Chapter 14

The Algebra of Quadratics

Contents

106. The Beginnings

Can you think two numbers which sum to 10 and multiply together to give 24?

Can you think two numbers which sum to 10 and multiply together to give 25?

Can you think two numbers which sum to 10 and multiply together to give 26?

Problems like these appear on Babylonian clay tablets that date back to 2000 B.C.E. and solving them was likely of practical interest to scholars at the time, perhaps to work out the areas of rectangular fields with certain constraints on them.

> **Practice 106.1** Is there a rectangle of area 24 square feet and perimeter 20 feet? If so, what are the dimensions of the rectangle?

These problems can be tricky to solve. You may have thought of 4 and 6 as two numbers that sum to 10 and have product 24. You may have thought it a bit sneaky to choose two numbers that are the same—5 and 5—that sum to 10 to give a product 25. And you may be struggling to find two numbers that sum to 10 with a product of 26.

Figuring out when problems like these can be solved and, when solvable, developing systematic methods finding their solutions became a fascination of scholars throughout the millennia.

Some beginning solution techniques were developed by both the Babylonians and Egyptians of 4000 years ago (discussions on these problems are also found on ancient Egyptian papyrus scrolls), and by Greek scholars and Chinese scholars of 2300 years ago. More robust techniques were described by seventh-century Indian mathematician Brahmagupta and by ninth-century Persian scholar al-Khwārizmī, who used his methods and notation of algebra (*al-jabr*). But it still took another 600 years before the mathematics behind these problems was fully developed and described.

You may have heard of the "quadratic formula." It summarizes the results of all this arduous work.

That formula was developed in 1594 by Belgian scholar Simon Stevin, but not quite in the form you may have been forced to memorize in school. French scholar René Descartes published that version of the formula in his 1673 book *La Géométrie*.

(By the way … You do not need to memorize the quadratic formula. Thinking of doing so reminds me of some personal misery in 10^{th}-grade math class giving me pangs of dread!)

Let’s lean into the opening problem of this section.

Can you think two numbers which sum to 10 and multiply together to give 24?

The answer is YES, and the numbers in mind are 4 and 6.
But let’s pretend we are not aware of this solution.

The algebraic way to attend to this problem is to give the two unknown numbers we seek names.

Let’s follow schoolbook convention of using the letters x and y. We seek two numbers that satisfy these two equations simultaneously.

$$x + y = 10$$

$$xy = 24$$

The algebraic techniques of section 85 don’t work here (the second equation is not a linear equation)—hence the challenge for scholars over the millennia. But we can try the school substitution technique of section 86.

Let’s add $-x$ to each side of the first equation to get

$$y = 10 - x$$

$$xy = 24$$

Now we can see that the second equation wants us to find a value x that make this sentence true:

$$x(10 - x) = 24$$

Another way to write this is

$$10x - x^2 = 24$$

or as

$$x^2 - 10x + 24 = 0$$

(Check this second claim as I was a bit swift going from the previous line to that one. Did I get it right?)

So, our original problem has come to trying to find for value x that makes

$$x^2 - 10x + 24 = 0$$

a true statement. (And once we find such a value for x, then setting y as $10 - x$ gives us the second number we seek.)

Practice 106.2

a) Show that setting x to be 4 makes $x^2 - 10x + 24 = 0$ a true statement. What is the associated value of y?

b) Show that setting x to be 6 also makes $x^2 - 10x + 24 = 0$ a true statement. What is the associated value of y?

We've made the original problem look scarier than how it started out, I know. But this is what scholars eventually realized: all the problems being discussed throughout the millennia could be turned into the same one type of problem.

Try to find a value for x that makes an equation of the form

$$ax^2 + bx + c = 0$$

for some given numbers a, b, and c a true statement.

Scholars became fascinated in trying to solve equations of this form.

Physicists became fascinated too: A Story about Gravity

What's your gut instinct?

> A lacrosse ball and a tennis ball are close to being the same size and they are both the same shape, but a lacrosse ball is much heavier.
>
> If you drop the two balls from the same height at the same time, which will hit the ground first?

If you try this experiment, it is very hard to tell which lands first—the motion is quick. So, you must rely on your intuition. (Or perhaps you can drop the two balls from the top of a very tall object and figure out a way to handle the effects of air resistance?)

Greek philosopher Aristotle (ca. 350 BCE), and other scholars of his time, believed that heavier objects fall through the air with faster acceleration than lighter objects. It seems reasonable to suspect that gravity has a "greater effect" on heavier objects than lighter ones and so causes them to increase in speed at a faster rate as they fall. This line of reasoning stayed in place some 2000 years.

But then, Italian physicist Galileo Galilei (1564-1642) questioned this reasoning.

Galileo said to imagine two objects of the same size and shape, but of different masses, being dropped from the Leaning Tower of Pisa at the same time. If gravity has a stronger effect on the heavier object, then it would land first, taking the shorter amount of time of the two to reach the ground.

Now attach a very light string between the two objects, ostensibly making them one object which is of greater mass than either object individually. Gravity should then have an even stronger effect on this system. So, the two linked objects dropped from the Tower should fall to the ground in a shorter time still (and land at the same time as one object!).

But how do the two objects now "know" they are attached as one object and should fall faster?
This doesn't make sense!

Galileo concluded then that all objects of the same size and shape must fall through the air in unison, irrespective of their masses. (Air resistance does cause objects to fall at different rates according to their shapes: a sheet of paper is slower to fall than a paperclip of the same mass, for instance. But it is not gravity causing this variation.)

Letters at the time and some biographical accounts suggest that Galileo considered dropping objects off from the Leaning Tower of Pisa to try verifying that their accelerations to the ground were the same, but there is no evidence to indicate that he actually did this.

But we do know that he instead followed the clever idea of rolling objects of different masses down a ramp. Their descent down is still due to gravity, but their motion is much slower and can be carefully measured.

Galileo did verify that the acceleration of an object due to gravity is the same for all objects, irrespective of mass. Aristotle was wrong, and scholars since him for 2000 years were wrong too.

> **A Really Cool Video:** *In 1971, Apollo 15 Commander David Scott performed a live television demonstration while on the surface of the Moon. He simultaneously dropped a hammer and a feather to show, in the absence of an atmosphere, they do indeed accelerate to the ground at identical rates.*
>
> *You can see the video here.* https://www.youtube.com/watch?v=5C5_dOEyAfk

Let me take some liberties and make up some fanciful data that illustrates what you do see if you follow Galileo's approach.

Suppose we have a 100-foot-tall tower, and we drop an object from that height. Let's say it takes 2.5 seconds for it to reach the ground. (I am making up this number.) We observe the height of this falling object every half a second.

Let t represent the time passed since dropping the object, measured in seconds, and h the height of the object at each time, measured in feet, and suppose we obtain this (absurdly fanciful) data.

time	t	0.0	0.5	1.0	1.5	2.0	2.5
height	h	100	96	84	64	36	0

We want to obtain information about the acceleration of the object as it falls.

Physics class taught us that acceleration is the rate of change of velocity, where velocity is the rate of change of position (in our case, heights)

Since the data is based on regular time differences, the velocity of the falling object can be studied by taking the differences in height values. And that's the first row of a difference table!

And acceleration is the change of velocity over regular time intervals, and that would be differences again!

	100	96	84	64	36	0
an indicator of velocity →	-4	-12	-20	-28	-36	
an indicator of acceleration →	-8	-8	-8	-8		

Like my unrealistic data here, Galileo did see constant second differences in his data showing a constant rate of acceleration due to gravity. He also saw that the value of acceleration was the same for all objects. He did indeed overturn the thinking of Aristotle.

And Galileo went further …

We know from our work on patterns, Section 102, that constant second differences indicate that the data has a formula given by "1, n, and n^2." Except here we are not working with the unknown n, but with the time t as the underlying variable.

In seeing constant double differences, Galileo showed that the changing height of a falling object is given by a formula involving 1, t, and t^2. That is, he established:

$$\textbf{height } \boldsymbol{h} \textbf{ at time } \boldsymbol{t} \quad = \quad \boldsymbol{at^2 + bt + c}$$

for some numbers a, b, and c.

> Problems in physics about the motion objects under the effect of gravity are problems of the same type studied since the time of Babylonians and Egyptians 4000 years ago.

We have another context requiring a deep understanding of expressions of the form $ax^2 + bx + c$.

Studying expressions of this form is considered a fundamental part of the algebra curriculum. But I personally love discussing this material not for the stated importance of this topic as a piece of content, but for the powerful—and delightful—mathematical thinking it offers. We get to think like mathematicians and learn how to avoid hard work.

I am looking forward to getting into it with you.
And I promise:

No feelings of dread!

MUSINGS

Musing 106.3 We started this section with these three questions:

Can you think two numbers which sum to 10 and multiply together to give 24?

Can you think two numbers which sum to 10 and multiply together to give 25?

Can you think two numbers which sum to 10 and multiply together to give 26?

Using some graphing technology …

a) Draw on the same set of axes the graphs of

$$x + y = 10$$

and

$$xy = 24$$

Do you see the two solutions $x = 4$, $y = 6$ and $x = 6$, $y = 4$?

b) Draw on the same set of axes the graphs of

$$x + y = 10$$

and

$$xy = 25$$

What solution(s) do you see?

c) Draw on the same set of axes the graphs of

$$x + y = 10$$

and

$$xy = 26$$

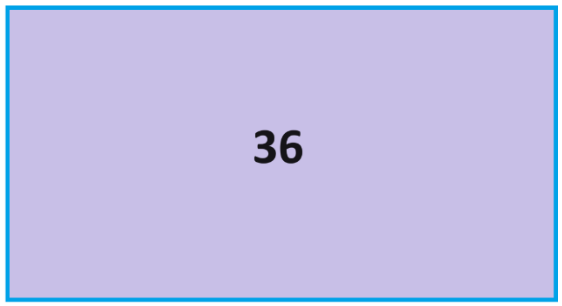
What solutions do you see?

Musing 106.4 At what acceleration do objects, in the absence of air resistance, fall to the ground?
Care to look at the accepted value of acceleration due to gravity on Earth?
What's the value of acceleration due to gravity on the Moon?

107. The Name *Quadratic*

Here’s a rectangle. I’ll tell you that its area is 36 square units.

What can you tell me about the rectangle?



The fact is that you really can’t tell me anything about the figure other that “it is a rectangle of area 36 square units.” It might be a 4-by-9 rectangle or a 2-by-18 rectangle or a $4\frac{1}{2}$-by-8 rectangle or a 0.02-by-1800 one. You just don’t know.

But suppose I now add one more piece of information about this rectangle, that it possesses the most **symmetry** a rectangle could possibly possess.

This fact about symmetry now conducts magic: it transforms uncertainty into certainty. You can now say that the rectangle is, for sure, a 6-by-6 square.

This little example illustrates the power of symmetry in mathematics. Recognizing symmetry in a scenario often collapses unknown information into crystalline precise information.

> **Mathematicians value the power of symmetry.**
> **Symmetry is a mathematician’s friend!**

To illustrate this power, let’s answer a classic textbook problem found in almost every school algebra book across the planet within the chapter on quadratics. We can answer the challenge without me having yet explained what the word “quadratic” means nor providing any schoolbook mathematics about quadratics.

Example: A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. What should the dimensions of the rectangle be to obtain a pen of largest possible area?

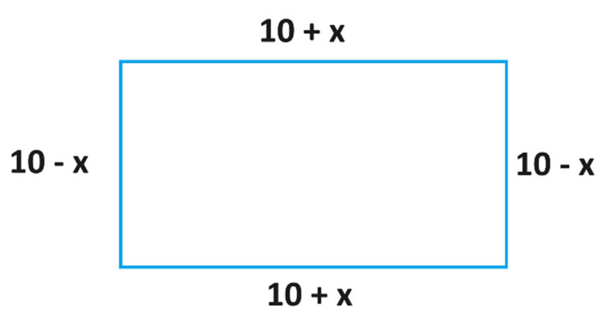
perimeter = 40 m

(Is there a farmer who operates this way?)

Answer: If symmetry is our friend, we might guess that a square pen is the answer to the problem. That is, that a10 meter by 10 meter pen, of perimeter 40 meters, gives a rectangle of largest possible area, 100 square meters.

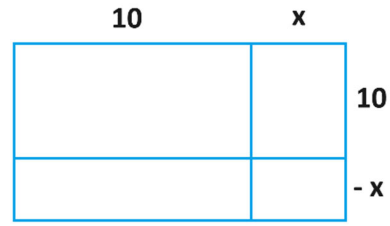
Can we prove that this is the case?

Any other rectangle we make out of the fencing will have one side longer than 10 meters and the other shorter than 10 meters. Let's label the side lengths by measures that indicate how far our rectangle deviates from being a ten-by-ten square. Let's say one side is of length $10 + x$ meters and the other $10 - x$ meters for some number x. (The perimeter is still 40 meters.)



perimeter = 40 m

Using the area model, we see that the area of this pen is



$$\begin{aligned}(10-x)(10+x) &= 100-10x+10x-x^2 \\ &= 100-x^2\end{aligned}$$

square meters.
That is, the area of any other pen is 100 square meters minus x^2 square meters. It is sure to be less that 100 square meters if x is a value other than 0.

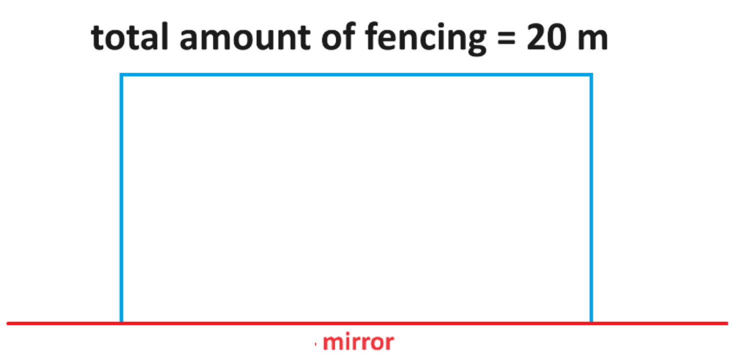
So, yes, the rectangular pen of maximal area occurs by choosing x to be 0 and making a ten-by-ten square pen.

The takeaway here is that basing our thinking on the most symmetrical rectangle of all we could consider, a square, and looking at all other rectangles as deviations from it, led us to mathematics that we could readily handle.

Symmetry made the problem tractable.

Practice 107.1 *Now for an even less realistic problem*

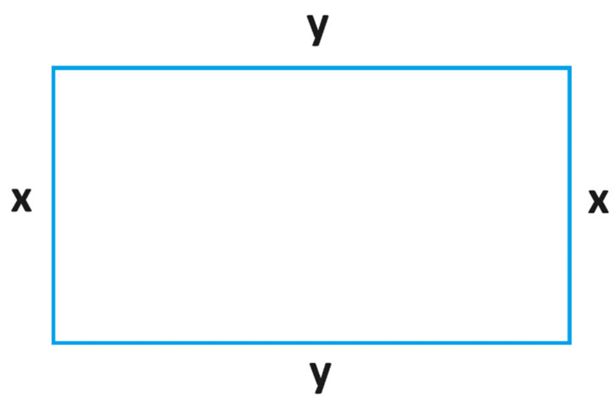
A farmer has a huge, mirrored exterior wall along one side of her barn. With 20 meters of fencing, she wants to make a rectangular pen with the mirrored wall serving as one side of the pen.



a) When the farmer makes such a pen, what will she see in the reflection?
What shape will the mirror image and the real fencing make together? Of what perimeter?

b) Explain why the farmer should make a rectangular pen that is half a square to get a pen of largest possible area with her 20 meters of fencing.

Most schoolbooks have students solve our first 40-meter fencing problem by introducing two symbols for the side lengths of the rectangle—x and y, of course—and writing down some equations that describe the situation mathematically.



perimeter = 40 m

We have

$$2x + 2y = 40$$

and we wish to make the area of the rectangle, $x \times y$, as large as possible.

Multiplying both sides of the equation we have by $\frac{1}{2}$ gives

$$x + y = 20$$

which can be rewritten

$$y = 20 - x$$

The area of the rectangle is thus

$$\text{Area} = \ x(20 - x) = -x^2 + 20x$$

which is an expression of the form $ax^2 + bx + c$ with $a = -1$, $b = 20$, and $c = 0$.

Students are taught techniques to handle expressions of this form and are then expected to use those techniques from this point forward.

Our farmer problem is of the classic type studied throughout millennia since the time of the Babylonians.

And we solved this problem by focusing on a square.

Scholars realized that a focus on squares was the key to solving **all** of these problems!

The Latin for "to have made into a square" is *quadratus*, and these problems became known as "those which can be solved by the quadratus method." Today we call them **quadratic** equation problems and any expression of the form

$$ax^2 + bx + c$$

is called a **quadratic** expression.

Practice 107.2 The prefix *quad*- means "four."

A square has four sides. Actually, all **quad**rilaterals have four sides.

A **quad**rangle is a figure that has four angles. (Courtyards outside of buildings, with four right-angled corners are often called quadrangles rather than quadrilaterals.)

To **quad**ruple a quantity is to make it four times as big.

A **quad**ruped is an animal with four feet.

Can you think of other English words that begin with "quad"? Are they all associated with the number four in some way?

At face value, it is not at all clear why an expression of the form $ax^2 + bx + c$ should be associated with the number four and be given a name involving "four-ness." But as we shall see in the next section, the symmetry and four-ness of a square is going to be mighty powerful friend.

MUSINGS

Musing 107.3 Do you have an algebra textbook at hand? If so, does its chapter on quadratic equations discuss the origin of the name "quadratic"?

Does your textbook also include a problem about a farmer needing to make a rectangular pen with a fixed length of fencing?

Musing 107.4 I am looking for two numbers that sum to 100 and have the largest possible product.

a) What would you guess the two special numbers to be? (What would be the most "symmetrical" answer to this problem?)

b) In answering part a) you might suggest that 50 and 50 are be the two numbers summing to 100 with the largest possible product. Prove that this guess is correct!

(To do this, notice that for any two other numbers that sum to 100, one will be smaller than 50 and the other larger than 50. Represent the numbers as $50 - x$ and $50 + x$, for some value x. Now examine the product of these two quantities.)

Musing 107.5 Aleksandra was given this challenge.

Find two numbers that differ by 100 *and have product* 5069*.*

She reasoned: "Symmetry is my friend. A symmetrical solution would have the two numbers the same. So let me represent the numbers by how different they are from being the same."

She wrote two numbers differing by 100 as $n - 50$ and $n + 50$.

How do you think she proceeded from there?

Musing 107.6 Find two numbers whose sum is 100 and whose product is 2491.

Musing 107.7 Back to these three questions:

Can you think two numbers which sum to 10 and multiply together to give 24?

Can you think two numbers which sum to 10 and multiply together to give 25?

Can you think two numbers which sum to 10 and multiply together to give 26?

Let's answer then with a symmetry mindset and write the two numbers we are thinking of as $5 - x$ and $5 + x$.

a) What is the product of $5 - x$ and $5 + x$?

b) Can this product equal 24? If so, what value of x makes that so?

c) Can this product equal 25? If so, what value of x makes that so?

d) Can this product equal 26? If so, what value of x makes that so?

108. The Power of a Square

This section is purely algebraic. We'll focus on solving equations that involve quadratic expressions. Such equations are called—no surprise—**quadratic equations**.

But let's set up this this experience like a video game by presenting a series of difficulty levels to master to gradually build up our confidence. We'll be pros at solving quadratic equations by the end of level 6.

Here goes!

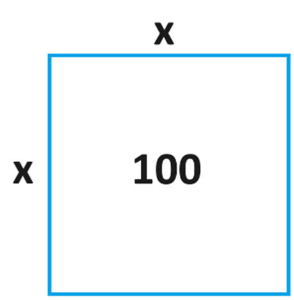
LEVEL 1 QUADRATIC EQUATIONS

Here's our first level 1 problem.

> **Problem:** Kindly solve $x^2 = 100$.

In reading the given equation out loud—"x square equals one hundred"—we hear the word *square*. This problem wants us to focus on a square, literally!

We seek the side length of a square, being called x, whose area is 100 square units.



As a problem in geometry, there is only one answer: x must have the value 10.

But we know we can extend our arithmetic beyond the literalness of geometry and positive numbers to recognize that there is a second value x could adopt to give arithmetic truth, namely, x could also be -10.

The quadratic equation $x^2 = 100$ has two solutions.

$$x = 10 \text{ or } -10$$

Problem: Please solve $p^2 = 49$.

This problem calls the unknown quantity p rather than x. No big deal.

Something squared is 49 . That something must be 7 or its negative version, -7.

$$p^2 = 49$$
$$p = 7 \text{ or } -7$$

And that's about it for level 1!

Of course, there are always ways to through in some extra twists just to keep you on your toes.

Problem: Please solve $64x^2 = 25$.

It seems compelling to multiply each side of the equation by $\frac{1}{64}$ so that it reads

$$x^2 = \frac{25}{64}$$

The quantity x is squared to give $\frac{25}{64}$. Consequently,

$$x = \frac{5}{8} \text{ or } -\frac{5}{8}$$

Problem: Kindly solve $q^2 = 7$.

The quantity q is squared to give the value 7, which is not a perfect square. Not a problem. We can just say that q is either $\sqrt{7}$ or its negative version $-\sqrt{7}$.

$$q = \sqrt{7} \text{ or } -\sqrt{7}$$

Problem: Please solve $x^2 = -9$.

We're out of luck here. No number multiplied by itself will give a negative result. To be clear ….

Recall from section 25 that a positive number times a positive number is positive number and that a negative number times a negative number is also positive. So, if x is positive or negative, $x \times x$ is sure to be positive. Also, in the case where x is neither positive nor negative, x is 0, we have that 0×0 still fails to be negative.

This quadratic equation has no solutions.

Practice 108.1 We've seen examples of quadratic equations that have 2 solutions and an example of a quadratic equation with 0 solutions.

Give an example of a quadratic equation with exactly 1 solution.

Practice 108.2 Kindly solve each of these quadratic equations.

a) $2x^2 = 50$
b) $y^2 + 5 = 14$
c) $100x^2 = 1$
d) $4p^2 = 0.25$
e) $a^2 + 7 = 7$
f) $x^2 = 20$
g) $x^2 = -20$

LEVEL 2 QUADRATIC EQUATIONS

We're ready for a notch of increased difficulty.

Problem: Please solve $(x+3)^2 = 25$.

The key here is to "step back" from the problem to see that the given equation is simply stating

Something squared is 25

This is a level 1 statement! The "something" must be either 5 or -5.
And our something is the quantity $x+3$.

So we have

$x+3=5$ or $x+3=-5$

Subtracting 3 from each side of each equation gives

$x=2$ or $x=-8$

Great!

Problem: Please solve $(w-6)^2 = 36$.

Again, the given equation reads "something squared is 36." We deduce

$w-6=6$ or $w-6=-6$

Adding 6 to each side of each equation gives

$w=12$ or $w=0$

Comment: Most people "speed up" the writing process by simply presenting

$$(w-6)^2 = 36$$

$$w-6 = 6 \text{ or } -6$$

$$w = 12 \text{ or } 0$$

(Notice the middle line and notice how we added 6 to each side of it to obtain the third line.)

Comment: Many people "speed up" the writing process even more by using the $\pm$ symbol for the second line.

$$(w-6)^2 = 36$$

$$w-6 \quad = \quad \pm 6$$

But this often creates pickles. If we add 6 to each side of what we see it's tempting to then write

$$w \quad = \quad \pm 12$$

But the original equation has solutions 12 and 0, not 12 and -12.

My advice …

Don't use the $\pm$ symbol and take the extra half a second to write the word "or."

Okay. Continuing on …

Problem: Please solve $(2x+4)^2 = 4$.

Something squared is 4 and so we have

$2x + 4 = 2$ or -2

We want the value of x, so let's subtract 4 throughout

$2x = -2$ or -6

Multiplying through by $\frac{1}{2}$ then gives

$x = -1$ or -3.

Done!

Practice 108.3 Kindy solve

a) $(4\text{x}-6)^2=-7$
b) $(4\text{x}-6)^2=0$
c) $(4\text{x}-6)^2=4$
d) $(4\text{x}-6)^2=5$

Practice 108.4 Please solve

a) $(y+1)^2-2=23$
b) $4(p-2)^2-16=0$
c) $9+\left(34x-77\frac{1}{2}\right)^2=0$
d) $\left(x-\sqrt{2}\right)^2=5$

LEVEL 3 QUADRATIC EQUATIONS

Let's now go up another level—quite a level it shall seem!

When one sees a level 3 problem for the very first time it is shocking. It looks completely different from those in levels 1 and 2.

Problem: Can you solve $x^2 + 6x + 9 = 25$?

This is an opportunity to again engage in the first two fundamental steps for solving a problem.

STEP 1: Be your honest human self and acknowledge your human reaction to the problem.

If the problem looks scary, gulp and say, "This problem looks scary." If the problem looks intriguing, be intrigued. If it looks confusing and you don't know what to do, say "I don't know what to do!"

Mathematics is a human enterprise, made by humans, for humans. So, be your honest human self.

Next, take a deep breath and move to …

STEP 2: DO SOMETHING! Anything!

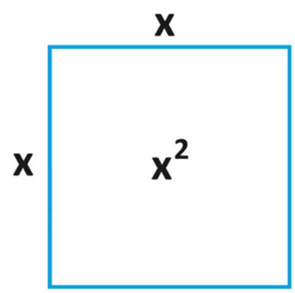
If you need to take a break to let your brain mull on the problem, take a break. Or maybe you can underline some words in the question or circle all the vowels you see. Or perhaps you can draw a picture or rewrite the question backwards. Just do something! You will be surprised how powerful taking a first piece of action—of any kind! —can be in getting one past an emotional impasse.

So, what can we do with what is being asked of us? What can we do with the given equation?

$$\boldsymbol{x^2 + 6x + 9 = 25}$$

It does look scary.

But if I read it out loud, I do hear the word "square" again. So, as an attempt to DO SOMETHING, I can at least draw a picture of a square to match the x^2 term I see.

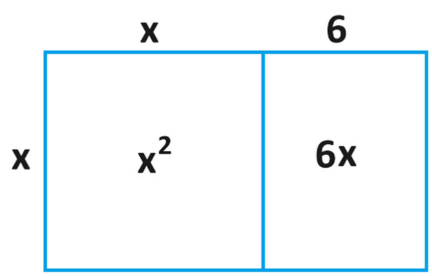


Great! We did something!

What's a next something we can do?

I see the $6x$ term in the equation. (There's also a 9, but let's not worry about that yet.)
Could I add to the picture a piece that represents $6x$?

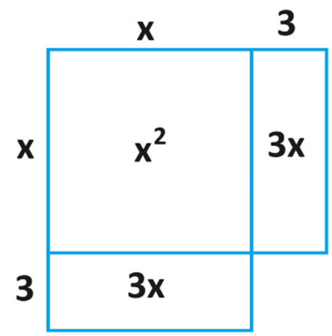
How about this?



But this makes me nervous: we've lost the symmetry of our square.

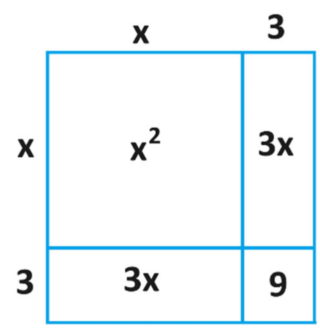
Is there are way to add an area of $6x$ to the picture, but to do so in a symeterical way? We might not create a full square, but can we at least head in the direction of doing so? (After all, we do have another piece of area 9 to add to the picture coming up.)

Try splitting the piece of area $6x$ into two pieces of area $3x$ and place those pieces as follows.



This feels good! And it also feels utterly compelling to complete the picture to make a square!

And, lo and behold, that missing corner piece is a three-by-three square of area 9, just as we need.



Our pieces of areas x^2 and $6x$ and 9 fit together to make a picture of an $x+3$ by $x+3$ square.

The original problem wants x^2+6x+9 to equal 25. And we've just shown that this is the same as wanting $(x+3)^2$ to be 25.

So, we just need to solve

$$(x+3)^2 = 25$$

which is a level 2 problem. We have

$$x+3 = 5 \text{ or } -5$$
$$x = 2 \quad \text{or } -8.$$

And one can double check that both $x=2$ and $x=-8$ really do make x^2+6x+9 equal to 25.

This is the challenge of level 3.

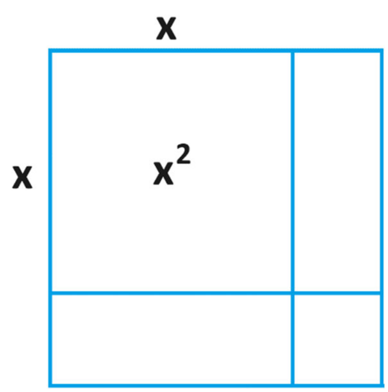
Try to recognize complicated-looking expressions as level 2 problems in disguise.

Let's practice this.

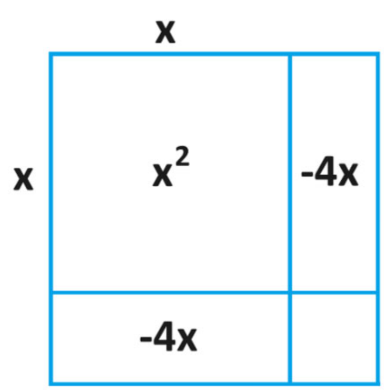
Problem: Kindly solve $x^2 - 8x + 16 = 49$.

We have an x^2 term, so let's draw a square.

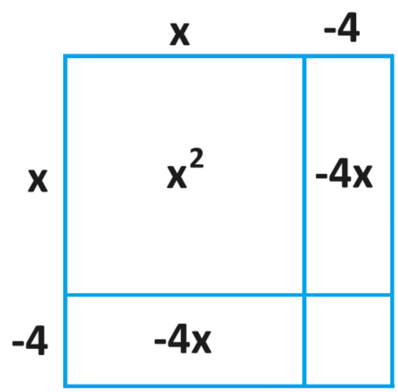
Actually … since we know we are going to add on to this picture to make an even larger square, let's just go ahead and draw a big square divided into four pieces with symmetry right away.



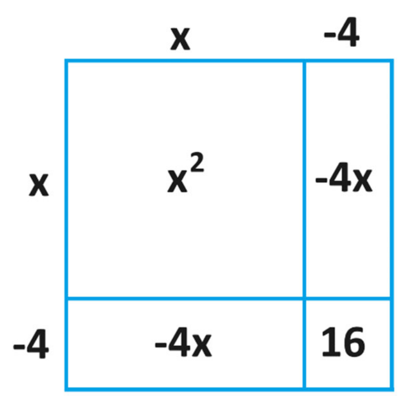
We need to include an "area" of $-8x$. (Remember, we are pushing our diagrams beyond geometry to speak general truth about all arithmetic. It's the area model of section 24.)
We do this in a symmetrical way.



These pieces must come from multiplying x with -4.



The final piece that completes the square is $(-4) \times (-4) = 16$, which is precisely the value the problem wanted!



We're seeing that $x^2 - 8x + 16$ is the same as $x - 4$ squared, and the equation $x^2 - 8x + 16 = 49$ is just the equation

$$(x - 4)^2 = 49$$

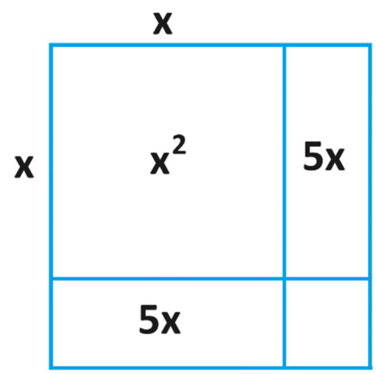
That's a level 2 problem!

$$x - 4 = 7 \quad \text{or} \quad -7$$

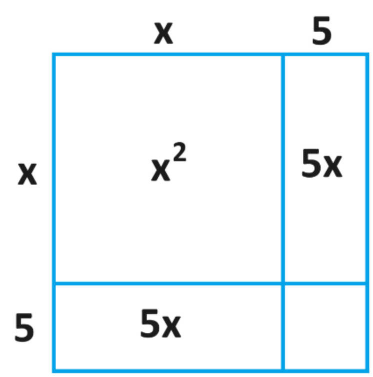
$$x = 11 \quad \text{or} \quad -3$$

Problem: Please solve $x^2 + 10x + 25 = 33$

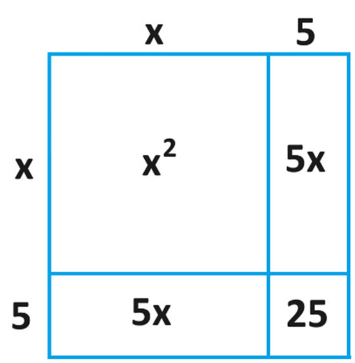
Let me draw the square with an x^2 piece and the $10x$ piece split in two.



Each of these pieces must come from multiplying x by 5.



The final piece that completes the square is $5 \times 5 = 25$, which is precisely the number we need!



So, the quantity $x^2 + 10x + 25$ is really $x + 5$ as a square, in disguise, and we're really being asked to solve the level 2 problem

$$(x + 5)^2 = 33$$

The number 33 is awkward, it doesn't have a nice square root. Be we can write our answers in terms of the square root of 33 and its negative.

$x + 5 = \sqrt{33}$ or $-\sqrt{33}$

Subtracting 5 gives

$x = \sqrt{33} - 5$ or $-\sqrt{33} - 5$.

Comment: Some people are fussy and prefer to write numbers that are a sum of integers and awkward numbers, such as square roots, with the integers mentioned first and the awkward parts second.

So, some book authors (or test writers) might expect us to present these solutions as

$x = -5 + \sqrt{33}$ or $-5 - \sqrt{33}$

It's just fussing with style of presentation, not an issue of mathematics.

Practice 108.5 Kindly consider solving each of these equations.

a) $p^2 - 6p + 9 = 9$
b) $x^2 - 4x + 4 = 1$
c) $x^2 - 20x + 100 = 7$
d) $r^2 - 16r + 64 = -2$
e) $x^2 + 2\sqrt{5}x + 5 = 36$
f) $x^2 - 2\sqrt{2}x + 2 = 19$

Practice 108.6: Solve for x giving your answer in terms of A and B.

$x^2 + 2Ax + A^2 = B^2$

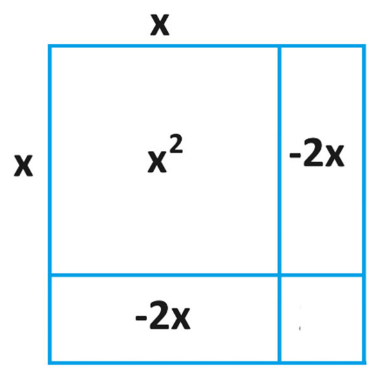
LEVEL 4 QUADRATIC EQUATIONS

Consider this problem.

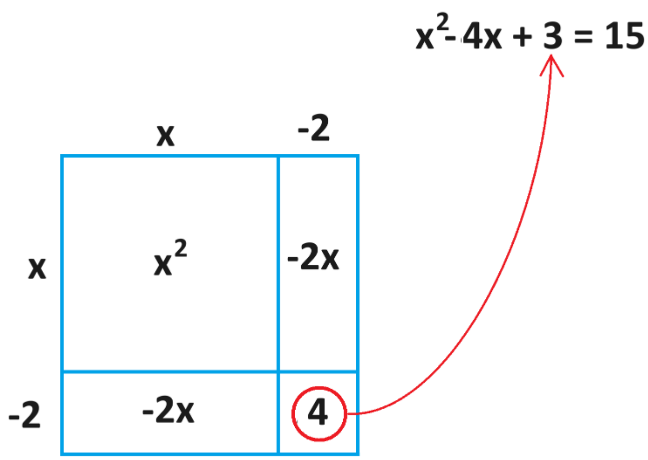
Problem: Please solve $x^2 - 4x + 3 = 15$.

This looks like a level 3 problem. Is there a hidden difficulty notch of difficulty to this one?
Let's find out!

Start by drawing the square. We have an x^2 piece and an "area" of $-4x$ split into two equal pieces.



Each of these pieces comes from multiplying x by -2, which then leads to number 4 as final piece of the square.



And that's a problem. We have a mismatch!

The square wants the number 4, but the problem given to us presents the number 3.

What can we do?

Here's a piece of life advice:

> If there is something in life you want, make it happen!
> (And be prepared to deal with the consequences.)

Can we turn that 3 into a 4?

Sure. Let’s just add 1 to it!

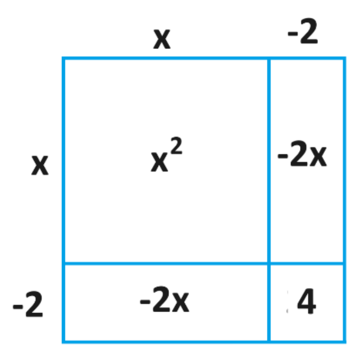
But there are consequences. If we add 1 to the left side of an equation, we must do the same to the right side as well.

$$x^2 - 4x + \overset{+1}{3} = \overset{+1}{15}$$

Now we have the equation

$$x^2 - 4x + 4 = 16$$

with the left side perfectly matching $x - 2$ as a square.



$$x^2 - 4x + 4 = 16$$

And this is great! We have the level 2 equation

$$(x-2)^2 = 16$$

Solving, we get

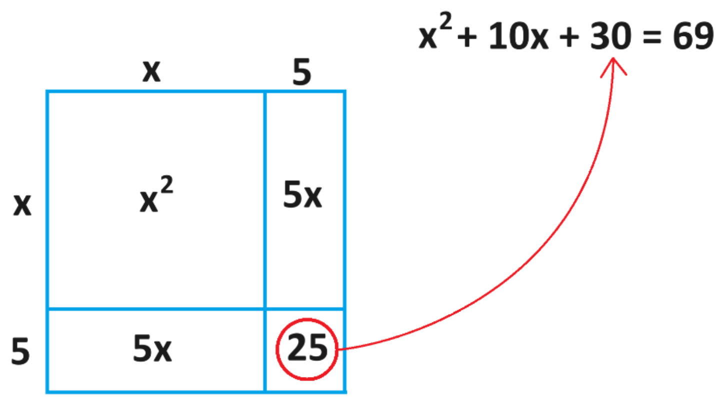
$$x - 2 = 4 \text{ or } -4$$

$$x = 6 \text{ or } -2$$

Done!

Problem: Please solve $x^2 + 10x + 30 = 69$.

Let’s draw the square with an x^2 piece and two $5x$ pieces, and not be phased by the mismatch.



The square “wants” the number 25, but the problem presented the number 30 instead.
Let’s change that 30 to 25 by subtracting 5 from the left and dealing with the consequences.

$$x^2 + 10x + \overset{-5}{30} = \overset{-5}{69}$$

The equation $x^2 + 10x + 25 = 64$ is a much friendlier because we now recognize it as

$$(x+5)^2 = 64$$

This is a level 2 problem.

$x + 5 = 8$ or -8

$x = 3$ or -13

All is just grand!

Practice 108.7: Please solve

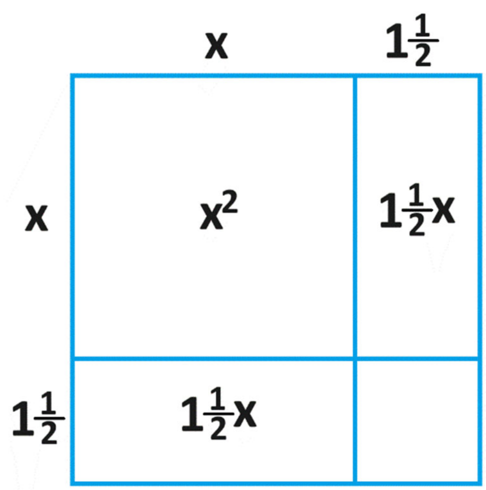
a) $f^2 + 8f + 15 = 80$
b) $w^2 - 22w + 90 = -31$
c) $x^2 - 6x = 3.$

LEVEL 5 QUADRATIC EQUATIONS

What do you notice about this next problem?
Can you see something I've been secretly avoiding all this time?

Problem: Solve $x^2 + 3x + 1 = 5$.

If we try the "quadratus method" (the square method) for this equation, we find ourselves dealing with fractions. And the fractions here are awkward.



Practice 108.8: Optional Push on with this problem and do work with fractions. Show that the square method eventually gives the solutions $x = 1$ or $x = -4$.
(The square method will never let you down if you care to push on with it.)

But is there a way we can avoid awkward fractions? That is, can we be mathematicians and work to avoid hard work?

The problem lies with the middle term $3x$ with 3 being an odd number. This term does not split into two nicely.

Is there a clever way we can make the middle term in $x^2 + 3x + 1$ an even number of xs?

Idea 1: *Add x to both sides of the equation and try to solve instead*

$$x^2 + 4x + 1 = 5 + x$$

This is a grand idea, but I have the feeling with an x appearing on the right, I'll be getting "solutions" of the form

$$x = \text{something still involving } x\text{s}$$

That's not really a solution.

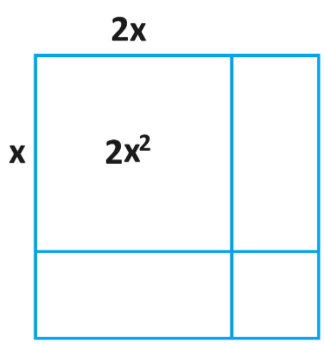
Hmmm.

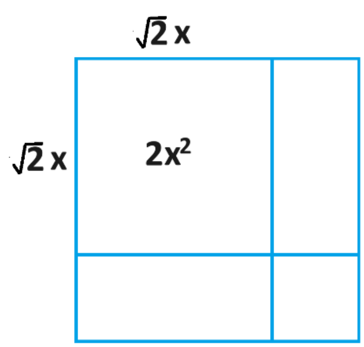
Idea 2: *Double everything and solve instead*

$$2x^2 + 6x + 2 = 10$$

This gives $6x$ as a middle term that splits nicely.

But if I draw the square, I am not sure what to do with the first $2x^2$ piece. I could break symmetry to stay with whole numbers (but that is not good as symmetry is my friend) or I could preserve symmetry and work with the number $\sqrt{2}$ (but that seems harder than working with fractions!)





Hmm.

Check: Do you see that $\sqrt{2}x \times \sqrt{2}x$ equals $2x$?

So, what can we do? Both our ideas were good. They just turned out not to be helpful.

This is the nature of doing real mathematics. One often has brilliant ideas that, sadly, turn out not to be helpful.

The thing to do in such moments is to mull and think and wait for a next brilliant idea to come your way. This might take minutes, hours, days, even months or years for a particularly tough problem!

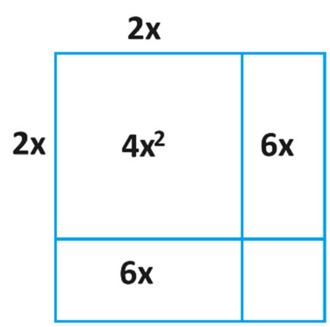
After some mulling and thinking it might occur to you to try the following.

Idea 3: *Instead of doubling everything, try multiplying everything by four! Work with the equation*

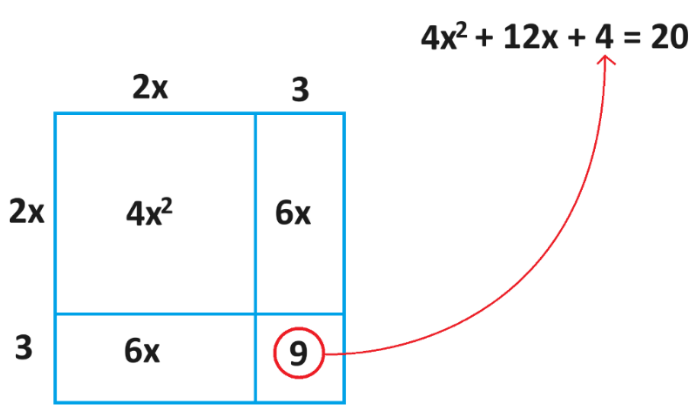
$$4x^2 + 12x + 4 = 20$$

This is lovely.

The first term, $4x^2$, is a nice perfect square and the middle term uses an even number and splits nicely.



Something times $2x$ makes $6x$, so we have side lengths of 3, and the final piece of the area to complete the square is 9.



Adding 5 to both sides of the equation handles the mismatch

$$4x^2 + 12x + 9 = 25$$

And this equation is really just "$2x + 3$ as a square makes 25" in disguise.

$$(2x + 3)^2 = 25$$

We're back to level 2!

$$2x + 3 = 5 \text{ or } -5$$

Subtracting 3 throughout …

$$2x = 2 \text{ or } -8$$

Multiplying by $\frac{1}{2}$ throughout …

$$x = 1 \text{ or } -4$$

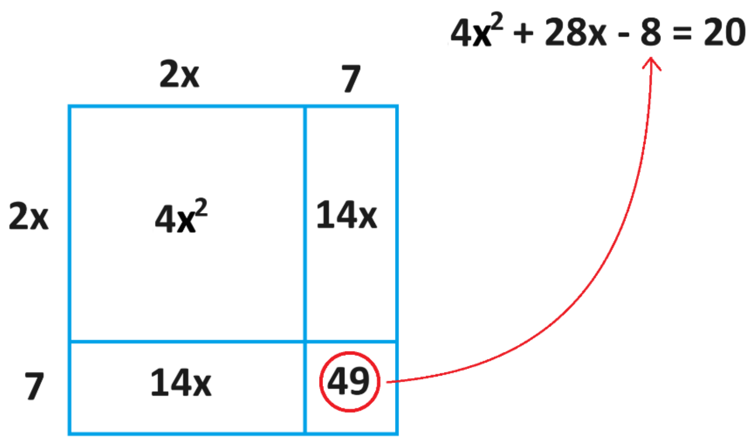
Wow!

Multiplying through by 4 to handle an odd number of xs in the equation unlocked the problem!

Problem: Kindly solve $x^2 + 7x - 2 = 5$.

We have a middle term involving an odd number of xs. We can avoid fractions by multiply through by 4 to solve instead

$$4x^2 + 28x - 8 = 20$$



The square method shows we really want the number 49, not -8. Let's add 57 to each side of our equation and work with

$$4x^2 + 28x + 49 = 77$$

The number on the right is awkward, but we can handle it.

We have

$(2x + 7)^2 = 77$

$2x + 7 = \sqrt{77}$ or $-\sqrt{77}$

$2x = \sqrt{77} - 7$ or $-\sqrt{77} - 7$

$x = \frac{\sqrt{77}-7}{2}$ or $\frac{-\sqrt{77}-7}{2}$

or, some books will prefer us to write $x = \frac{-7+\sqrt{77}}{2}$ or $\frac{-7-\sqrt{77}}{2}$.

Practice 108.9: Solve as many of these as you have the patience for.

a) $w^2 - 5w + 6 = 2$
b) $x^2 + 9x + 1 = 11$
c) $p^2 + p + 1 = 0.75$
d) $x^2 = 10 - 3x$
e) $x^2 - x - 1 = 2\frac{3}{4}$
f) $x^2 + 3 = 9$

LEVEL 6 QUADRATIC EQUATIONS

We're here. The final level!

And this next problem is chock full of issues to contend with, starting with a number attached to the x^2 term. (Another feature I've been sidestepping up to now.)

Problem: Please solve $3x^2 + 5x + 1 = 9$.

In the last level we multiplied equations through by 4 to create a $4x^2$ term within our equations. And we could deal with that because $4x^2$ is a nice perfect square: it is $2x$ times $2x$.

But this problem has the term $3x^2$, which is not a nice perfect square. Can we make it one?

Yes! Let's multiply the equation through by 3 and work with

$$9x^2 + 15x + 3 = 27$$

Great! The first term is $9x^2 = (3x) \times (3x)$, a nice square.

But next we seen we have the middle term $15x$, and odd number of xs.

We can fix that by multiplying through by 4. Great!

A Worry! Will doing so ruin the perfect square we just created out front?

Let's find out.

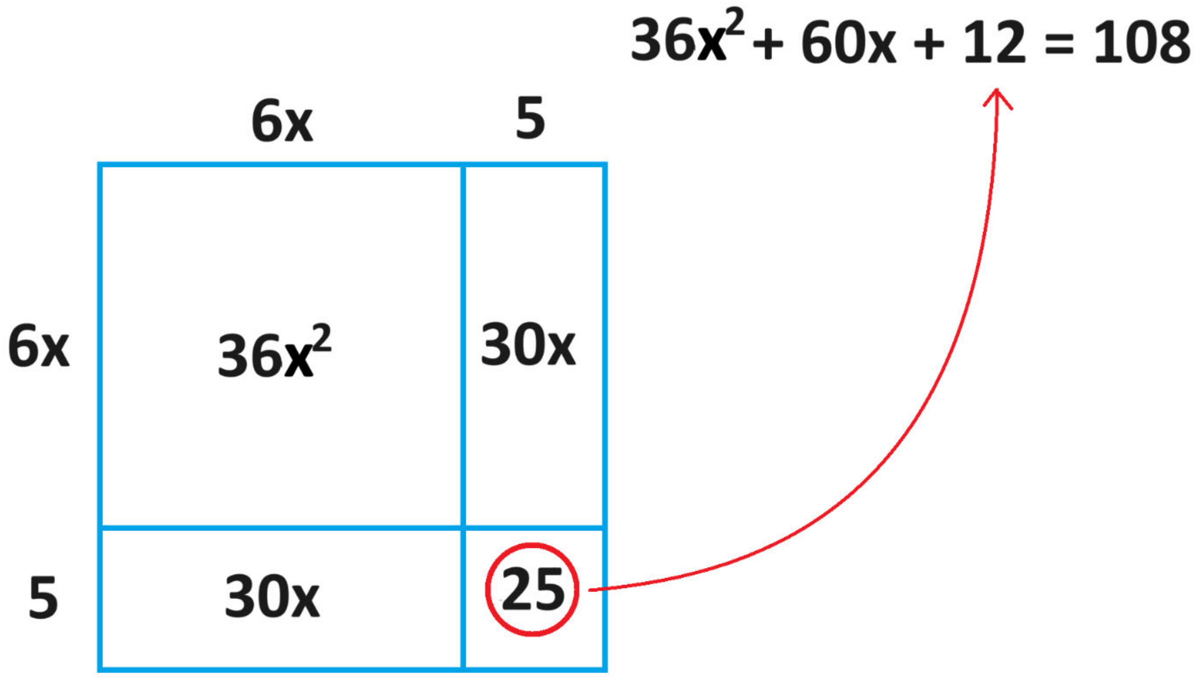
Multiplying $9x^2 + 15x + 3 = 27$ through by 4 gives

$$36x^2 + 60x + 12 = 108$$

Phew! $36x^2 = (6x) \times (6x)$ is still a nice perfect square. Nothing has been ruined!

Practice 108.10 Explain why multiplying a perfect square number by 4 is certain to give another perfect square. (Perhaps start by considering a square number n^2 and then play with $4n^2$ to show it is also a square number.)

Now things are good for the square method.



(Did you catch that we need to multiply $6x$ by 5 to get $30x$?)

The square wants the number 25, so let's add 13 to both sides and work with

$$36x^2 + 60x + 25 = 121$$

And why did we do all this crazy work? To recognize $36x^2 + 60x + 25$ as a square. It is a $6x + 5$ by $6x + 5$ square and we've turned our problem into a level 2 problem

$$(6x + 5)^2 = 121$$

We get

$6x + 5 = 11$ or -11

$6x = 6$ or -16

$x = 1$ or $-\frac{8}{3}$

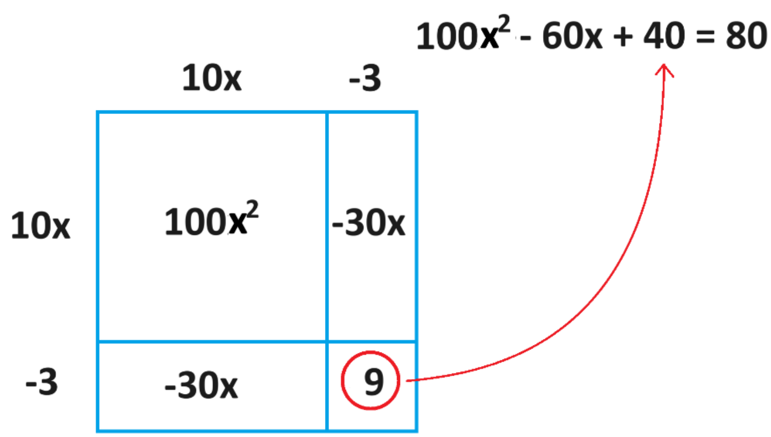
Problem: Solve $5x^2 - 3x + 2 = 4$.

Let's multiply through by 5 to make a perfect square up front.

$$25x^2 - 15x + 10 = 20$$

Now let's multiply through by 4 to handle the odd number of xs within the middle term.

$$100x^2 - 60x + 40 = 80$$



The square method shows we want the number 49. Let's subtract 31 from each side.

$$100x^2 - 60x + 9 = 49$$

And this is a level 2 problem

$(10x - 3)^2 = 49$

$10x - 3 = 7$ or -7

$10x = 10$ or -4

$x = 1$ or -0.4

Practice 108.11 Solve as many of these as you have the patience for.

a) $3x^2 + 3x + 1 = 19$
b) $-3x^2 + 3x + 1 = 19$
c) $\alpha^2 - \alpha + 1 = \frac{7}{4}$
d) $10k^2 = 1 + 10k$
e) $2x^2 = 9$
f) $4 - 3x^2 = 2 - x$

Practice 108.12 Consider

$$4x^2 + 6x + 3 = 1$$

Does this quadratic equation look like it will have problems when solving it? Does it have problems?

Solve the equation obviating any difficulties you encounter.

Here's the upshot of this section

EVERYTHING IS LEVEL 2!

We have illustrated that every quadratic equation

$$ax^2 + bx + c = d$$

is just a level 2 question in disguise.

$$(something)^2 = number$$

And we can solve level 2 equations with relative ease. It's the power of a square!

MUSINGS

Musing 108.13 Solving the equations below will have you multiplying though by 4 multiple times—if you decide to avoid fractions. Give them a try if you like. (You might also detect structure in the answers.)

a) $x^2 + x = 2$.
b) $2x^2 + x = 3$.
c) $4x^2 + x = 5$.
d) $8x^2 + x = 9$.
e) $16x^2 + x = 17$.

Musing 108.14 a) What are the solutions to this equation?

$$(x - 5)(x + 3) = 0$$

b) Show that the equation in part a) is a quadratic equation in disguise.

Practice 108.15
a) Design a quadratic equation that just has one solution, the number 4.
a) Design a quadratic equation that has two solutions, both of which are negative numbers.
c) Design a quadratic equation with 2 and 10 as solutions.

Practice 108.16 Consider the expression

$$(3x - 2)^2 + 6$$

with the unspecified number x.

What value for x gives the smallest possible value to $(3x - 2)^2 + 6$? How do you know?

Practice 108.17 Find one solution to

$$(x + 1)^3 = 27$$

Practice 108.18 a) Solve $x^2 + 5x + 13 = 49$

Now, can you "see through" each of these next challenges?

b) Please solve w+$5\sqrt{w} + 13 = 49$.

c) Please solve $u^4 + 5u^2 + 13 = 49$

MECHANICS PRACTICE

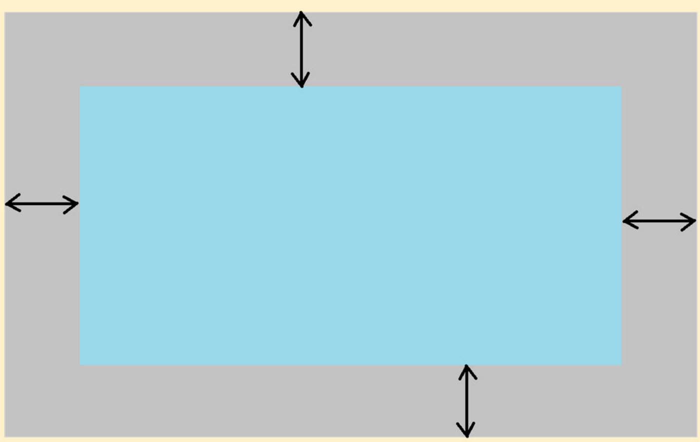
Practice 108.19 Did you try all the practice problems throughout this section? There are quite a few of them!

Practice 108.20

a) A rectangle is twice as long as it is wide. Its area is 30 square meters. What are the dimensions of the rectangle?

b) A rectangle has one side 4 meters longer than the other. Its area is 30 square meters. What are the dimensions of the rectangle?

Practice 108.21 I would like to create a stone border of uniform width around a 10 foot by 8 foot swimming pool.



I have enough stone to make border whose total area is 63 square feet.

What is the largest border width I could create?

Practice 108.22 Two consecutive odd numbers (that is, two odd numbers that differ by 2) have product 30,275. What are the two odd numbers?

109. The Famous Quadratic Formula

Many textbook authors would be surprised at the approach I’ve taken to solving quadratic equations.

For example, in solving

$$5x^2 - 3x + 2 = 4$$

the standard first step would be to subtract 4 from each side of the equation and put it in the form $ax^2 + bx + c = 0$, that is, to create the number zero on the right side of the equation. Doing so would give us this equation.

$$5x^2 - 3x - 2 = 0$$

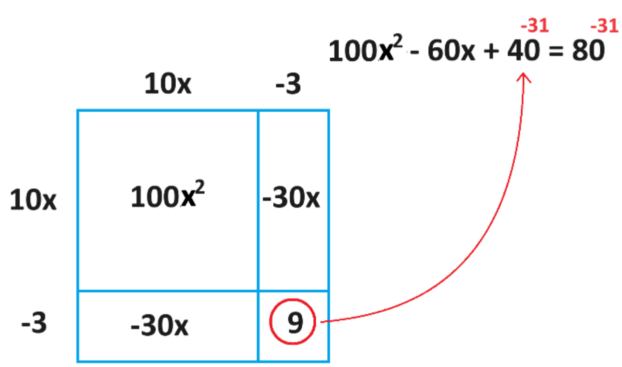
Instead, our approach in this example is to leave the numbers as non-zero and, worse, make them bigger by multiplying them all by 5.

$$25x^2 - 15x + 10 = 20$$

And to contend with the odd number of xs for the middle term, we make numbers larger still by multiplying through by 4.

$$100x^2 - 60x + 40 = 80$$

It seems we’ve made our initial equation more complicated than simpler! But really, we’ve made matters beautifully simpler by creating a perfect square and a level 2 equation.



We see

$$100x^2 - 60x + 9 = 49$$

as

$$(10x - 3)^2 = 49$$

which we solved as the final problem of the last section.

This technique of using squares is the very work scholars developed over the millennia to solve quadratic problems. It is also the heart of the formula written down by Simon Stevin in 1594 and René Descartes in 1673 for solving quadratic equations as culmination of this work.

Please now allow me to derive the famous **quadratic formula** for you. (Curriculum authors would be shocked if I didn't present it.) It's just a matter of repeating the work we've been doing all along but applying it now to an abstract example.

Let's solve this quadratic equation via our square method.

$$ax^2 + bx + c = 0$$

Here a, b, and c are numbers, and we shall assume we did make a first move of subtracting a number from both sides of a given equation to make the right-hand number 0. (Descartes assumed we did this in his book, and this is probably why curriculum authors expect us to do that first too.)

We don't know anything about the number a. It might already be a perfect square, or it might not be.

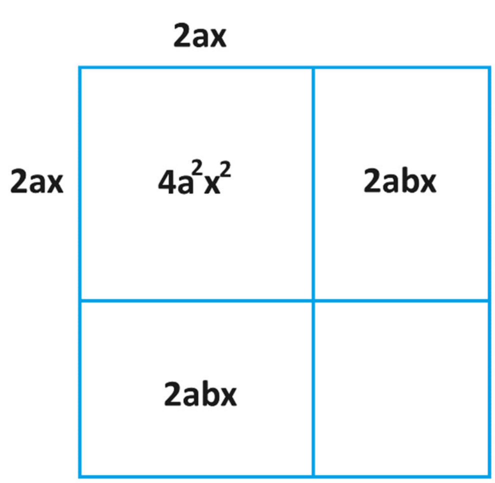
But to cover ourselves, let's multiply the equation through by a to make sure we have a perfect square up front.

$$a^2x^2 + abx + ac = 0$$

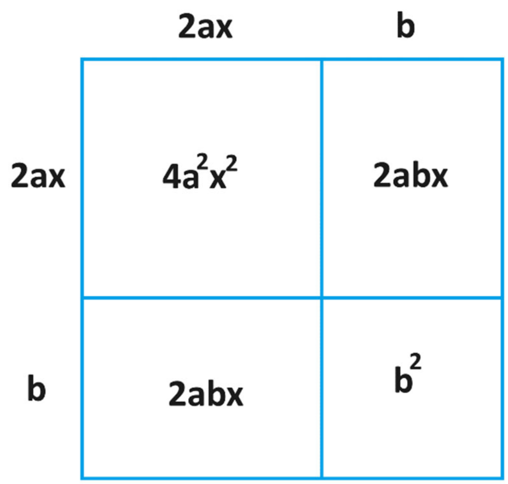
The middle term might or might not be easily split into two equal parts. To cover ourselves there too, let's multiply through by 4 as well.

$$4a^2x^2 + 4abx + 4ac = 0$$

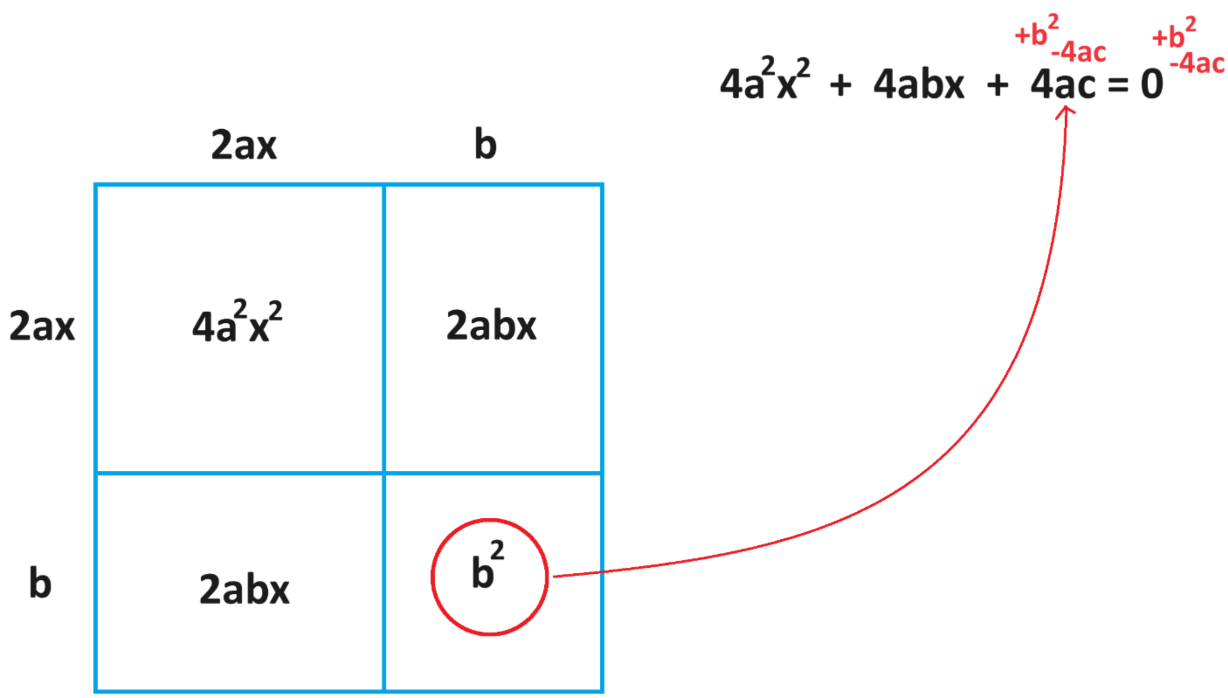
Let's start drawing our square.



Multiplying $2ax$ by b gives us $2abx$.



But we have a mismatch.



The square wants a b^2 piece. So, let's subtract $4ac$ from each side of our equation and add in a b^2 to each side. We get

$$4a^2x^2 + 4abx + b^2 \;=\; b^2 - 4ac$$

The picture shows that $4a^2x^2 + 4abx + b^2$ is $2ax + b$ as a square.

We have a level 2 problem.

$$\boldsymbol{(2ax + b)^2 = b^2 - 4ac}$$

Something squared is $b^2 - 4ac$ and so our something is the square root of that or the negative version of that square root.

$$2ax + b = \sqrt{b^2 - 4ac} \quad \text{or} \quad -\sqrt{b^2 - 4ac}$$

Adding $-b$ throughout gives

$$2ax = \sqrt{b^2 - 4ac} \ - b \ \text{ or } -\sqrt{b^2 - 4ac} \ - b$$

Given the expected style convention, we write this as

$$2ax = -b + \sqrt{b^2 - 4ac} \ \text{ or } -b - \sqrt{b^2 - 4ac}$$

Multiplying through by $\frac{1}{2a}$ (most people call this "dividing through by $2a$") then gives our final solutions.

$$\boldsymbol{x = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \textbf{or} \quad \frac{-b-\sqrt{b^2-4ac}}{2a}}$$

People like to combine these two solutions by using a $\pm$ symbol in the middle to indicate we can have a $+$ sign or a $-$ sign.

The Famous Quadratic Formula

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

There it is! It's just the result of the square method applied to $ax^2 + bx + c = 0$.

Example: Solve the problem given at the opening of this section

$$5x^2 - 3x + 2 = 4$$

via the quadratic formula.

Answer: The quadratic formula assumes we have made the right side of the equation equal to 0. To use it we must rewrite the equation as

$$5x^2 - 3x - 2 = 0$$

To use the formula, we set $a = 5$, $b = -3$, and $c = -2$.

We get the solutions,

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot (5) \cdot (-2)}}{2 \cdot 5}$$

and this looks messy and scary!

The square root symbol comes with a vinculum, so let's work out what's under that vinculum first.

$$(-3)^2 - 4 \cdot (5) \cdot (-2) = 9 - (-40) = 49$$

So, our solutions are

$$x = \frac{3 \pm \sqrt{49}}{10}$$

which are

$$x = \frac{3+7}{10} \text{ or } \frac{3-7}{10}$$

giving $x = 1$ or $x = -\frac{2}{5}$, just as we had last section

Practice 109.1: Please solve

$$3x^2 + 5x + 1 = 9$$

using the quadratic formula and then again by using the square method. (Of course, you should get the same answers each time!)

I personally prefer the square method for solving quadratic equations, drawing squares on the side of the page. I understand what to do and there is nothing for me to memorize.

But it is a slower enterprise. Many people like the quadratic formula because it is speedy to use, which is important for exams and tests that insist students get answers to questions with speed.

Let's be honest, though. If your goal is just to get a numerical answer to a problem and not to exhibit any understanding or ingenuity, then just ask a computer assistant or algebra system to solve the quadratic equation for you. We live in the 21st century after all.

Also, schoolbooks give the impression that all quadratic equations involve "nice" numbers: whole numbers and simple fractions. But equations that arise from real world scenarios are usually "messy" and thus difficult to work with. For example, solving

$$9.782x^2 - 3.067x + 15.285 = 0$$

via the square method would not be fun. Using the quadratic formula would be a little more manageable, but still not fun.

Computer algebra systems really do have a good place in mathematical life.

Practice 109.2 Solve whichever of these quadratic equations take your fancy, using whatever method you like. (Or solve some twice with two different methods!)

a) $6x^2 - x + 10 = 11$
b) $30x^2 - 17x = 2$
c) $x^2 - 4x + 4 = 0$
d) $2x^2 + 5 = 11x$
e) $93x^2 - 117 = 0$
f) $x^2 + x + 1 = 2$
g) $x^2 + x + 1 = 1$
h) $x^2 + x + 1 = 0$

MUSINGS

Musing 109.3 In this problem, m and n are fixed numbers and x is the unknown value.

Consider the equation

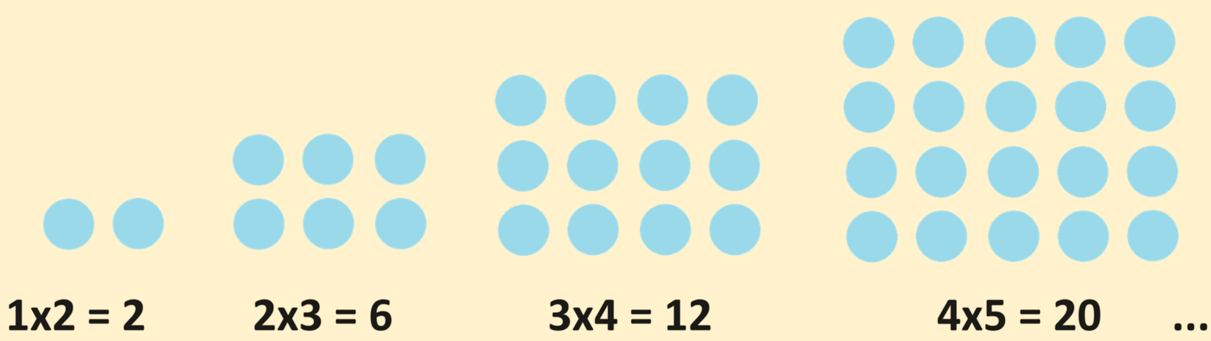
$$x^2 - (m+n)x + mn = 0$$

a) Show that $x = m$ is s solution to this equation. Show that $x = n$ is too.

b) Use the quadratic formula to solve this equation and show that it gives these values as two solutions.

MECHANICS PRACTICE

Practice 109.4 The sequence of **oblong numbers** begins 2, 6, 12, 20, 30, 42, 56, …



a) What is the 100th oblong number in this list?

b) The number 5402 is an oblong number. What is its position in the list?

Practice 109.5 Find a number p so that $1 + \frac{1}{p}$ equals p.

110. Unexpected Quadratic Problems

Quadratic equations can sneak up on us.

We saw this in the opening section of the chapter. Who would naturally suspect that the motion of falling objects is governed by a quadratic expression?

Algebra textbooks are particularly clever in creating problems that turn out to be quadratic in nature. These problems, like the farmer and her fencing, are typically not real-world relevant.

Here's another such example.

> **Example:** I ride my bike along a straight stretch of road. The road is 120 km long and I ride at a constant speed.
>
> I then ride my bike back along the same stretch of road, again at a constant speed, but this time 10 km/hr faster than I did before.
>
> I was one hour quicker on my return journey.
>
> What were my two riding speeds?

Do you remember the two key steps to problem solving?

> **Step 1**: Have an emotional reaction.

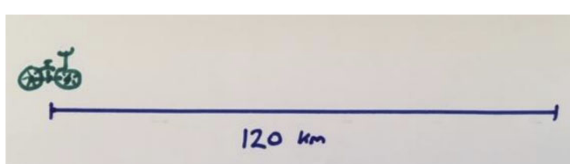
Take a deep breath and then

> **Step 2:** DO SOMETHING! Anything!

Let me give you my take on this problem.

I admit that the question feels a bit scary to me. It feels like it is lacking some information.

But in order to just to do something, let me draw a picture.
I'll draw a straight stretch of road (and label is as 120 km long) and a bicycle.

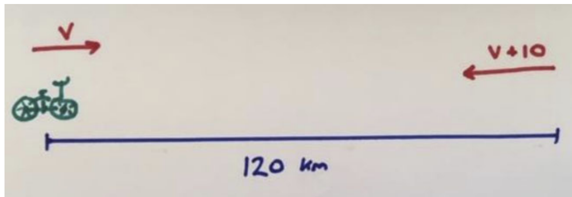


Great first step!

What more can I do?

The question is about a constant speed and a second constant speed that is 10 km/hr faster.

Let's call the speeds v km/hr and $v + 10$ km/hr and indicate them in the picture too.



Looking good!

What next?

The question does mention a time difference. Can we get some details about times?

Well, when moving at a constant speed

$$\text{velocity} = \frac{\text{distance covered}}{\text{time taken}}$$

Multiplying both sides of this equation by the denominator gives

$$\text{time taken } \times \text{velocity} = \text{distance covered}$$

and so, with one more algebraic step we see

$$\text{time taken} = \frac{\text{distance covered}}{\text{velocity}}$$

Covering 120 km at a constant speed of v km/hr will take

$\frac{120}{v}$ hours.

For the return journey, covering 120 km at a speed of $v + 10$ km/hr, will take

$\frac{120}{v+10}$ hours.

And this is 1 hour quicker.

So, we have

$$\frac{120}{v+10} = \frac{120}{v} - 1$$

What sort of equation is this?

I don't like the fractions, so let's multiply though by v and by $v + 10$. We get

$$v \times (v+10) \times \frac{120}{v+10} \quad = \quad v \times (v+10) \times \left(\frac{120}{v} - 1\right)$$

That is,

$$v \times 120 = (v+10) \times 120 - v \times (v+10)$$

That is,

$$120v = 120v + 1200 - (v^2 + 10v)$$

That is,

$$0 = 1200 - v^2 - 10v$$

That is,

$$v^2 + 10v = 1200$$

Whew! (Did you follow each of those steps?)

So, now we have a quadratic equation in v.

But I've forgotten what the question was!

Looking back

What were my two riding speeds?

Okay. So, we need to work out the value of v and of $v + 10$. And we have a quadratic equation we can solve to find v. (So, yes! This was a quadratics problem!)

The square method has me work with

$$v^2 + 10v + 25 = 1225$$

which is

$$(v + 5)^2 = 1225$$

(Check this! I saw the picture just in my head and wrote this down.)

Consequently,

$$v + 5 = 35 \text{ or } -35$$

and

$$v = 30 \text{ or } -40$$

Me riding 40 km/hr backwards is unrealistic for this problem, so I deduce that I must have first ridden at a constant speed of 30 km/hr to first cover the 120 km of road (and have done so in 4 hours) and then have ridden at a constant speed of 40 km/hr back (to cover the length again in just 3 hours).

We got there (in just over two pages of work!)

Practice 110.1: Xavier took part in a speed banana-eating contest.

He was given one dozen bananas to eat.

He ate his first 6 bananas at a constant rate, and his second 6 bananas at another constant, but slower, rate.

Some details:

- It took him 5 seconds longer to eat his second 6 bananas compared to his first 6.
- His banana-eating rate was 0.1 banana per second less for his second 6 bananas compared to his first 6.

How long did it take Xavier to eat all 12 bananas?

Hint: Let t represent the number of seconds it took Xavier to eat the first six bananas. He thus ate those bananas at a rate of $\frac{6}{t}$ bananas per second. The rate at which he ate his next 6 bananas was $\frac{6}{t+5}$ bananas per second.

The Three Opening Questions – Yet Again!

We opened this chapter with three questions.

Can you think two numbers which sum to 10 and multiply together to give 24?

Can you think two numbers which sum to 10 and multiply together to give 25?

Can you think two numbers which sum to 10 and multiply together to give 26?

You analyzed these questions graphically in problem 106.3 and you completely solved them using the power of symmetry in problem 107.7. That's my favorite way!

Recall:

The most "symmetrical" pair of numbers that sum to 10 are 5 and 5.

So, let's call our two numbers $5 - x$ and $5 + x$ to reflect how much they deviate from this symmetry.

Their product is then

$$(5 - x)(5 + x) = 25 - x^2$$

We see that the product is "twenty-five take away something" and that something is never negative.

Choosing $x = 1$ give a product of 24 and our two numbers are 4 and 6.

Choosing $x = 0$ give a product of 25 and our two numbers are 5 and 5.

There is no value of x that will give a product of 26.

This approach does not typically appear in textbooks.

Here's the approach that does.

Call the two numbers x and y. We have

$$x + y = 10$$

Suppose we want their product to be 24.

$$xy = 24$$

Observe that we have $y = 10 - x$ and so we can rewrite this product as

$$x(10 - x) = 24$$

That's

$$10x - x^2 = 24$$

which can be rewritten

$$x^2 - 10x + 24 = 0$$

Now use either the square method of quadratic equation to see this has solutions

$x = 4$ or 6.

Practice 110.2 Show that the square method for this example yields the equation

$$(x - 5)^2 = 1$$

If we examine the case for a product of 25, show that the square method yields the equation

$$(x - 5)^2 = 0$$

If we examine the case for a product of 26, show that the square method yields the equation

$$(x - 5)^2 = -1$$

What are the solutions in each of these cases?

I personally think involving the quadratic equation offers the least insight as to what is going on with these three little questions.

But sums and products problems abound in textbook quadratics chapters for students to practice their quadratic might. One can solve these in this expected schoolbook way, for sure. But it is fun to think of solving them too by taking advantage of symmetry.

If two numbers sum to 30*, try calling them* $30 - x$ *and* $30 + x$*.*

If two numbers differ by 30*, try called them* $x - 15$ *and* $x + 15$ *(that is, fifteen below a central number and fifteen above that central number.)*

And so on.

After all, the entire story of quadratics is that of the power of symmetry.

MUSINGS

Musing 110.3 Find the smallest possible positive value k for which the equation $x + \frac{1}{x} = k$ has a solution for a value x.

Musing 110.4 Two numbers sum to 1000. That is the largest product they could have?
How do you know?

MECHANICS PRACTICE

Practice 110.4 A stretch of highway road is 292.5 miles long.

I normally drive at a constant highway speed. But I realized that if I were to drive 20 miles per hour faster, I could cover this stretch of highway in two hours less time.

What is my normal highway driving speed and would I be speeding if I did try to save two hours of driving time?

Practice 110.5
a) Are there two numbers that differ by 198 and multiply to give a million?
b) Are there two numbers that sum to 198 and multiply to give a million?
(Did you catch that these are both YES/NO questions?)

111. How Many Solutions?

The square method shows that every quadratic equation is really a level 2 quadratic problem in disguise.

$$(\text{something})^2 = \text{a number}$$

And we can readily see the number of solutions the quadratic equation will have from inspection.

Example:

1. Each of these equations will have 2 solutions:

$(2x-3)^2 = 16$ $\quad (x+4)^2 = 100$ $\quad (5w-7)^2 = 19$

2. Each of these equations will have just 1 solution:

$(2x-3)^2 = 0$ $\quad (x+4)^2 = 0$ $\quad (5w-7)^2 = 0$

3. Each of these equations will have 0 solutions:

$(2x-3)^2 = -4$ $\quad (x+4)^2 = -33$ $\quad (5w-7)^2 = -1$

Well … this presumes something that we've taken for granted all along.

Are we sure point 1 is correct?

Sure, there are two numbers that multiply together to give 16, namely 4 and -4. Could there be a third one?

Sure, there are two numbers that multiply together to give 100, namely 10 and -10. Could there be a third one?

Sure, there are two numbers that multiply together to give 19, namely $\sqrt{19}$ and $-\sqrt{19}$. Could there be a third one?

How do we know the answer is NO in each case?

Maybe a quadratic equation could have more than two solutions?

All is good and clear with points 2 and 3.

Point 2: We proved back in section 52 that if two numbers multiply together to give the answer 0, it must be because at least one of the numbers was itself zero. (People call this the **zero product property** or something similar.)

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

It follows from this that 0 has "only one square root."

If $x^2 = 0$, then $x \cdot x = 0$. We must then have $x = 0$.

Thus, a level 2 quadratic problem of the form $(\text{something})^2 = 0$ can only proceed to yield 1 solution.

Point 3: We showed in section 108 that:

No number squared will give a negative answer.

Consequently, a level 2 quadratic problem of the form $(\text{something})^2 = \text{negative}$ cannot have any solutions.

So, the only question that remains is Point 1.

Are we certain that

$$x^2 = a\ positive\ number$$

has precisely 2 solutions?

Practice 111.1

a) Consider the equation $x^2 = 16$. We can rewrite this as

$$x^2 - 16 = 0$$

Rewrite this equation yet another way using the Difference of Two Squares from section 99.

Deduce that $x = 4$ and $x = -4$ are indeed the only two possible solutions to $x^2 = 16$.

b) Consider the equation $x^2 = 100$.

Rewrite the equation yet using the Difference of Two Squares and explain why $x = 10$ and $x = -10$ are the only two possible solutions to it.

c) Consider the equation $x^2 = 19$.

Explain why $x = \sqrt{19}$ and $x = -\sqrt{19}$ are the only two possible solutions to it.

The Difference of Two Squares formula and the Zero Product Property explain why a level 2 quadratic equation of the form

$$(\text{something})^2 = \text{positive}$$

will give 2 solutions, no more and no less.

Practice 111.2 Please determine how many solutions each of the quadratic equations will have. (No need to actually find the solutions.)

a) $(79x + 33)^2 = -987$

b) $(2x - 7)^2 = 0$

c) $(5 - x)^2 = 9$

d) $x^2 + 6x + 9 = 444$

e) $x^2 - 4x + 1 = 100$

f) $3x^2 - 5x - 7 = 20$

Comment: Algebra textbooks will have students answer questions like these by looking at the quadratic formula. The square method, to me, seems like a more natural way to proceed here.

MUSINGS

Musing 111.3 Consider the quadratic equation

$$ax^2 + bx + c = 0$$

(written with the right side equal to 0 as typical algebra textbooks expect).

a) The quadratic formula shows that the solutions of this equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Explain why the quadratic equation is sure to have 2 solutions if $b^2 - 4ac$ is greater than zero, 1 solution if $b^2 - 4ac$ is equal to zero, and 0 solutions if $b^2 - 4ac$ is less than zero.

b) We showed in section 109 that the equation $ax^2 + bx + c = 0$ is equivalent to the Level 2 quadratic equation

$$(2ax + b)^2 = b^2 - 4ac$$

Explain why the quadratic equation is sure to have 2 solutions if $b^2 - 4ac$ is greater than zero, 1 solution if $b^2 - 4ac$ is equal to zero, and 0 solutions if $b^2 - 4ac$ is less than zero.

Comment: I don't want to imply that knowing how many solutions a quadratic equation will have is at all important. If, for some reason you do need to know this, just solve the equation with the square method or with the quadratic formula and see how many solutions you get! No need to memorize "$b^2 - 4ac$" as something important here. (Though textbooks do think this expression is important for this level of work and call it the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$. They will test students on knowing it.)

MECHANICS PRACTICE

Practice 111.4
a) How many different rectangles are there of perimeter 100 units and area 600 square units?
b) How many different rectangles are there of perimeter 100 units and area 625 square units?
c) How many different rectangles are there of perimeter 100 units and area 650 square units?