Images of Straight Lines

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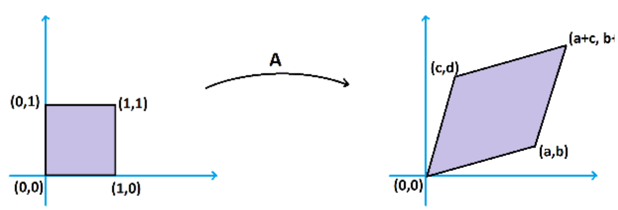
Suppose a transformation of the plane is represented by a matrix

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

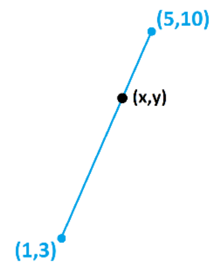
When we studied the geometric interpretation of the matrix, we examined the image of the fundamental unit square in the plane. We pinpointed the image of the four points $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, and assumed the straight edges of the square were mapped to straight edges connecting these image points.



Is this truly the case?

OUR CHALLENGE: Prove that a transformation given by a matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ maps a line segment connecting two points to the line segment that connects the image of those two points.

For example, consider this line segment:



To go from $(1,3)$ to $(5,10)$ we need to shift four units to the right and seven units upward. Any point (x, y) on this line is some fraction of this shift. Thus, we have

$$\begin{aligned} x &= 1 + 4t \\ y &= 3 + 7t \end{aligned}.$$

[Check: For $t = 0$ this is the point $(1,3)$ and for $t = 1$ this is the point $(5,10)$. Yes!]

Then

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1+4t \\ 3+7t \end{pmatrix} = \begin{pmatrix} a+3c+(4a+7c)t \\ b+3d+(4b+7d)t \end{pmatrix}$$

which is a point on the line segment connecting

$$\begin{pmatrix} a+3c \\ b+3d \end{pmatrix} = A\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

to

$$\begin{pmatrix} 5a+10c \\ 5b+10d \end{pmatrix} = A\begin{pmatrix} 5 \\ 10 \end{pmatrix},$$

as hoped.

There is nothing special about the endpoints $(1,3)$ and $(5,10)$ here.