Chapter 13

Trusting Patterns and Not Trusting Patterns

Contents

101. Trusting Patterns

If you trust patterns, what would you predict for the next number in this sequence of numbers?

2 5 8 11 14 17 20 ___

No doubt you are noticing that the numbers increase by 3 from one to the next suggesting that the next term of the sequence will be 23.

2 5 8 11 14 17 20 23
3 3 3 3 3 3 3

Of course, the there is no reason to believe that the differences will forever be 3, but if one is forced to make a guess for the next number, 23 seems like a good one.

Not all sequences have a constant difference between terms but could still appear to have structure to them.

Example: Make a reasonable guess as to the next number in this sequence.

2 3 6 11 18 27 38 ___

Answer: This time we notice that the "second differences," the difference of the differences between terms, appears to be constant. We can guess that the next difference after 11 will be $11 + 2 = 13$, and thus the term after 38 in the original sequence will be $38 + 13 = 51$.

2 3 6 11 18 27 38 51
1 3 5 7 9 11 13
→ 2 2 2 2 2 2

We’ve seen a sequence with constant first differences and one with constant second differences. A sequence such as

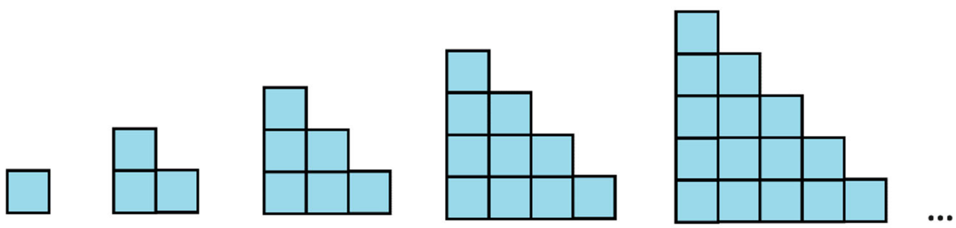
17 17 17 17 17 17 17 ...

appears constant from the get-go.

Practice 101.1 Make an intelligent guess as to the next number in this sequence.

3 5 11 21 35 __

Practice 101.2 Consider the following sequence of diagrams each made of squares 1 unit wide.



If the implied geometric pattern continues …

a) What would be the area of the tenth figure?
b) What would be the perimeter of the tenth figure?

Practice 101.3 Show that the third differences seem to be constant for the following sequence. Use this to make an educated prediction for the next number in the sequence.

0 2 20 72 176 350 612 ___

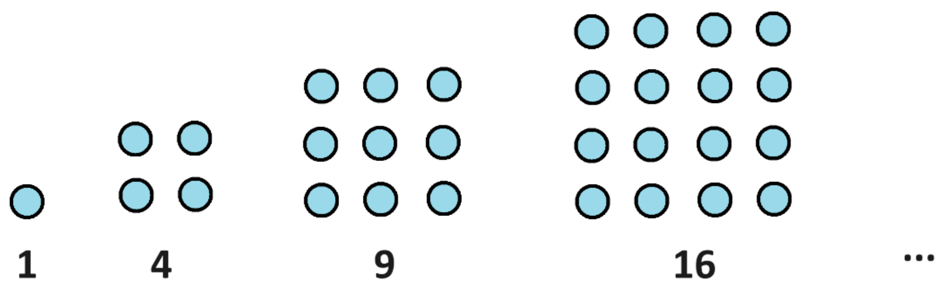
Practice 101.4 Consider the sequence of the doubling numbers:

1 2 4 8 16 32 64 128 ...

Is there a row of differences that is constant?

Practice 101.5

a) The sequence of square numbers begins $1, 4, 9, 16, 25, 36, 49, \ldots$.
(The nth square number is $n \times n = n^2$.)



Is there a row in the “difference table” of the square numbers that is constant?

b) The sequence of cube numbers begins $1, 8, 27, 64, 125, 216, 343, \ldots$.
(The nth cube number is $n \times n \times n = n^3$.)

Is there a row in the difference table of the cube numbers that is constant?

Practice 101.6 Use differences to make an educated guess as to the next element of this sequence:

-1 4 7 8 7 4 -1 ____

Leading Diagonals

Here's a sequence that begins $0, 16, 50, 108, \ldots$ and its entire **difference table**: the row of the differences between terms, the row of the differences of the differences, the row of the differences of the differences of the differences, and so on.

If we are trusting patterns, if looks like the third row of differences is constant and that the table is nothing but zeros thereafter.

0 16 50 108 196 320 486 700 …
16 34 58 88 124 146 214 …
18 24 30 36 42 48 …
6 6 6 6 6 …
0 0 0 0 …
0 0 0 …
0 0 …
$\ddots$

The numbers at the front of each row form the **leading diagonal** of the table. In this example, the leading diagonal of the table is $0, 16, 18, 6, 0, 0, 0, \ldots$.

Practice 101.7 Here is the leading diagonal of a difference table.

1 __ __ __ __ __ __ __ …
0 __ __ __ __ __ __ …
2 __ __ __ __ __ …
4 __ __ __ __ …
5 __ __ __ …
1 __ __ …
2 __ …
2 …
$\ddots$

Fill in all the blank spaces of the table.

Working on Problem 101.7, we see that the second entry of the top row (the second entry of the blue sequence) must “go up from 1 by 0” and so be 1.

The second entry on the first row of differences must go up from 0 by 2 and so be 2.

This then allows us to deduce that the third entry of the top row is “1 up by 2” and is 3.

And so on.

One can reconstruct the entire table this way.

1 1 3 — — — — — …
0 2 — — — — — …
2 — — — — — …
4 — — — — …
5 — — — …
1 — — …
2 — …
2 …
⋱

Practice 101.8 What sequence has 0, 0, 1, 0, 0, 0, 0, 0, 0, … as its leading diagonal?

0 — — — — — — — …
0 — — — — — — …
1 — — — — — …
0 — — — — …
0 — — — …
0 — — …
0 — …
0 …
⋱

Practice 101.9 What sequence has nothing by 1s for its leading diagonal?

1 — — — — — — — …

1 — — — — — — …

1 — — — — — …

1 — — — — …

1 — — — …

1 — — …

1 — …

1 …

⋱

We're seeing that the leading diagonal of a difference table encodes all the information in the table, including its top line, the original sequence.

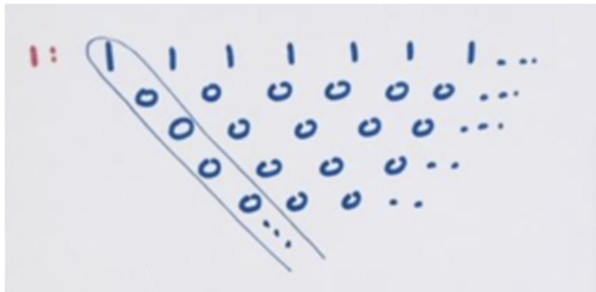
To know the leading diagonal of a sequence is to know everything about the sequence and its table.

Practice 101.10 Going back to problem 101.5 …

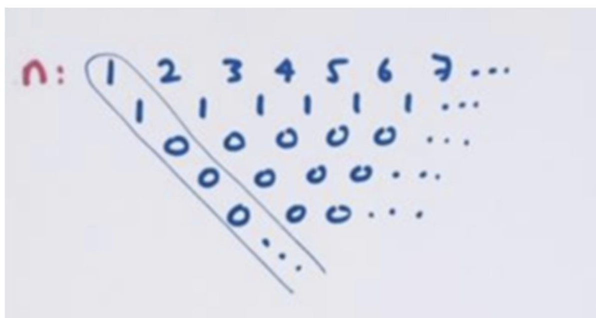
a) What is the leading diagonal of the sequence of square numbers: 1, 4, 9, 16, …?
b) What is the leading diagonal of the sequence of cube numbers: 1, 8, 27, 64, …?

Getting to Know Some Standard Leading Diagonals

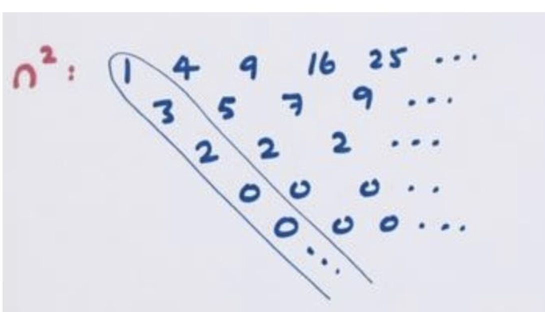
Here's the constant sequence of 1s and its leading diagonal. (I've started drawing these things by hand!)

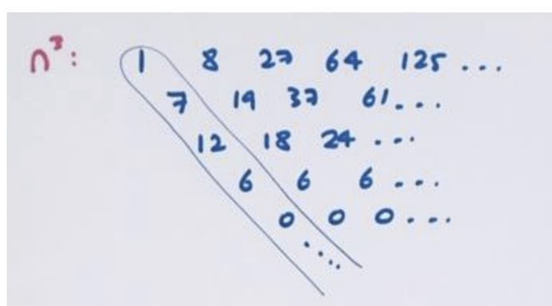


And here's the sequence of counting numbers and its leading diagonal. The nth counting number is just n.



And if you tried problem 101.10 you would have found the following leading diagonals for the square numbers (with nth term n^2) and the cube numbers (with nth term n^3).





Practice 101.11 Do you care to work out the leading diagonal of the sequence of fourth powers $1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, \ldots$.? (Here the nth term of the sequence is n^4.)

MUSINGS

Musing 101.12
a) What is the leading diagonal of the sequence of tripling numbers: 1, 3, 9, 27, 81, 243, 729, ….?

b) What is the leading diagonal of the sequence of quadrupling numbers:
1, 4, 16, 64, 256, 1024, 4096, ….?

c) Make as guess as to what the leading diagonal of the sequence of quintupling numbers
1, 5, 25, 125, 625, 30125, …. is going to be. Is it?

Musing 101.13 The leading diagonal of a sequence is 1, 10, 100, 1000, 10000, …, the powers of ten. What is the sequence?

102. Getting Formulas for Patterns you Trust

Here's an idea.

Consider the sequence given by the formula $n^2 + n$.

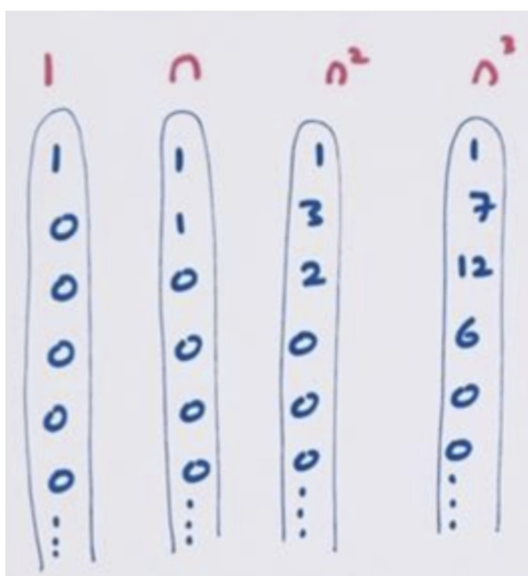
Its first term is	$1^2 + 1 = 1 + 1 = 1$
Its second term is	$2^2 + 2 = 4 + 2 = 6$
Its third term is	$3^2 + 3 = 9 + 3 = 12$
Its fourth term is	$4^2 + 4 = 16 + 4 = 20$

and so on.

$n^2 + n$: **2 6 12 20 30 42 56 72 90 110 ...**

Is the leading diagonal of this sequence at all related to the leading diagonals for the n^2 sequence and the n sequence?

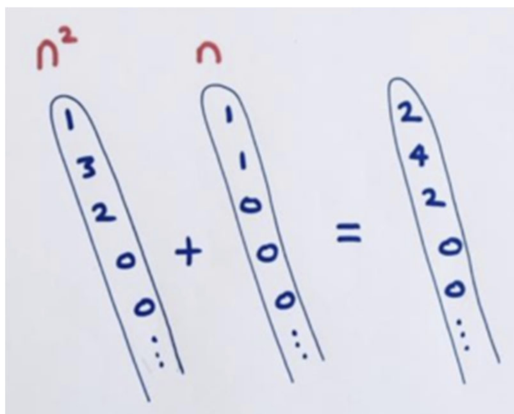
Practice 102.1 Compute the leading diagonal of the sequence with terms given by $n^2 + n$. Does it relate in any way with any of the standard leading diagonals? Perhaps just with the diagonals for n and n^2?



Try this before reading on.

Here's the leading diagonal for the sequence with terms given by $n^2 + n$.

Did you catch this connection to the two standard diagonals?



The leading diagonal for the sequence $n^2 + n$ is "the sum" of the leading diagonal for n^2 and the leading diagonal for n.

The top entries for n^2 and n add to make the top entry for $n^2 + n$.
The second entries for n^2 and n add to make the second entry for $n^2 + n$.
The third entries for n^2 and n add to make the third entry for $n^2 + n$.
The fourth entries for n^2 and n add to make the fourth entry for $n^2 + n$.
And so on.

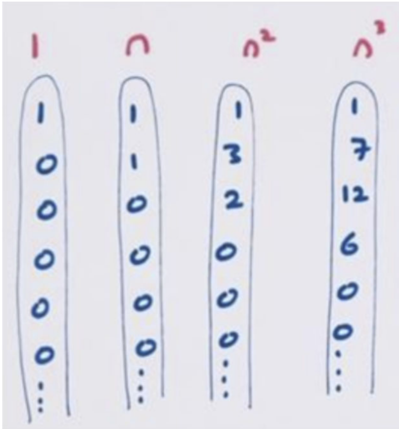
Comment: I am starting to get a bit loose with my language. Rather than write "the leading diagonal of the sequence with terms given by the formula $n^2 + n$," for instance, I am just writing "the leading diagonal of $n^2 + n$." I hope the meaning of this shorthand is clear enough.

This example suggests a strategy for finding a formula for a given sequence.

When handed a sequence of numbers …

1. Compute its difference table and find its leading diagonal.

2. Try to recognize that leading diagonal as a combination of some of the standard leading diagonals we know. This will suggest a possible formula for the sequence of numbers.



There is no reason to believe that our candidate formula must work. But since we are only told a finite number of terms of the sequence, we can …

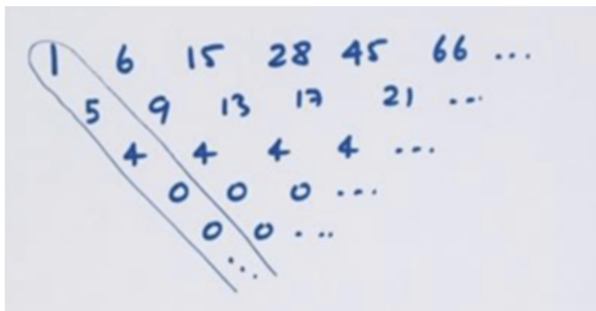
3. Check if the candidate formula works for each of the numbers handed to us.

Practice 102.2 OPTIONAL
Do you care to add to our picture the leading diagonals for n^4 and n^5 and n^6.
(The answer can be no!)

Example: Find a formula that seems to fit the numbers in this sequence.
Use your formula to make an educated guess as to what the seventh number in the sequence might be.

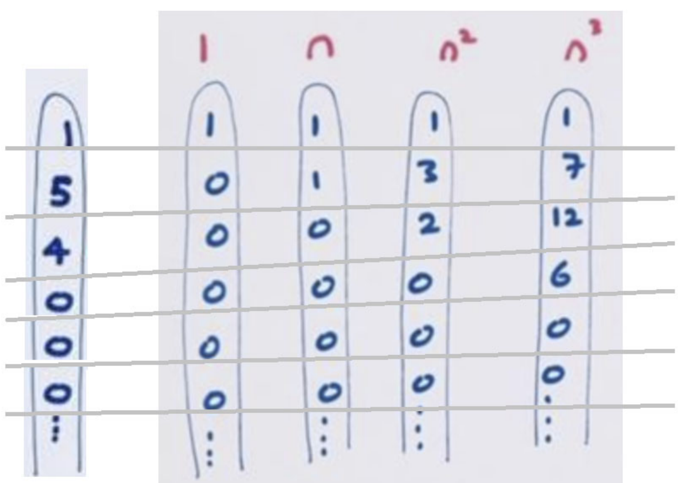
1 6 15 28 45 66 ...

Answer: Here's the difference table of the sequence.



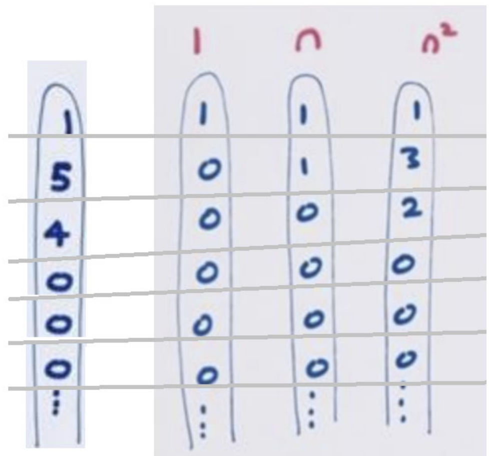
Since we're playing the game of trusting patterns, let's assume the pattern of zeros continues.

Now we ask: Is the leading diagonal we see—1, 5, 4, 0, 0, 0, ….—a combination of our standard diagonals?

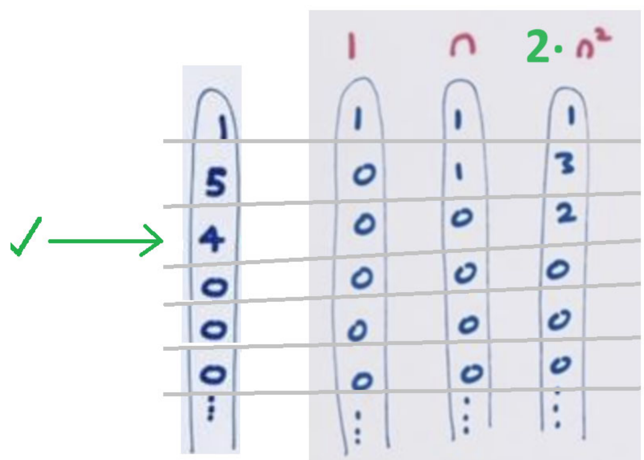


Observe that our leading diagonal has 0s in the fourth position onwards. It seems unlikely then that we will need the n^3 diagonal (or the diagonal from any higher power) since they have numbers other than 0 in those positions.

Let's see if we can just use the diagonals for 1, n, and n^2.

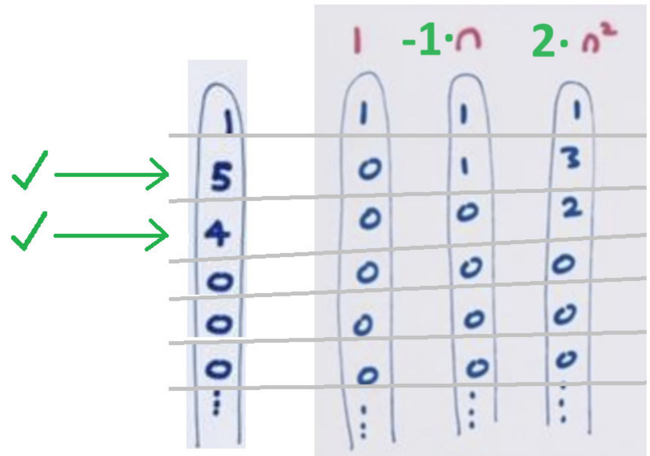


Our leading diagonal has the number 4 in the third position and only the diagonal for n^2 has a non-zero entry in that third spot. It looks like we will need two copies of n^2 to make the third row match up.



But now look at the second row.
We need the number 5. But we've got 6 from double three.

It looks like we need -1 copies of the n diagonal.



So, right now the third row is good:

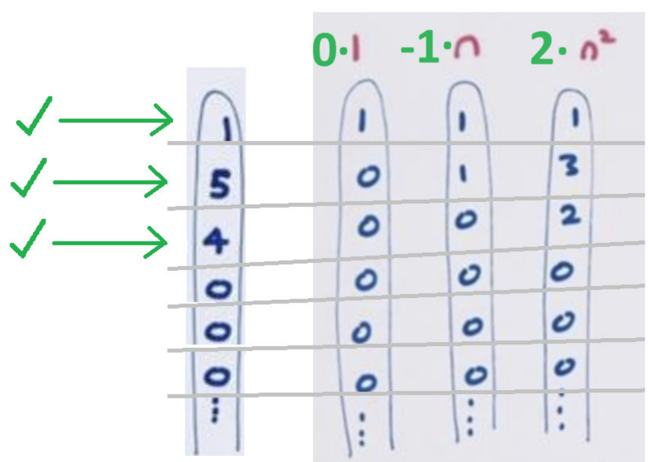
> 4 is two copies of 2 and negative one copies of 0.

The second row is good:

> 5 is two copies of 3 and negative one copies of 1

Now we have to attend to the top row.

We want the number 1 and right now we have two copies of 1 and negative one copies of 1, which does make 1. We lucked out! We don't need any copies of the first diagonal.



Question: Can you see that each row of zeros is "good" too? (Luckily, combinations of zero still give zero.)

We have a candidate formula for our sequence: $2n^2 + (-1) \cdot n$, which most people write in terms of subtraction.

$$2n^2 - n$$

Does this formula work?

For $n = 1$ it gives $\quad 2 \times 1 - 1 = 1$
For $n = 2$ it gives $\quad 2 \times 4 - 2 = 6$
For $n = 3$ it gives $\quad 2 \times 9 - 3 = 15$
For $n = 4$ it gives $\quad 2 \times 16 - 4 = 28$
For $n = 5$ it gives $\quad 2 \times 25 - 5 = 45$
For $n = 6$ it gives $\quad 2 \times 36 - 6 = 66$

It's good!

To finish off the question, we now predict,

For $n = 7$ we'll have $\quad 2 \times 49 - 7 = \mathbf{91}$

Practice 102.3 Practice this technique to show that the following sequence of numbers seems to fit the formula $n^2 - 3n + 4$.

2 2 4 8 14 22 33 ...

Practice 102.4 Use our technique to show that the following sequence of numbers seems to fit the formula $n^3 - 3n^2 + 4n - 1$.

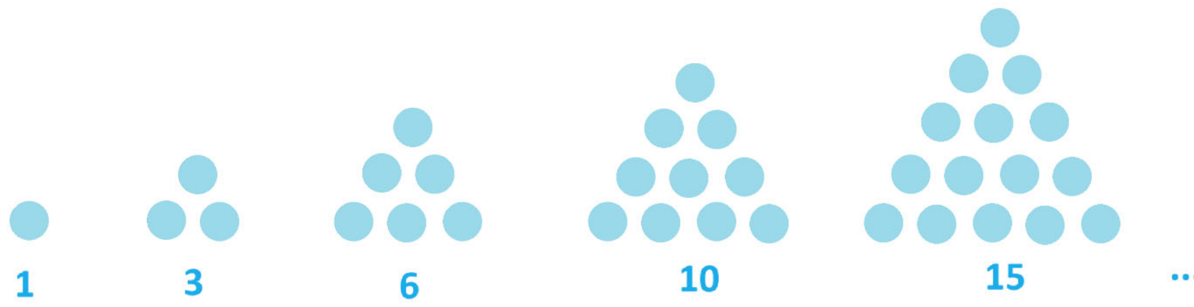
0 2 10 30 68 130 222 ...

What does the formula predict for the 100th number in the sequence?

Practice 102.5 Write down a formula that seems to the nth term of this sequence.

3 3 3 3 3 3 3 3 3 ...

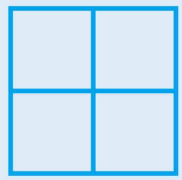
Practice 102.6 We disused the **triangular numbers** in Chapter 2 and even described a formula for them back in those early days.



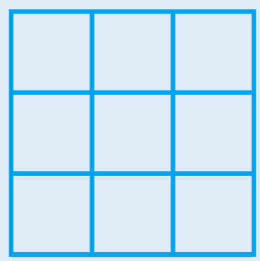
Rediscover that formula for the triangular numbers using our difference methods. (Be open to the appearance of fractions!)

MUSINGS

Musing 102.7 One can find $\mathbf{5}$ squares it this two-by-two grid of squares: **four** one-by-one squares and **one** two-by-two square.



a) Can you see 14 squares in this three-by-three grid: **nine** one-by-one squares, **four** two-by-two squares, and **one** three-by-three square?



b) Can you identify $\mathbf{30}$ squares in a four-by-four grid and $\mathbf{55}$ squares in a five-by-five grid?

c) How many squares are there to be found in a six-by-six grid?

There is $\mathbf{1}$ one square in a one-by-one grid.
We have so far, the sequence of numbers

1 5 14 30 55 ...

d) Trusting patterns, make a prediction for the next number in the sequence.

e) Trusting patterns, find a formula for the numbers in this sequence.

f) Make a guess as to the sum of the first one hundred square numbers.

$$1 + 4 + 9 + 16 + 25 + 36 + \cdots + 9{,}801 + 10{,}000$$

MECHANICS PRACTICE

Practice 102.8 Find formulas that fit the numbers in as many of these sequences you have the patience for.

a) $\mathbf{5 \quad 8 \quad 11 \quad 14 \quad 17 \quad 20 \quad 23}$ ...

b) $\mathbf{1 \quad 3 \quad 15 \quad 43 \quad 93 \quad 171 \quad 283}$ ...

c) $\mathbf{1 \quad 0 \quad 1 \quad 10 \quad 33 \quad 76 \quad 145 \quad 246 \quad 385}$ ...

d) $\mathbf{3 \quad 3 \quad 7 \quad 21 \quad 51 \quad 103 \quad 183 \quad 297}$ ...

e) $\mathbf{0 \quad 9 \quad 24 \quad 45 \quad 72 \quad 105 \quad 144 \quad 189 \quad 240}$ ...

f) $\mathbf{6 \quad 24 \quad 60 \quad 120 \quad 210 \quad 336}$ ...

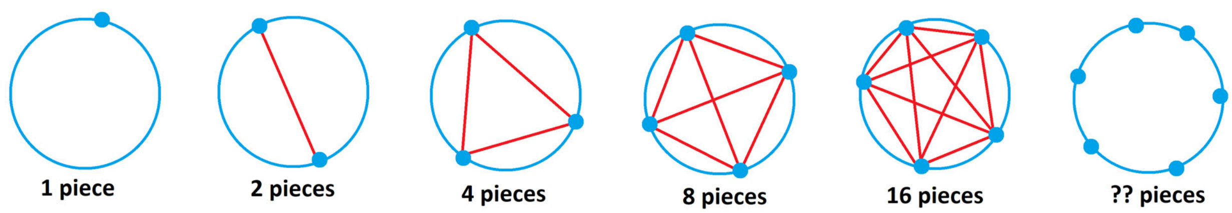
g) $\mathbf{230 \quad 275 \quad 324 \quad 377 \quad 434 \quad 495}$ ...

103. But we Can’t Trust Patterns

We’ve already seen that one should be wary of patterns. For instance, in Chapter 2 we saw that prime numbers can be tricksters.

Here’s my favorite example of compelling pattern that turns out to be utterly false.

> *Draw circles with 1 dot, 2 dots, 3 dots, and so on, marked on their rims.*
>
> *Within each circle, connect each pair of dots with a line segment to divide the circle interior into pieces (like chopping up a circular pizza).*
>
> *How many pieces result?*



When one plays with this puzzle, the pattern 1 piece, 2 pieces, 4 pieces, 8 pieces, 16 pieces, appears.

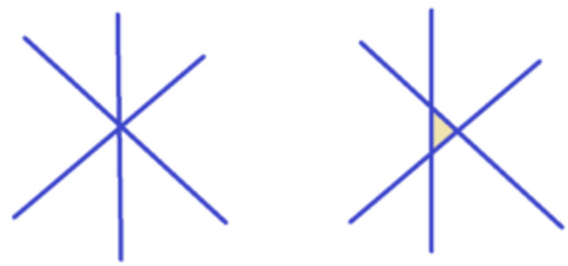
> **Practice 103.3** Do you count 16 pieces for the circle with five dots on its rim? Some of them are quite slim.

It is utterly compelling to guess that the next picture, with 6 dots on the rim of a circle, will show 32 pieces.

> **Practice 103.4** Draw six dots on the boundary of a large circle. Connect each pair of dots with a line segment. (There should be five segments “emanating” from each dot.)
>
> Count the number of pieces formed. What count do you get?

If you are careful with your work, you will either see 30 or 31 pieces. (Watch out for “false” regions that arise from lines unnecessarily crossing each other near the boundary dots.)

Why the possible difference in counts? Well, it depends on whether you placed your dots symmetrically about the circle. If you did that, you would have three lines all passing through the center of the circle and count only 30 regions. If, on the other hand, you distributed the dots unsymmetrically, those three lines won’t meet at a common point and create one extra region for you.



Practice 103.5 How many pieces result from a diagram with **7** dots on the rim of a circle? (Assume the dots are distributed about the boundary so that no three line-segments pass through a common point. That is, let’s assume we have the largest number of pieces possible.)

What’s the (maximal) count of regions for **8** dots? **9** dots?

Mathematics is the practice and art of looking for structure and pattern, and then explaining why those patterns are or are not true. (Mathematics then works to extend and connect structures found to new realms and new heights.)

Mathematicians will likely be excited if they detect a pattern in a scenario and deeply motivated to push and probe and think deeply about the situation even further because of the pattern. But they will never trust a patten they see until they can find and describe a logical reason for it.

As we have just seen, patterns can be tricksters!

MUSINGS

Musing 103.6 If you tried problem 103.5 you will see that the (maximal) count of regions for increasing numbers of dots on the boundary of a circle follows this sequence of numbers:

1 2 4 8 16 31 57 99 163 ...

These numbers are not the doubling numbers. But is there another pattern to them?

a) Use a difference table to predict the next number in the sequence. (Check if you get 256 for the next number in the sequence. This is the maximal number of regions that arise with ten dots on the boundary of the circle.)

b) Care to write a formula for these “piece numbers” based on the leading diagonal you see in part a)?

Of course, there is no reason to believe that the pattern we are seeing here with the fourth differences being constant is a true pattern. This pattern might eventually break down too! (Unless you have a reason to believe it won’t. Do you?)

104. Messing with Peoples' Minds

If asked to predict the fifth number in this sequence, most people would say 10.

$$2 \quad 4 \quad 6 \quad 8 \quad \_\_$$

We can even give a formula for this sequence: the nth term of the sequence seems to be $2n$. Check:

For $n = 1$ we get $2 \times 1 = 2$
For $n = 2$ we get $2 \times 2 = 4$
For $n = 3$ we get $2 \times 3 = 6$
For $n = 4$ we get $2 \times 4 = 8$

And this does suggest that the 5^{th} term is $2 \times 5 = 10$.

Practice 104.1
a) Consider the formula

$$2n \quad + \quad \frac{7}{24} \times (n-1) \times (n-2) \times (n-3) \times (n-4)$$

What does this expression give if n is 1?
What does it give for $n = 2$? For $n = 3$? For $n = 4$?

Finally, what does it give if n is 5?

This practice problem shows that we could well argue that the next term of our sequence is 17. We even presented a formula to "prove" we are right!

$$2 \quad 4 \quad 6 \quad 8 \quad \underline{17}$$

Practice 104.2

a) Show that the formula

$$2n \quad + \quad \frac{-13}{24} \times (n-1) \times (n-2) \times (n-3) \times (n-4)$$

suggests that -3 is the natural next number after 2, 4, 6, and 8.

2 4 6 8 -3

b) Write down a formula that suggests that $1\frac{1}{2}$ is the natural next number to write down after 2, 4, 6, and 8.

2 4 6 8 $1\frac{1}{2}$

Practice 104.3 Write down a formula in terms of n that gives the numbers 3, 6, and 9 for n equal to 1, 2, and 3, respectively, and the number 127 for $n = 4$.

3 6 9 127

These practice problems show that if you have a formula that works for the first few terms of a sequence you can always to add an extra expression to the formula that

a) does not affect the first few terms of the sequence
b) makes the next term of the sequence any crazy number you want it to be!

This can really mess with people's sense of order and structure!

Practice 104.4 The nth square number is given by n^2.

Write down a formula in terms of n that gives the first four square numbers 1, 4, 9, 16, and 25 for n equal to 1, 2, 3, 4, and 5 in turn, and then gives 37 for $n = 6$.

1 4 9 16 25 <u>37</u>

This work shows that one should never trust patterns on face value.

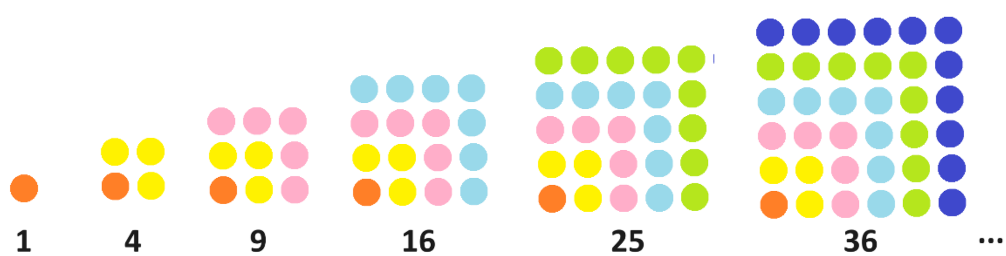
Even if you have a formula that "justifies" the pattern for the numbers you see, we now know that it is not the only possible formula that works, and that the sequence could continue in many crazy ways (and in ways that can still be "justified" with formulas).

Of course, when mathematicians detect a pattern, they are excited by it and are motivated by it. But they won't trust it until they have solid logical reasons to believe it is true.

Practice 104.5 We saw that the square numbers have differences that are the odd numbers (which then have a constant difference of 2).

1 4 9 16 25 36 ...
3 5 7 9 11 ...
2 2 2 2 ...

Is this a real pattern? In going from one square number to the next, do the values really increase by the odd numbers one after another?



MUSINGS

Musing 104.6 Would you dare give 17 as the next number on this "intelligence quiz" question even if you could write a formula to justify your answer?

2 4 6 8 ?

MECHANICS PRACTICE

Practice 104.7 a) Use leading diagonals to find a formula that fits this sequence of numbers.

-1 1 5 11 19 29 ...

b) What are the values of the first seven terms of the sequence given by this formul5

$$n^2 - n - 1 + \frac{-42\frac{1}{2}}{720}(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$$

Practice 104.8 Create a formula that generates this sequence of numbers.

1 2 3 4 $\sqrt{2}$

105. How to Spell your Name in Math

My name is JAMES.

The first letter in my name is **J**, which is the $\mathbf{10}$th letter of the alphabet.
The second letter in my name is **A**, which is the $\mathbf{1}$st letter of the alphabet.
The third letter in my name is **M**, which is the $\mathbf{13}$th letter of the alphabet.
The fourth letter in my name is **E**, which is the $\mathbf{5}$th letter of the alphabet.
The fifth letter in my name is **S**, which is the $\mathbf{19}$th letter of the alphabet.

Consequently, the set of numbers that represents my name is:

10 1 13 5 19

I am now going to write down a formula that generates each of these numbers in turn.

The formula is going to look shockingly scary. You will likely have an emotional reaction in response to seeing it, but—deep breath—it will be okay. We'll work our way through it.

Here's the formula for JAMES. It comes in five "chunks" added together.

$$\mathbf{10}\times\frac{(n-2)(n-3)(n-4)(n-5)}{(-1)(-2)(-3)(-4)} + \mathbf{1}\times\frac{(n-1)(n-3)(n-4)(n-5)}{(1)(-1)(-2)(-3)} + \mathbf{13}\times\frac{(n-1)(n-2)(n-4)(n-5)}{(2)(1)(-1)(-2)}$$
$$+ \mathbf{5}\times\frac{(n-1)(n-2)(n-3)(n-5)}{(3)(2)(1)(-1)} + \mathbf{19}\times\frac{(n-1)(n-2)(n-3)(n-4)}{(4)(3)(2)(1)}$$

Some questions:

- Do you see the numbers 10, 1, 13, 5, and 19 within this ghastly formula?
- Do you see that each chunk is a whole number times a fraction?
- Do you see that the denominator of each fraction is a product of four numbers?
- Try putting $n = 3$ into the formula. Do you see that only the third chunk "survives" and that all the other chunks have value 0?

Here's my answer to that fourth question. If n is the number 3, then the formula reads

$$\mathbf{10} \times \frac{(3-2)\overbrace{(3-3)}^{0}(3-4)(3-5)}{(-1)(-2)(-3)(-4)} \;+\; \mathbf{1} \times \frac{(3-1)\overbrace{(3-3)}^{0}(3-4)(3-5)}{(1)(-1)(-2)(-3)} \;+\; \mathbf{13} \times \frac{(3-1)(3-2)(3-4)(3-5)}{(2)(1)(-1)(-2)}$$
$$+\; \mathbf{5} \times \frac{(3-1)(3-2)\underbrace{(3-3)}_{0}(3-5)}{(3)(2)(1)(-1)} \;+\; \mathbf{19} \times \frac{(3-1)(3-2)\underbrace{(3-3)}_{0}(3-4)}{(4)(3)(2)(1)}$$

In four of the chunks, the fraction has a numerator composed of the product of four numbers with one of the numbers being 0. So, four of the fraction numerators are 0.

$$\mathbf{10} \times \frac{0}{(-1)(-2)(-3)(-4)} \;+\; \mathbf{1} \times \frac{0}{(1)(-1)(-2)(-3)} \;+\; \mathbf{13} \times \frac{(3-1)(3-2)(3-4)(3-5)}{(2)(1)(-1)(-2)}$$
$$+\; \mathbf{5} \times \frac{0}{(3)(2)(1)(-1)} \;+\; \mathbf{19} \times \frac{0}{(4)(3)(2)(1)}$$

So, when $n = 3$, our formula becomes an expression with only the third term "surviving."

$$\mathbf{0} \;+\; \mathbf{0} \;+\; \mathbf{13} \times \frac{(3-1)(3-2)(3-4)(3-5)}{(2)(1)(-1)(-2)} \;+\; \mathbf{0} \;+\; \mathbf{0}$$

- That third term is a whole number times a fraction. Do you see that the denominator of the fraction was designed to perfectly match the numerator of the fraction for this case with $n = 3$?

- Do you see that for $n = 3$ we are getting the value

  $$0 \;+\; 0 \;+\; 13 \times 1 \;+\; 0 \;+\; 0$$

  which is 13?

Practice 105.1 Conduct similar reasoning to show that the formula gives the value 10 for $n = 1$.

$$\mathbf{10} \times \frac{(\boldsymbol{n}-\mathbf{2})(\boldsymbol{n}-\mathbf{3})(\boldsymbol{n}-\mathbf{4})(\boldsymbol{n}-\mathbf{5})}{(-\mathbf{1})(-\mathbf{2})(-\mathbf{3})(-\mathbf{4})} + \mathbf{1} \times \frac{(\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}-\mathbf{3})(\boldsymbol{n}-\mathbf{4})(\boldsymbol{n}-\mathbf{5})}{(\mathbf{1})(-\mathbf{1})(-\mathbf{2})(-\mathbf{3})} + \mathbf{13} \times \frac{(\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}-\mathbf{2})(\boldsymbol{n}-\mathbf{4})(\boldsymbol{n}-\mathbf{5})}{(\mathbf{2})(\mathbf{1})(-\mathbf{1})(-\mathbf{2})}$$
$$+ \mathbf{5} \times \frac{(\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}-\mathbf{2})(\boldsymbol{n}-\mathbf{3})(\boldsymbol{n}-\mathbf{5})}{(\mathbf{3})(\mathbf{2})(\mathbf{1})(-\mathbf{1})} + \mathbf{19} \times \frac{(\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}-\mathbf{2})(\boldsymbol{n}-\mathbf{3})(\boldsymbol{n}-\mathbf{4})}{(\mathbf{4})(\mathbf{3})(\mathbf{2})(\mathbf{1})}$$

Show that it also gives the value 1 for $n = 2$, and the value 5 for $n = 4$, and the value 19 for $n = 5$.

This formula is indeed "spelling" my name! For n taken to be 1, 2, 3, 4, and 5 in turn it gives 10, 1, 13, 5, and 19, the letters of my name, in response.

We can see how this formula was crafted.

- It's a formula in n that comes in five "chunks," one for each letter of JAMES we are trying to encode.

- Each chunk is a whole number (representing a letter of my name) times a fraction.

- The numerator of each fraction in a chunk is designed to be 0 for all values of n, except one: the one that corresponds to the position number of the letter in my name.

  For example, E is the **fourth** letter in my name and the fraction in the chunk for the letter E has numerator $(n-1)(n-2)(n-3)(n-5)$. This vanishes if n is 1, 2, 3, or 5, but "survives" if n is 4.

- The denominator of each fraction is designed to make the fraction equal to 1 for when the chunk survives.

  For example, focusing on E still, if n is the value 4, then $(n-1)(n-2)(n-3)(n-5)$ is the number $(3)(2)(1)(-1)$. We made this the denominator of the fraction in that fourth chunk.

One can craft a formula that "spells" any name—or word—you like!

Practice 105.2 Family back in Australia know me as JIM.

a) Analyze the formula below. Make sure you understand how it "spells" my familial Aussie name.

$$10 \times \frac{(n-2)(n-3)}{(-1)(-2)} \;+\; 9 \times \frac{(n-1)(n-3)}{(1)(-1)} \;+\; 13 \times \frac{(n-1)(n-2)}{(2)(1)}$$

b) We can make this formula a little friendlier by evaluating the denominators. Do you see that this formula can be rewritten as follows?

$$5(n-2)(n-3) - 9(n-1)(n-3) + \frac{13}{2}(n-1)(n-2)$$

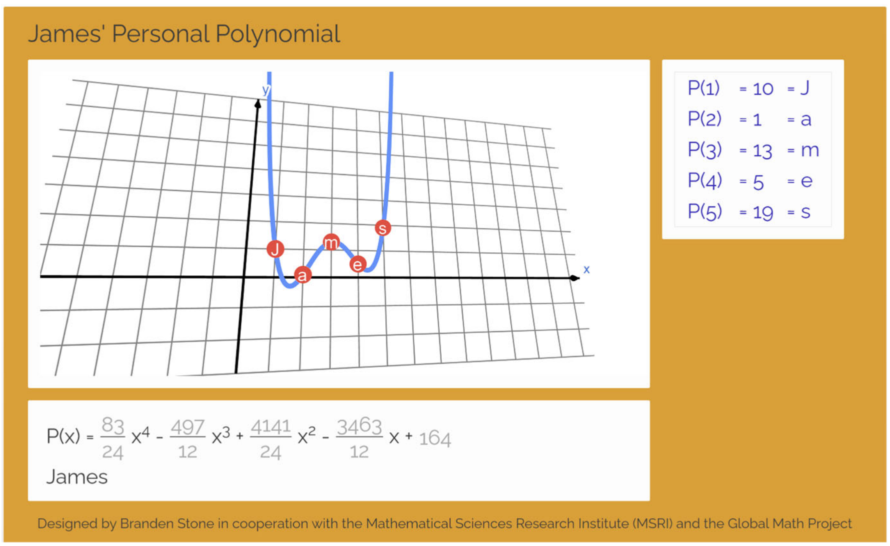
Practice 105.3 Write down a formula that spells the word CHEESE.

Practice 105.4 What's the formula that spells your name?

Personal Polynomial Website

Colleagues at the Simons Laufer Mathematical Sciences Institute (Mathematical Sciences Research Institute) create a widget that computes for you the formula that spells your name. It even draws the graph of the equation that spells your name! Check it out at globalmathproject.org/personal-polynomial.

The site also contains videos explaining the mathematics again.



Heads Up: This website uses the letter x for the unknown and displays the formula in "function" notation. We'll learn about this notation in some later chapters, but you can probably guess what it all means right away.

MUSINGS

Musing 105.5 We can push the technique of this section to create a formula for general data sets.

For instance, consider this data set of values.

x	3	6	8	12
y	9	11	33	5

Here's an equation that matches the values given in the table. It comes in four chunks.

$$y = 9 \times \frac{(x-6)(x-8)(x-12)}{(-3)(-5)(-7)} + 11 \times \frac{(x-3)(x-8)(x-12)}{(3)(-2)(-6)}$$

$$+ 33 \times \frac{(x-3)(x-6)(x-12)}{(5)(2)(-4)} + 5 \times \frac{(x-3)(x-6)(x-8)}{(9)(6)(4)}$$

a) Analyze this equation and make sure you understand why it does the job.

b) Create an equation that fits the data in this table.

x	7	10	11
y	3	-5	8

Comment: This technique for writing down an equation that fits some given data is due to French mathematician Joseph-Louis Lagrange (1736-1813). It is today called Lagrange Interpolation.

Musing 105.6 We saw in Section 82 how to find the equation of the line that passes through two given points.

a) Use the techniques of that section to show that line that passes through the points $(1,10)$ and $(6,20)$ has equation $y = 2x + 8$.

b) Does Lagrange Interpolation give the same equation when you use it to find an equation that fits this data table?

x	1	6
y	10	20

c) Use Lagrange interpolation to find the equation of the line that passes through the points $(0,30)$ and $(500, -100)$.