Use a Plug-In Document Camera

This is my preferred option. I share my screen on Zoom and write on paper as per usual.



Prices for plug-in document cameras seem to vary considerably. My IPEVO "Ziggi" wasn't expensive when I bought it several years ago. Some people like the INSWAN INS-1 Tiny 8MP USB document camera.



Use your SmartPhone as a Document Camera

Turn your phone into a document camera for screen-sharing on Zoom!

Here are some videos on this:

<https://youtu.be/v8IT9c3Ju2E>

<https://youtu.be/6Il008KwcUE>

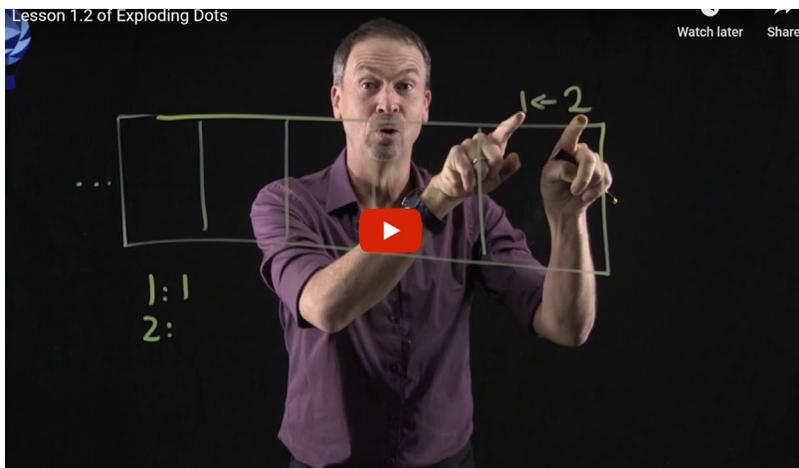
<https://youtu.be/NAOALweAyc>

Just type “Make your smartphone into a document camera,” or some such, in YouTube to find more advice.

If you have a flexible tripod for your phone, then no need for soup cans or clamps on light stands!

Make a Light Board

Those who know my work know I use a light board too.



I made an investment and purchased a pre-made one at www.learning.glass/product-line/, but these are far from cheap.

But there are videos on the internet showing how to build your own (inexpensive) ones if you are game. Here's one, for example. <https://youtu.be/L1au1JxMSaA>



Lesson 2 Teacher Guide

Wild Machines

A focus on base ten and why we humans are drawn to it.

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Further Resources for this Lesson

Videos: Explorations 1 and 2 of *Exploding Dots* at www.globalmathproject.org.

Web App: Interactive online play at www.explodingdots.org. Collect KAPOWS!



Part 1: DÉJÀ VU

Are you and your students ready for another mind-reading trick?

This time share with your class the following six groups of numbers and ask each student to silently think of a number between 1 and 26.

GROUP A	GROUP B	GROUP C
1 4 7	2 5 8	3 4 5
10 13 16	11 14 17	12 13 14
19 22 25	20 23 26	21 22 23
GROUP D	GROUP E	GROUP F
6 7 8	9 10 11	18 19 20
15 16 17	12 13 14	21 22 23
24 25 26	15 16 17	24 25 26

As before, have the following conversation with a student.

“Shankar, tell me the groups in which your secret number lies.”

“It’s in groups A, C, and F.”

“Ahh. Your secret number is 22.”

Given the work of last lesson, your students might suspect you are looking at the top left values in each group mentioned and adding them to find secret number. And you are! Shankar’s secret number 22 appears in the groups with corner numbers 1, 3, and 18 and indeed $1 + 3 + 18 = 22$.

GROUP A	GROUP B	GROUP C
① 4 7	2 5 8	③ 4 5
10 13 16	11 14 17	12 13 14
19 22 25	20 23 26	21 22 23
GROUP D	GROUP E	GROUP F
6 7 8	9 10 11	⑱ 19 20
15 16 17	12 13 14	21 22 23
24 25 26	15 16 17	24 25 26

Shankar’s secret number was 22.

Have students check that this trick works for each of their own secret numbers too.

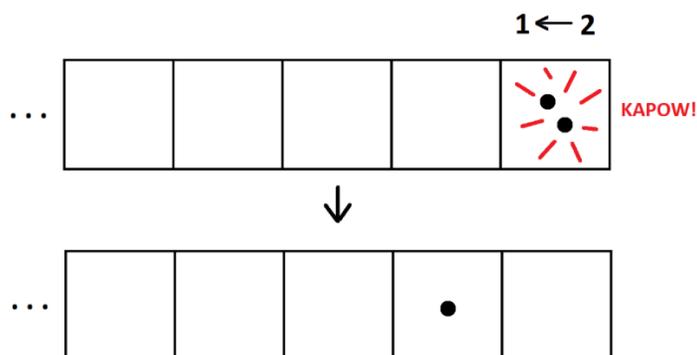
And, of course, the question now is: Why is this working? What’s special about the numbers 1 and 2, 3 and 6, and, 9 and 18?



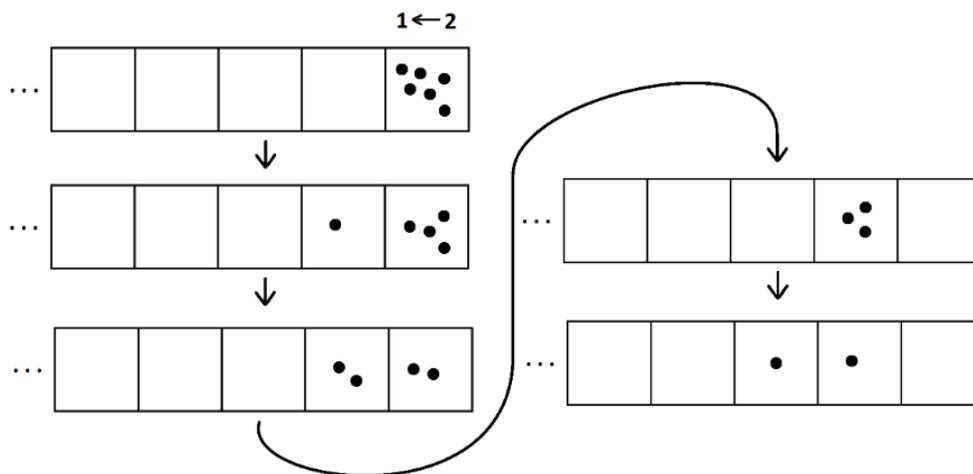
Part 2: GOING BEYOND THE $1 \leftarrow 2$ MACHINE

Recall the $1 \leftarrow 2$ machine from last time. It consists of a row of boxes extending as far to the left as we please. One places dots in this machine—always in the rightmost box—and the exciting part is that

Whenever there are two dots in a box, they explode and disappear – KAPOW! – to be replaced by one dot, one box to the left.

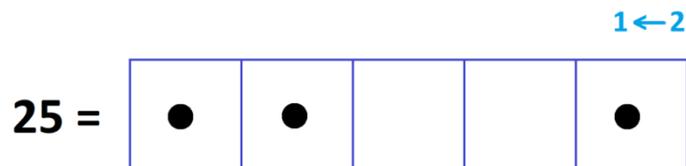


For example, putting six dots into the box—either all at once or one at a time—eventually leads to the $1 \leftarrow 2$ machine code 110.





Hours of fun can be had working out the $1 \leftarrow 2$ machine codes for numbers. And we can perform mind-reading tricks with these codes too!

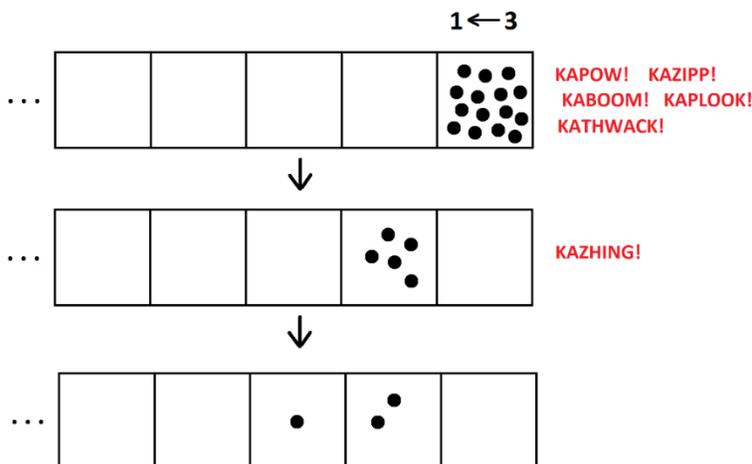


1: 1	9: 1001	17: 10001	25: 11001
2: 10	10: 1010	18: 10010	26: 11010
3: 11	11: 1011	19: 10011	27: 11011
4: 100	12: 1100	20: 10100	28: 11100
5: 101	13: 1101	21: 10101	29: 11101
6: 110	14: 1110	22: 10110	30: 11110
7: 111	15: 1111	23: 10111	31: 11111
8: 1000	16: 10000	24: 11000	

But let me continue the story of these machines.

After playing with the $1 \leftarrow 2$ machine for a while, I suddenly had a flash of insight. Instead of playing with a $1 \leftarrow 2$ machine, I realized I could also play with a $1 \leftarrow 3$ machine. (Again, this is written and read backwards: a “three-one “machine.) Now, whenever there are *three* dots in a box, they explode away to be replaced with one dot, one box to the left.

Here’s what happens to fifteen dots in a $1 \leftarrow 3$ machine.





First there are five explosions in the first box, with each explosion making a dot in the second box to the left. Then three of those dots explode away. This leaves behind two dots and makes one new dot one place to the left. We thus see the code 120 for fifteen in a $1 \leftarrow 3$ machine.



What is the $1 \leftarrow 3$ machine code for the number thirteen?

(Do you see it is 111?)

As we saw with the number fifteen it is possible have two dots in a box that don't explode away. (We need three dots in a box for an explosion.) So this means that the codes for numbers in a $1 \leftarrow 3$ machine can involve the digits 0, 1, and 2.



Here's the list of the $1 \leftarrow 3$ machine codes for the first ten numbers. Please double check that I got the codes right!

1:	1	6:	20
2:	2	7:	21
3:	10	8:	22
4:	11	9:	100
5:	12	10:	101



Which number has $1 \leftarrow 3$ machine code 222?

(Hint: It's one of the numbers that appears in a card for this lesson's opening mind-reading trick.)

And, continuing my story, let me say I had hours of fun playing with numbers in a $1 \leftarrow 3$ machine.



But then ...

Another flash of insight!

Instead of doing a $1 \leftarrow 3$ machine, I realized I could do a $1 \leftarrow 4$ machine!

And then ...

Another flash of insight!

Instead of doing a $1 \leftarrow 4$ machine, I could do a $1 \leftarrow 5$ machine!

And then ...

Another flash of insight!

Instead of doing a $1 \leftarrow 5$ machine, I could do a $1 \leftarrow 6$ machine!

And then ...

I decided to go wild!

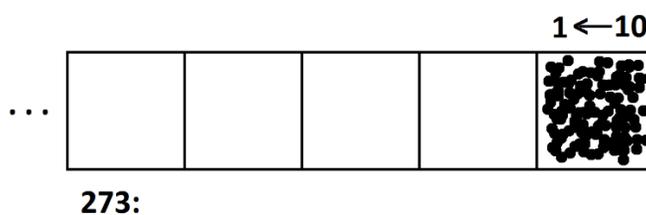


Part 3: THE $1 \leftarrow 10$ MACHINE

Okay. Let's go all the way up to a $1 \leftarrow 10$ machine and put in 273 dots in it.

Here goes!

What is the secret $1 \leftarrow 10$ machine code for the number 273?



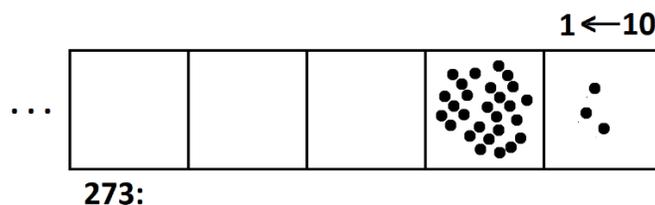
I thought my way through this by asking a series of questions.

Will there be any explosions? Are there any groups of ten that will explode? Certainly!

How many explosions will there be initially? Twenty-seven.

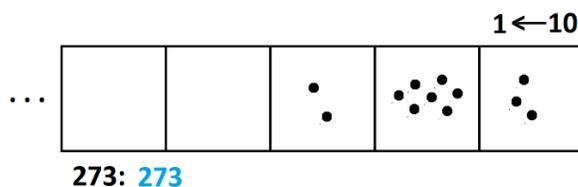
Any dots left behind? Yes. Three.

Okay. So, there are twenty-seven explosions, each making one dot one place to the left, leaving three dots behind.



Any more explosions? Yes. Two more.

Any dots left behind? Seven left behind.





Look at what we have:



The $1 \leftarrow 10$ code for two hundred seventy-three is ... 273.

Whoa!

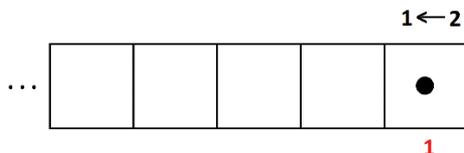
Something curious is going on!

What is the natural big question to ask?



Part 4: EXPLAINING THE MACHINES

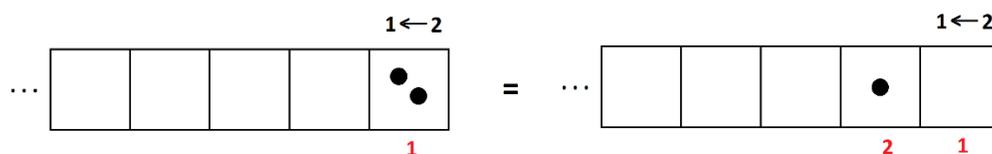
We did manage to explain what was going on with the $1 \leftarrow 2$ machines last lesson. Recall this machine is set so that dots in the rightmost box are always worth one



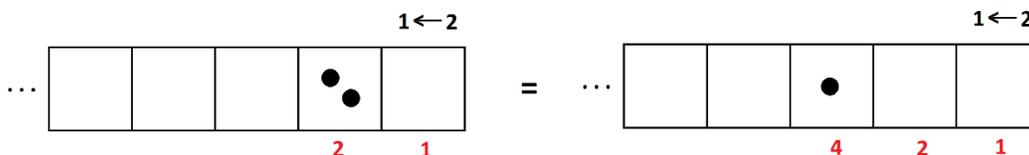
and as we keep adding dots to the machine, we follow the rule

Whenever there are two dots in any one box they “explode,” that is, disappear, and are replaced by one dot, one place to their left.

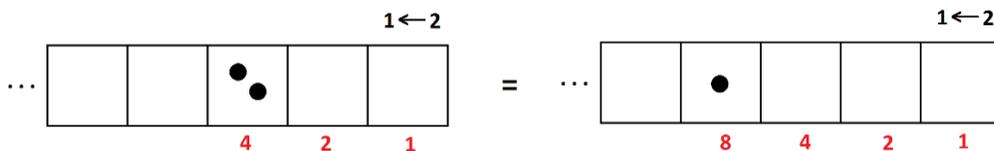
This means, with an explosion, two dots in the rightmost box are equivalent to one dot in the next box to the left. And since each dot in the rightmost box is worth 1, each dot one place over must be worth two 1’s, that is, 2.



And two dots in this second box are equivalent to one dot, one place to the left. Such a dot must be worth two 2’s, that is, worth 4.



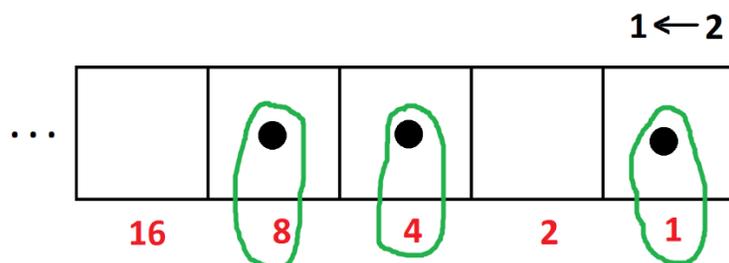
And two 4’s makes 8 for the value of a dot the next box over.



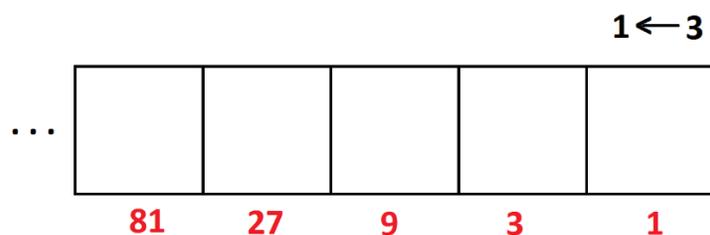
And two 8’s make 16, and two 16’s make 32, and two 32’s make 64, and so on.



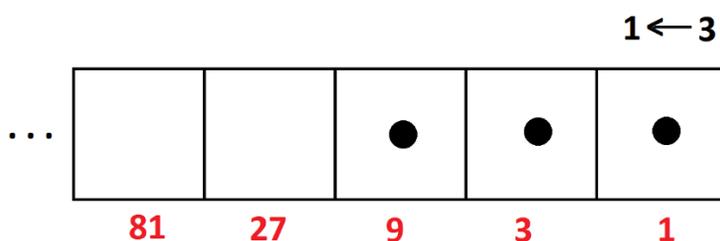
For example, the code for thirteen is 1101 and we can see this is correct by looking at the values of the dots in this machine.



The same idea must be at play for the $1 \leftarrow 3$ machine. Here dots in the rightmost box again are each worth one, but now *three* dots in any one box are equivalent to one dot, one place to the left. We get the dot values in this machine by noting that three 1's is 3, and three 3's is 9, and three 9's is 27, and so on.



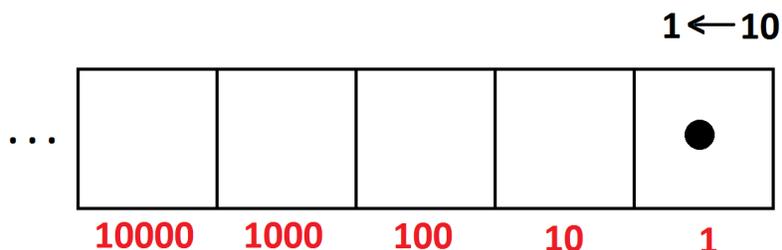
At one point, we said that the $1 \leftarrow 3$ code for thirteen is 111. And we see that this is correct: one 9 and one 3 and one 1 does indeed make thirteen.



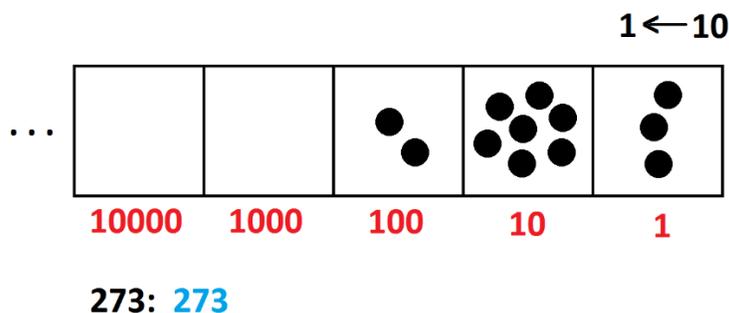
The $1 \leftarrow 3$ machine codes for numbers are called *ternary* or *base three* representations of numbers. Only the three symbols 0, 1, and 2 are ever needed to represent numbers in this system.



And finally, for a $1 \leftarrow 10$ machine, we see that ten ones make 10, ten tens make 100, ten one-hundreds make 1000, and so on. A $1 \leftarrow 10$ has ones, tens, hundreds, thousands, and so on, as dot values.



We saw that the code for the number 273 in a $1 \leftarrow 10$ machine is 273, and this is absolutely correct: 273 is two hundreds, seven tens, and three ones.



In fact, we even speak the language of a $1 \leftarrow 10$ machine. When we write 273 in words, we write

273 = two hundred seventy three

We literally say, in English at least, two HUNDREDS and seven TENS (that “ty” is short for “ten”) and three.

So, through this story of dots and boxes we have discovered *place-value* and *number bases*: base two, base three, base ten, and so on. And society has decided to speak the language of the base ten machine.



Why do you think we humans have a predilection for the $1 \leftarrow 10$ machine? Why do we like the number ten for counting?



One answer could be because of our human physiology: we are born with ten digits on our hands. Many historians do believe this could well be the reason why we humans have favored base ten.

There are some cultures on this planet that have used base twenty. Why might they have chosen that number, do you think?

In fact, there are vestiges of base twenty thinking in western European culture of today. For example, in French, the number 87 is spoken and written as *quatre-vingt-sept*, which translates, word for word, as “four twenties seven.” In the U.S., the famous Gettysburg address begins: “Four score and seven years ago.” The word *score* is an old word for “twenty” and so Abraham Lincoln was saying “four-twenties and seven years ago.” That’s 87 years.

I happen to know that Martians have three fingers on each of two hands. What base do you think they might use in their society? Probably base six.

All right. The point of this lesson has been made. We have discovered base ten place value for writing numbers and seen their context in the whole story of place value. We humans happen to like base ten in particular, because that is the number of fingers most of us seem to have.



Part 5: EXPLAINING THE OPENING PUZZLER

Recall the scenario.

Think of a number between 1 and 26. Look for your number on each of these cards.

GROUP A	GROUP B	GROUP C
1 4 7	2 5 8	3 4 5
10 13 16	11 14 17	12 13 14
19 22 25	20 23 26	21 22 23
GROUP D	GROUP E	GROUP F
6 7 8	9 10 11	18 19 20
15 16 17	12 13 14	21 22 23
24 25 26	15 16 17	24 25 26

Add together the top left number on each of the cards on which your number appears. What do you notice?

You will have likely noticed that the sum of top left numbers of the cards in which a chosen number appears always sums to the chosen number.

For example, the number 14 appears in groups B, C, and E with respective top left numbers 2, 3, and 9, and $2 + 3 + 9 = 14$. And 5 appears in groups B and C with top left numbers 2 and 3, respectively, and $2 + 3 = 5$.

 *The top left numbers of the cards are 1 and 2, 3 and 6, and, 9 and 18. Also, “2” is two copies of 1, and “6” is two copies of 3, and “18” is two copies of 9. What Exploding Dots machine might be at play here, do you think?*

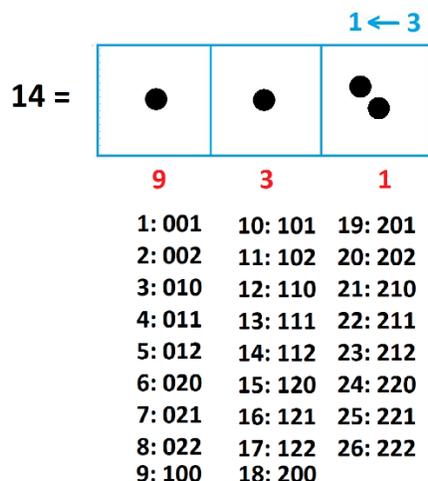
Write the codes of the numbers 1 through 26 in the machine you are thinking of.

Can you now explain what is happening?



GIVING IT AWAY

Here are the codes of the numbers 1 through 26 in a $1 \leftarrow 3$ machine, with leading zeros inserted to make each code three digits long. (Can you see that the number 27 has a code four digits long?)



Now look at the numbers in each group.

Group A contains all the numbers with 1 in the rightmost place of its code and group B all the numbers with a 2 there.

Group C contains all the numbers with 1 in the middle place of its three-digit code and group D all the numbers with a 2 there.

Group E contains all the numbers with 1 in the leftmost place of its code and group F all the numbers with a 2 there.

Also:

The top left number of group A is 1 and of group B double 1.

The top left number of group C is 3 and of group D double 3.

The top left number of group D is 9 and of group F double 9.

The trick works from the fact that, using a $1 \leftarrow 3$ machine, we see that each number from 1 to 26 is a sum of the numbers 1 and 2, 3 and 6, and 9 and 18.



It seems irresistible now to try to use a $1 \leftarrow 4$ machine to make a mind-reading trick. Can you create one?



SOMETHING TOTALLY OPTIONAL

The fun never ceases! If some of your students are game, they might want to delve into this wild exploration.

Extra Wild Exploration:

Here's yet another mindreading trick. It works exactly the same way as the previous tricks (all the mind-reader has to do is again look at the top left corner numbers to each group with a YES answer and sum those numbers to determine the secret number) but this time, a participant is asked to think of a number between -10 and 21 .

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 8 12	-8 -4 9 12	4 -4 20 12	-2 -10 14 6	1 5 9 15
17 21 9 13	-7 -3 8 13	5 -3 21 13	-1 -9 15 7	-1 -5 -9 17
14 18 6 10	-10 -6 6 10	2 -6 18 10	2 -6 18 10	3 7 11 19
15 19 7 11	-9 -5 7 11	3 -5 19 11	3 -5 19 11	-3 -7 13 21

Can you figure out why this trick works?

Hint: Can each number between -10 and 21 be written as a combination of the numbers that appear in the top left corners of the cards?



Adventures in Exploding Dots

Lesson 2 Optional Student Handout

Wild Machines

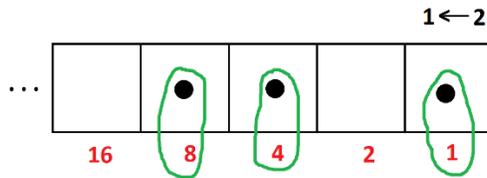
A focus on base ten and why we humans are drawn to it.



OPTIONAL PRACTICE PROBLEMS

In a $1 \leftarrow 2$ machine a pair of dots in any one box is equivalent to a single dot, one box to their left. As dots in the rightmost box are worth 1, dots in subsequent boxes are thus worth 2, 4, 8, 16, and so on.

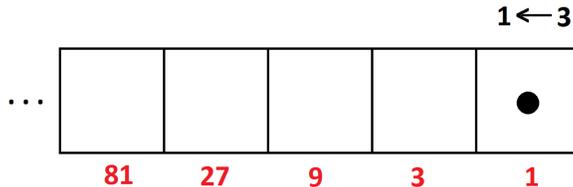
We can see that the $1 \leftarrow 2$ machine code 1101 for the number thirteen, for instance, is correct: thirteen is one 8, one 4, and one 1.



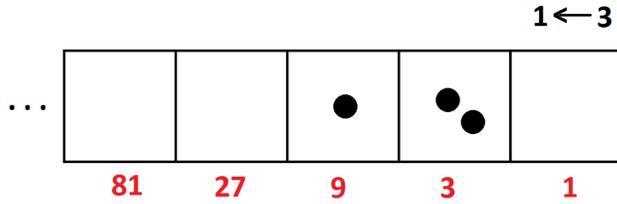
Here are some questions you might, or might not, want to try:

1. What number has $1 \leftarrow 2$ machine code 100101?
2. What is the $1 \leftarrow 2$ machine code for the number two hundred?

In a $1 \leftarrow 3$ machine, three dots in any one box are equivalent to one dot one place to the left. (And each dot in the rightmost box is again worth 1.) We get the dot values in this machine by noting that three 1's is 3, and three 3's is 9, and three 9's is 27, and so on.



3. a) What is the value of a dot in the next box to the left after the ones shown?
b) The $1 \leftarrow 3$ machine code for fifteen is 120. We see that this is correct as one 9 and two 3's does indeed make fifteen.

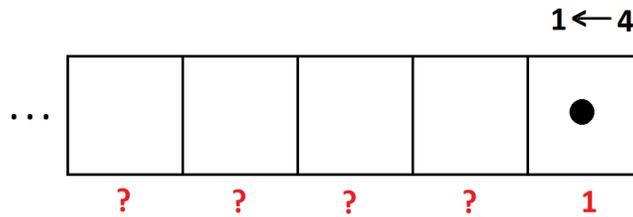


Could we also say that the $1 \leftarrow 3$ code for fifteen is 0120? That is, is it okay to put zeros in the front of these codes? What about zeros at the ends of codes? Are they optional?

Is it okay to leave off the last zero of the code 120 for fifteen and just write instead 12?

- b) What number has $1 \leftarrow 3$ machine code 21002?
- c) What is the $1 \leftarrow 3$ machine code for two hundred?

4. a) In the $1 \leftarrow 4$ system four dots in any one box are equivalent to one dot, one place to their left. What is the value of a dot in each box?



- b) What is the $1 \leftarrow 4$ machine code for twenty-nine?
- c) What number has 132 as its $1 \leftarrow 4$ machine code?

5. I happen to know that *Venutians* have six fingers on each of two hands. What base do you think they might use in their society?



MIND-READING TRICK

For your record, here's the mind-reading trick from today's lesson.

Share with a friend the following six groups of numbers.

GROUP A	GROUP B	GROUP C
1 4 7	2 5 8	3 4 5
10 13 16	11 14 17	12 13 14
19 22 25	20 23 26	21 22 23
GROUP D	GROUP E	GROUP F
6 7 8	9 10 11	18 19 20
15 16 17	12 13 14	21 22 23
24 25 26	15 16 17	24 25 26

Ask your friend to secretly think of a number between 1 and 26 and have him tell you in which groups his secret number lies.

Add together the top left corner numbers of each group he mentions. This sum equals his secret number!

EXAMPLE: If Shankar is thinking of the number 22, he will tell you that he sees his number in groups A, C, and F. And, indeed, $1 + 3 + 18$ equals 22!

GROUP A	GROUP B	GROUP C
① 4 7	2 5 8	③ 4 5
10 13 16	11 14 17	12 13 14
19 22 25	20 23 26	21 22 23
GROUP D	GROUP E	GROUP F
6 7 8	9 10 11	⑱ 18 19 20
15 16 17	12 13 14	21 22 23
24 25 26	15 16 17	24 25 26



SOMETHING TOTALLY OPTIONAL

The fun never ceases! If you are game, might you want to delve into this wild exploration?

Extra Wild Exploration:

Here's yet another mindreading trick. It works exactly the same way as the previous tricks (all the mind-reader has to do is again look at the top left corner numbers to each group with a YES answer and sum those numbers to determine the secret number) but this time a participant is asked to think of a number between -10 and 21 .

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E
16 20 8 12	-8 -4 9 12	4 -4 20 12	-2 -10 14 6	1 5 9 15
17 21 9 13	-7 -3 8 13	5 -3 21 13	-1 -9 15 7	-1 -5 -9 17
14 18 6 10	-10 -6 6 10	2 -6 18 10	2 -6 18 10	3 7 11 19
15 19 7 11	-9 -5 7 11	3 -5 19 11	3 -5 19 11	-3 -7 13 21

Can you figure out why this trick works?





Adventures in Exploding Dots

Lesson 2 Student Handout SOLUTIONS

Wild Machines

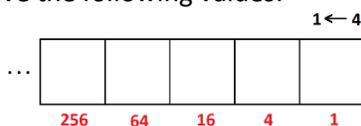
A focus on base ten and why we humans are drawn to it.



Solutions to Optional Practice

1. Thirty-seven. It's a 32 and a 4 and a 1.
2. 11001000
3. a) Each dot in the next box to the left is worth three 81's, that's 243.
b) Yes it is okay to insert a zero at the front of the code. This would say that there are no 27's, which is absolutely correct. Deleting the end zero at the right, however, is problematic. 120 is the code for fifteen (one 9 and two 3's) but 12 is the code for five (one 3 and two 1's).
c) One hundred and ninety one. (Two 81's, one 27, and two 1's.)
d) 21102

4. a) For a $1 \leftarrow 4$ machine, boxes have the following values:



- b) The number twenty-nine has code 131 in a $1 \leftarrow 4$ machine.
c) Thirty. (This is one more than the code for twenty-nine!)
5. Might Venutians use base twelve? This means they will need twelve different symbols for writing numbers.

By the way, have you noticed that we use ten different symbols – 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 – which we call *digits*. (We call our fingers *digits* t