Chapter 15

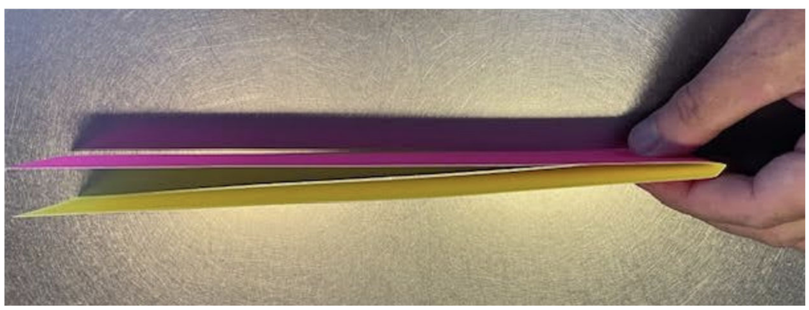
Exponents

Contents

112. Folding a Strip of Paper

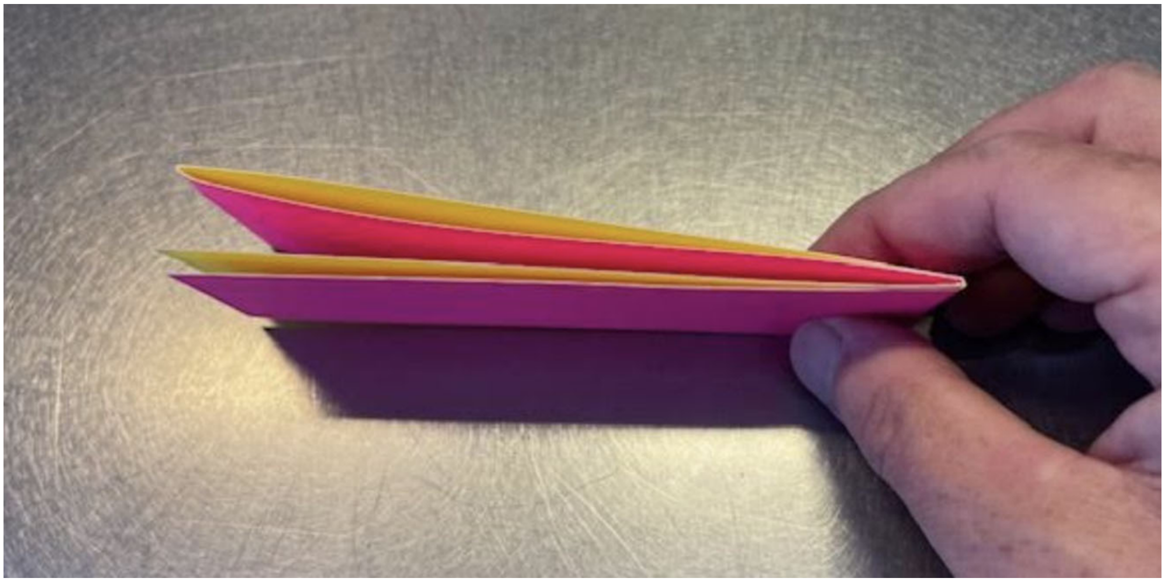
Here's a strip of paper.

Imagine folding the strip in half by picking up the right end and bringing it up and over to the left end.



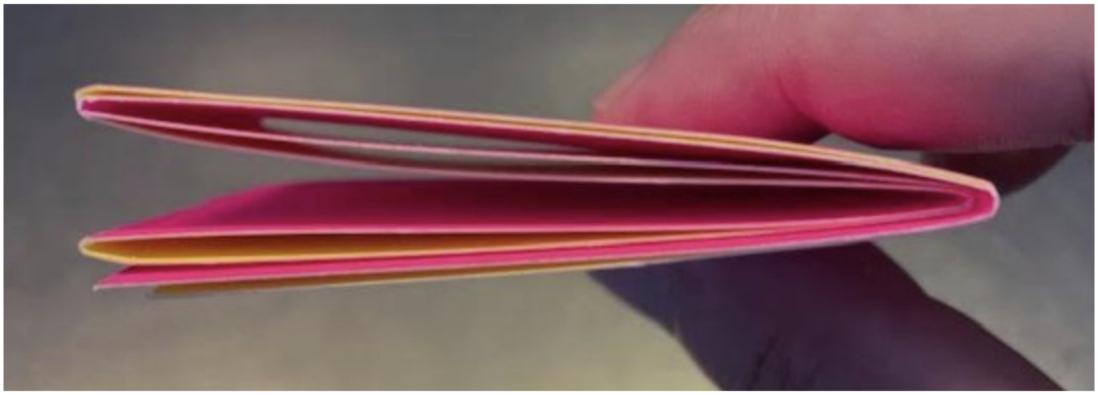
1 fold makes 2 layers

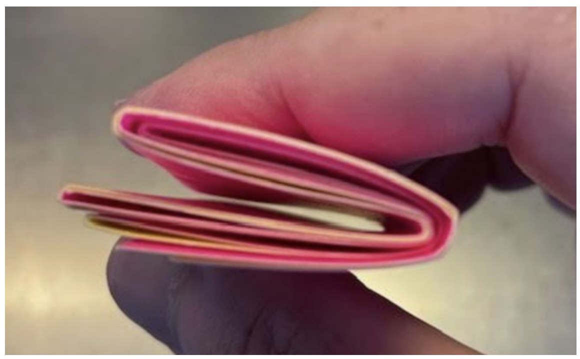
And imagine folding it in half again by picking up the right end.



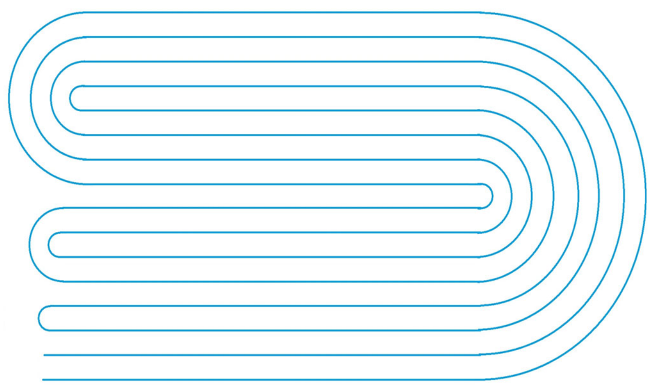
2 folds makes 4 layers

And to keep doing this.





3 folds make 8 layers

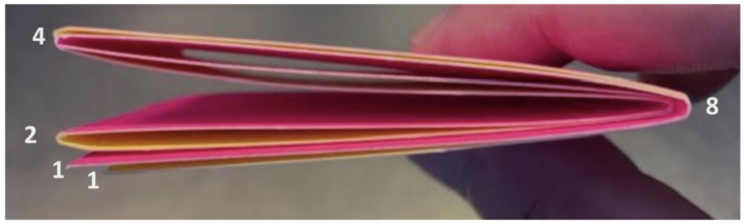


4 folds make 16 layers

Practice 112.1 How many layers will there be after five folds? After ten folds?

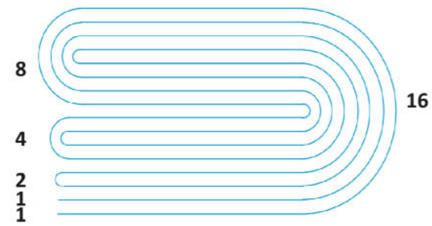
Folding a strip of paper and counting layers leads to the **doubling numbers** $1, 2, 4, 8, 16, 32, 64, \ldots$. We first met these numbers back in chapter 4 when we explored a mind-reading trick and examined binary codes in the $1 \leftarrow 2$ *Exploding Dots* machine. They arise here in paper folding because each fold conducted doubles the count of layers we see.

Practice 112.2 Examine these folding pictures.



1 + 1 + 2 = 4

1 + 1 + 2 + 4 = 8



1 + 1 + 2 + 4 + 8 = 16

What do you predict for the value of this sum? Are you correct?

$$1 + 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512$$

We have

$2 = 1 \times 2$

$4 = 1 \times 2 \times 2$

$8 = 1 \times 2 \times 2 \times 2$

$16 = 1 \times 2 \times 2 \times 2 \times 2$

$32 = 1 \times 2 \times 2 \times 2 \times 2 \times 2$

and so on.

To be clear, the number of layers we obtain after a certain number of folds matches the value of 1 doubled that many times. After folding a strip of paper in half 10 times, for instance, there will be

$$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1024$$

layers of paper.

Writing "1 × " in front of a product is not mathematically necessary and seems "clunky" when the core feature of the product is a single number being multiplied by itself multiple times. Let's just write nothing but ten 2s multiplied together for this example.

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1024$$

And as we saw in section 60, mathematicians have a notational shorthand for repeated multiplication. The notation uses superscripts and is called **exponential notation**.

In our example,

2^{10} means $\overbrace{2\times 2\times \ldots \times 2}^{10 \text{ times}}$

and this equals 1024.

In general,

> If n is a positive whole number, then for any number a we define a^n to be the quantity
>
> $$\overbrace{a \times a \times \cdots \times a}^{n \text{ times}}$$

We have

$$5^3 = 5 \times 5 \times 5 = 125$$

$$10^2 = 10 \times 10 = 100$$

and

$$1^{500} = \overbrace{1 \times 1 \times \cdots \times 1}^{500 \text{ times}} = 1.$$

Practice 112.3

a) What is the value of 0^4?

b) What is the value of $(\frac{1}{2})^3$?

c) What is the value of $(-1)^{403}$?

d) What is the value of $\left(\sqrt{3}\right)^6$?

We call the quantity a^n the number a raised to the nth **power**.

The meaning of a^1, a number raised to the first power, is a little cryptic.

$$a^1 = \overbrace{a \times a \times \cdots \times a}^{1 \text{ time}}$$

People interpret "the product of just one a" as just a.

$$a^1 = a$$

So,

$7^1 = 7$

$5033^1 = 5033$

and

$\left(-\frac{7}{\sqrt{3}}\right)^1 = -\frac{7}{\sqrt{3}}.$

The Story of Paper Folding

It was believed for centuries that one can fold a strip in paper in half only 7 or 8 times.

In 2002, then high school student Brittney Gallivan managed to fold a strip of toilet paper 2000 feet long in half twelve times, always folding from the same direction, right end to left end. (Others, such as the *MythBusters* team from a popular TV show have since folded large square pieces of paper in half from all four directions more than twelve times.)

Gallivan set a challenge to beat her record under the proviso that only a single strip of paper be used. In particular, no tape can be used to attach strips together.

In 2011, students at St. Mark's School in Boston created an object that is the equivalent of a single strip folded in half 13 times. It required ten miles of paper and use of MIT's "infinite corridor" as an indoor space with no breeze. They layered 64 sheets of paper on top of one another, taped the ends together in the exact pattern of a single strip of paper folded in half six times, and then completed an extra seven folds on that object.

Local media took this as breaking Gallivan's record. This was not the case: they used tape.

The folded object is in a glass case at St. Mark's School in Boston.

What is fascinating here is the dramatic effect of repeated halving with the rounding of the paper at the ends: ten miles of paper is reduced to an object about 5 feet long and 2 feet high.

Brittney wrote a book discussing her feat. In it, she gives a formula that models the expected final length of the object after folding given that paper necessarily rounds at each fold.

MUSINGS

Musing 112.4 A sheet of thin paper is about 0.05 millimeters thick.
The distance to the Moon is about $384{,}400$ km.

Barring physical constraints, how many folds does it take to repeatedly fold a (very long) strip of paper in half to obtain a wad high enough to reach the Moon?

Musing 112.5 Approximately, what is the area of your kitchen?

According to the internet, what is the area of the state or province in which you live?
According to the internet, what is the surface area of the Earth?

If your kitchen doubled in area every 24 hours (call that a *day*),

how many days will your kitchen cover your state? In how many days will your kitchen cover the surface of the Earth?

Musing 112.6 In Section 60 we mentioned that some authors define for a non-negative integer n and a general number a,

$$a^n = 1 \times \overbrace{a \times a \times \cdots \times a}^{n \text{ times}}$$

(They like to keep the "$1\ \times$" as part of the product.)

a) Is the value of 2^{10} still 1024 according to this alternative definition?

b) Is the value of 5033^1 still 5033, do you think, according to this definition?

c) Do you recall the reason why some authors prefer this definition?

We are not going to follow this definition of exponential notation in this chapter. That "$1\ \times$ " really is clunky for some of the work to come.

Moreover, we'll see that we can justify all the mathematics of exponents just beautifully with our less clunky definition.

Musing 112.7 A woman has five children. One August, she noticed that their ages (as a whole number of years) have a curious mathematical property:

For each day of the month, some combination her children have ages that add up to the day number.

What are the ages of her five children?

Musing 112.8 What is the first doubling number that begins with a seven? What's the second one and the third one?

$$1 \quad 2 \quad 4 \quad 8 \quad 16 \quad 32 \quad 64 \quad 128 \quad 256 \quad 512 \quad 1024 \quad \ldots$$

MECHANICS PRACTICE

Practice 112.9 Give three different examples of expressions of the form a^n with value 16.

Practice 112.10 If $(-1)^n$ equals 1, what can you say about the positive whole number n?

Practice 112.11 If a^{8088} equals 0, what can you say about the number a? How can you be sure?

Practice 112.12
a) Give an example of a number a where a^7 is larger than a.
b) Give an example of a number a where a^7 is smaller than a.
c) Give an example of a number a where a^7 is equal to a.

113. Pushing Paper-Folding until it Breaks

We have the definition

$$a^n = \overbrace{a \times a \times \cdots \times a}^{n \text{ times}}$$

This definition makes sense only for n a positive whole number $1, 2, 3, 4, \ldots$. (There are no restrictions on what type of number a can be.)

> **Practice 113.1** What is the value of a^0 according to this definition? What is the value of a^{-1}? (I hope you are objecting to this question as the definition makes absolutely no sense for $n = 0$ and $n = -1$!)

Nonetheless, we can use our physical model of paper folding to suggest what a^n could be if n is a new type of number beyond what the definition will allow.

Of course, we will go into this next exploration with our eyes open. We've learned that no one physical model "explains" everything about a particular piece of mathematics. But models can guide us to what the core mathematics at hand might be. Does paper-folding offer any guiding insights?

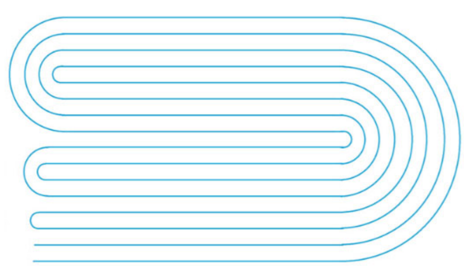
We've seen that the powers of two appear as the counts of layers in paper folding.

n folds create 2^n layers

1 fold makes $2^1 = 2$ layers

2 folds makes $2^2 = 4$ layers

3 folds make $2^3 = 8$ layers



4 folds make $2^4 = 16$ layers

Question: According to this model, what is the value of 2^0?
That is, how many layers of paper do we have if we conduct zero folds?

Well, if we conduct no folds, we'll have an untouched strip of paper still one layer thick.

0 folds "creates" 1 layer

The model suggests that

$$2^0 = 1$$

Now let's really push things.

Question: What does this model suggest as a value for 2^{-1}?

Hmm.
We need to fold a strip of paper in half -1 times. What could this mean?
What is the opposite of one fold?

A fold picks up the right end of the paper strip and brings it over to the left end, doubling the thickness of the strip. The reverse process would be to start at the left end of the strip and peel it apart, making a strip twice as long but half as thick.

Such peeling, such an "anti-fold" would change the strip from being 1 layer thick to $\frac{1}{2}$ a layer thick.

Paper folding suggests that

$$2^{-1} = \frac{1}{2}$$

Practice 113.2: Explain why 2^{-2} should have value $\frac{1}{4}$ according to this paper-folding model (with the act of peeling being consider the opposite of a fold).

Practice 113.3: Explain why, in general, 2^{-n} should have value $\frac{1}{2^n}$ according to this paper-folding model.

Many schoolbooks "explain" the meaning of negative powers by noticing a pattern.

$$
\begin{array}{ll}
\vdots & \\
2^3 = 8 & \\
 & \text{halve} \\
2^2 = 4 & \\
 & \text{halve} \\
2^1 = 2 & \\
 & \text{halve} \\
2^0 = 1 & \\
 & \text{halve} \\
2^{-1} = \frac{1}{2} & \\
 & \text{halve} \\
2^{-2} = \frac{1}{4} & \\
 & \text{halve} \\
2^{-3} = \frac{1}{8} & \\
\vdots &
\end{array}
$$

Paper folding is a physical manifestation of this pattern.
But as we saw in Chapter 13, patterns can be tricksters.

Practice 113.4 a) Use the difference table method of Section 102 to show that the sequence

$$\mathbf{1, 2, 4, 8, \ldots}$$

could be given by the four-chunk formula

$$-\frac{1}{6}(n-2)(n-3)(n-4) \; + \; (n-1)(n-3)(n-4)$$
$$-2(n-1)(n-2)(n-4) \; + \; \frac{4}{3}(n-1)(n-2)(n-3)$$

b) What does this formula give for $n = 0$? (That is, what does it suggest as the appropriate value for 2^0?)

c) What does this formula give for $n = -1$? (That is, what does it suggest as the appropriate value for 2^{-1}?)

Suppose we decide to trust patterns nonetheless and try pushing the paper-folding model further.

Question: What does paper-folding suggest for the value of $2^{\frac{1}{2}}$?

The issue we must contend with here is figuring out what "half a fold" could be.

Is this half a fold—a fold halfway across the paper? (Or is that a third of the way?)

Or this half a fold—half the motion of a fold? (Bending over 90 degrees rather than 180 degrees?)

It is hard to define what half a fold could possibly mean. And even if have an idea in mind, it seems even harder to answer: *And now how many layers of paper do we now have?*

Question: In the first picture, the left region is 1 layer thick and the right region is 2 layers thick. Should we take the average and say that $2^{\frac{1}{2}}$ represents 1.5 layers?

(Do you have any idea of how to define the number of layers there are in the second picture with a 90-degree bend?)

Interpreting half a fold is fraught!

Here's another approach.

Look at our data:

	0 folds	1 fold	2 folds	3 folds	4 folds	5 folds	...
Layers:	1	2	4	8	16	32	...

↑ half a fold?

The data seems to suggest that $2^{\frac{1}{2}}$ should, at the very least, have a value between 1 and 2. (Or is this just trusting patterns again?)

We might be bolder and say that since "one half" is halfway between 0 and 1 that maybe $2^{\frac{1}{2}}$ should be halfway between $2^0 = 1$ and $2^1 = 2$. That also suggests that $2^{\frac{1}{2}} = 1.5$.

But we are just flailing here.

The truth is that our paper-folding model, like all real-world attempts to fully illustrate math, is breaking down. It suggests meaningful values for 2^n for n a counting number, for n the opposite of a counting number, and for $n = 0$. But the model offers no real insight as to the value of 2^n for n any other type of number.

We are going to have to rely on mathematics to guide us beyond the real-world model.

But what mathematics?

MECHANICS PRACTICE

At present, we only have the basic definition of exponent notation at hand. For a number a and a positive whole number n we have:

$$a^n = \overbrace{a \times a \times \cdots \times a}^{n \text{ times}}$$

Practice 113.5 What is the value of 0^{38}?

Practice 113.6 The value of $(0.1)^n$ is a millionth. What's n?

Practice 113.7 What is the value of $10^2 \times 10^3$?

Practice 113.8 What is the value of $(2^3)^4$?

114. What we Require of Exponents

Let's keep playing with the powers of two since they are in our heads after paper folding.

For a counting number n we have

$$2^n = \overbrace{2\times 2\times \cdots \times 2}^{n \text{ times}}$$

(with the understanding that for $n = 1$ this is interpreted as $2^1 = 2$).

People describe 2^n as two **raised to the nth power** or as the **nth power** of two. The value n is called the **exponent** or the **power** in this expression and number 2 is the **base** number of the expression.

We defined exponentiation as "repeated multiplication." It has multiplication at its heart and so it seems natural to explore how exponentiation and multiplication interact.

Example: What is the value of $2^3 \times 2^5$?

Answer: Let's just write it out.

$$2^3 \times 2^5 = \;2 \times 2 \times 2 \quad \times \quad 2 \times 2 \times 2 \times 2 \times 2$$

We see a product of three 2s and a product of five 2s multiplied together.
The result is a product of eight 2s multiplied together.

$$2^3 \times 2^5 = 2^8$$

which equals 256.

Practice 114.1 Explain why the value of $10^{13} \times 10^{27}$ is sure to itself be a power of ten. Which power of ten?

Practice 114.2 Explain why the value of $917^{500} \times 917^{303}$ is sure to itself be a power of 917. Which power of 917?

We can see by writing products out in full (at least in our minds) that for any two counting numbers n and m we have

$$\mathbf{2^n \times 2^m = 2^{n+m}}$$

(And there is nothing special about the base number 2 here. This observation will hold no matter the base number.)

This seems like a fundamental behavior of exponentiation that lies at the heart of any general theory of exponents. So, let's see if we can develop such a theory based (ha!) on this key behavior of exponents.

> *Let's decide that* $2^m \times 2^n$ *should equal* 2^{m+n} *no matter what types of numbers* n *and* m *happen to be. What mathematics logically follows?*

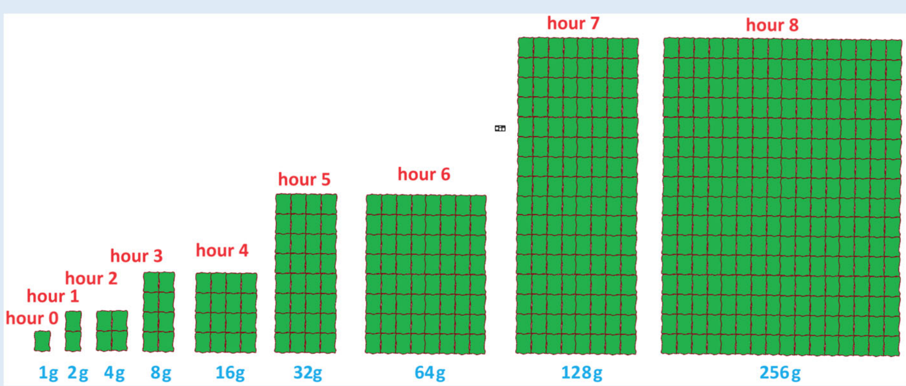
Let's explore this question.

ASIDE: A Deeper Motivation

There is another practical reason for wanting $2^{\mathrm{m}} \times 2^{\mathrm{n}}$ to always equal $2^{\mathrm{m+n}}$. It comes from thinking about "continuous multiplicative growth."

Imagine some growing green gooey biological culture that doubles in mass every hour.

Suppose we start with 1 gram of culture. Then after one hour we'll have 2 grams of culture, 4 grams after another hour, and so forth. It's mass at time n hours will be 2^n grams for each positive whole number n. (It's the initial one gram of mass has doubled n times. It has grown by a factor of 2^n.)



But this culture is continuously growing between the hour marks as well.

So, it seems natural to write for each time t hours that the mass of the culture is 2^t grams, even if t is not a whole number of hours and we're not sure what 2^t means. But 2^t will be some positive number representing the mass of the culture at time t.

Now let's think some more about what this scenario suggests.

We know …

Every three hours the culture doubles in mass three times. That is, it grows by a factor of 2^3.
Every five hours the culture doubles in mass five times. That is, it grows by a factor of 2^5.

So, if I watch the culture for 3 hours and then another 5 hours, then I will see the mass of the culture grow by a factor of 2^3 and then that result grow by a factor of 2^5. That is, overall it grows by a factor of $2^3 \times 2^5$.

But I will have been sitting there for a total of 8 hours, so the overall growth should actually be a factor of 2^8.

For this to be consistent, we need $2^3 \times 2^5$ to be the same as 2^8. (And we know this is the case for these nice whole numbers.)

But here's the kicker: This idea should hold for any two amounts of time I choose to watch the culture grow.

> If I watch the culture grow for m hours and then n hours, I will have sat there watching for a total of $m + n$ hours.
>
> The growth factor I witness could be expressed as $2^m \times 2^n$ and also as 2^{m+n}.
>
> These must be the same!
> And this must be the case even if m and n are not whole numbers of hours.

This imaginary biological example is showing us that it could well be useful to make sense of a quantity 2^t for all types of numbers t and have this quantity follow the behavior

$$2^m \times 2^n = 2^{m+n}$$

--always.

This is exactly what we are about to do.

Here's where we are at.

- For each positive whole number n we have defined 2^n to be

$$2^n = \overbrace{2\times 2\times \cdots \times 2}^{n \text{ times}}$$

So, we know $2^3 = 8$ and $2^{10} = 1024$ and $2^1 = 2$, and so on.

- We have observed the following fundamental behavior of (positive whole number) exponents.

$$2^n \times 2^m = 2^{n+m}$$

- We are declaring that for every number t, there is a quantity 2^t, even if t is not a positive whole number. The biological example suggests that 2^t will always be a positive number.

- We would like the fundamental property of exponents to continue to hold even if m and n are not positive whole numbers. Moreover, we are wondering if we can use the fundamental property to determine what the value of 2^t must be for different real numbers t.

Let's try!

Observation: Just as paper-folding suggests: $\mathbf{2^0 = 1}$.

Reason: Let's play with $2^m \times 2^n = 2^{m+n}$ choosing values for m and n that could be of benefit to us.

I know $2^3 = 8$ and we want to figure out 2^0.
Let's examine

$$2^3 \times 2^0 = 2^{3+0}$$

This is really

$$2^3 \times 2^0 = 2^3$$

which is

$$8 \times 2^0 = 8$$

So, something times 8 is 8. That something must be 1.

$$2^0 = 1$$

Practice 114.3: Would you come to the same conclusion looking at $2^9 \times 2^0$?

Observation: Just as paper-folding suggests: $\mathbf{2^{-1} = \frac{1}{2}}$.

Reason: Again, let's play with $2^m \times 2^n = 2^{m+n}$.

Let's examine

$$2^3 \times 2^{-1} = 2^{3+-1}$$

This is really

$$2^3 \times 2^{-1} = 2^2$$

I know the values of 2^3 and 2^2, so this reads

$$8 \times 2^{-1} = 4$$

So, something times 8 is 4. That something must be $\frac{1}{2}$.

$$2^{-1} = \frac{1}{2}$$

Practice 114.4: Establish that 2^{-2} must be $\frac{1}{4}$.

Practice 114.5: Establish that 2^{-3} must be $\frac{1}{8}$.

At this point, many textbooks will point out that

$$2^n \times 2^{-n} = 2^0$$

Since we've established that $2^0 = 1$, this reads

$$2^n \times 2^{-n} = 1$$

Multiplying each side of this equation by $\frac{1}{2^n}$ shows

$$2^{-n} = \frac{1}{2^n}$$

This holds no matter what type of number n might be.

Practice 114.6: At the end of section 112 we said that interpreting the value of 2^1 is a "little cryptic." Look at $2^3 \times 2^1$ to explain why 2^1 must be 2.

Up to this point, mathematics has agreed with everything our paper-folding model has suggested.

But paper-folding failed to give insight for a value to $2^{\frac{1}{2}}$.

What does the mathematics determine?

A typical first instinct is to look at something like

$$2^3 \times 2^{\frac{1}{2}} = 2^{3\frac{1}{2}}$$

Even though we know that $2^3 = 8$, making sense of $2^{3\frac{1}{2}}$ is just as difficult as trying to make sense of $2^{\frac{1}{2}}$. This natural attempt is of no help.

It's takes a flash of brilliance to realize that looking at $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ moves you forward.

$$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^1$$

Since $2^1 = 2$, this reads

$$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2$$

and we thus see that $2^{\frac{1}{2}}$ is a value that multiplies by itself to give the answer 2. That's the square root of 2, and we want the positive version of the square root (as the biological example suggests).

$$\mathbf{2^{\frac{1}{2}} = \sqrt{2}}$$

The value of $\sqrt{2}$ is about 1.414. So, the value of $2^{\frac{1}{2}}$ is, surprisingly, not halfway between the values of 2^1 and 2^2.

Question: What do you think of the interpretation: *Folding a strip of paper in half, half a time produces an object* $1.414\ldots$ *layers thick?* (Sounds like nonsense to me!)

Practice 114.7:

a) Show that $2^{\frac{3}{2}}$ must be $\sqrt{8}$ by looking at $2^{\frac{3}{2}} \times 2^{\frac{3}{2}}$.

b) Show that $2^{\frac{3}{2}}$ must be $2\sqrt{2}$ by looking at $2^{1} \times 2^{\frac{1}{2}}$.

c) Are $\sqrt{8}$ and $2\sqrt{2}$ the same number?

It seems natural to presume that our basic property of exponents extends to a product of three terms.

$$2^{m} \times 2^{n} \times 2^{r} = 2^{m+n+r}$$

And this does follow logically from the fact that "order does not matter" in multiplication and it does not matter in addition. So, we can multiply together just the first two terms first and see where that takes us.

$$\begin{aligned} 2^{m} \times 2^{n} \times 2^{r} &= (2^{m} \times 2^{n}) \times 2^{r} \\ &= 2^{m+n} \times 2^{r} \\ &= 2^{(m+n)+r} \\ &= 2^{m+n+r} \end{aligned}$$

In the same way our basic property of exponents extends to a product of four or more terms too.

We can make use of this.

Example: Show that $2^{\frac{1}{3}}$ is the **cube root** of 2. That is, it is a number that multiplies by itself three times to give the answer 2.

Answer: We have

$$2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 2^{1} = 2$$

According to my calculator, $2^{\frac{1}{3}}$ is about 1.260.

Check: Is $1.260 \times 1.260 \times 1.260$ close to 2?

For a positive whole number n, people call a number that multiplies by itself n times to give the answer a an **nth root of a** and denote it $\sqrt[n]{a}$. (People assume that n is at least 2.)

For $n = 2$ this is called a **square root** of a and people usually just write $\sqrt{a}$ rather than $\sqrt[2]{a}$.

For $n = 3$ this is called a **cube root** of a.

For example,

$\sqrt[3]{64} = 4$ since $4 \times 4 \times 4 = 64$

$\sqrt[6]{1{,}000{,}000} = 10$ since $10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1{,}000{,}000$

$\sqrt[401]{1} = 1$ since $1 \times 1 \times \cdots \times 1 = 1$

There is some confusion about our convention: Because we are using the geometry symbol √, all quantities must be positive, possibly 0, but never negative.

But you will see people write, for instance,

$$\sqrt[3]{-8} = -2$$

because $(-2) \times (-2) \times (-2) = -8$ and there is no option other than -2 for a number that multiplies by itself thrice to give -8. People want to loosen up the convention.

If a negative number has an nth root, then we may use the symbol √ to *state that root.*

Carrying on …

> **Practice 114.8:** Show that $2^{\frac{1}{5}}$ is the **fifth root** of 2. That is, it is a number that multiplies by itself five times to give the answer 2.

We're seeing that if n is a whole number bigger than or equal to 2 then

$$2^{\frac{1}{n}} = \sqrt[n]{2}$$

because multiplying $2^{\frac{1}{n}}$ by itself n times gives 2.

$$2^{\frac{1}{n}} \times 2^{\frac{1}{n}} \times \cdots \times 2^{\frac{1}{n}} = 2^{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}} = 2^1 = 2$$

Practice 114.9:

a) Show that $2^{\frac{3}{5}}$ is a value that multiplies by itself five times to give the value 2^3 and so is $\sqrt[5]{2^3}$.

b) Show that $2^{\frac{1}{5}}$ multiplied by itself three times gives $2^{\frac{3}{5}}$.

c) Argue now that $\sqrt[5]{2^3}$, the fifth root of the quantity two cubed, and $(\sqrt[5]{2})^3$, the fifth root of two then cubed, must be the same value.
(Can you also check that they are on a calculator? Do you see the right keys to press?)

This practice example shows how to give meaning to $2^{\frac{a}{b}}$ for any positive fraction $\frac{a}{b}$.

Schoolbooks might have students memorize that it can be interpreted both as "the bth root of 2^a" and as "the bth root of 2 raised to the ath power," but all that is important that we have no given it meaning—and if you need to know its value for a specific fraction $\frac{a}{b}$, just use a calculator!

But we saw in Sections 58 and 59 that not every number is a fraction. For example, we proved that $\sqrt{2}$ cannot be written as a fraction.

So, can we give a meaningful value to

$$2^{\sqrt{2}}$$

for instance?

Question: Type $2^{\sqrt{2}}$ into a calculator. Does the calculator assign it a value?

Here's the approach mathematicians—and your calculator—take to make sense of this quantity.

Start by noting that even though $\sqrt{2}$ has an infinitely long decimal expansion (without a repeating pattern)

$$\sqrt{2} = 1.41421356\ldots$$

we can approximate its value more and more accurately by a list of fractions

$$1$$

$$1.4 = \frac{14}{10}$$

$$1.41 = \frac{141}{100}$$

$$1.414 = \frac{1414}{1000}$$

$$1.4142 = \frac{14142}{10000}$$

$$\vdots$$

And we have a meaning for $2^{\frac{a}{b}}$ for each such fraction $\frac{a}{b}$.

$$2^1 = 2$$

$$2^{1.4} = 2^{\frac{14}{10}} = \sqrt[10]{2^{14}} \approx 2.639$$

$$2^{1.41} = 2^{\frac{141}{100}} = \sqrt[100]{2^{141}} \approx 2.657$$

$$2^{1.414} = 2^{\frac{1414}{1000}} = \sqrt[1000]{2^{1414}} \approx 2.665$$

$$2^{1.4142} = 2^{\frac{14142}{10000}} = \sqrt[10000]{2^{14142}} \approx 2.6651$$

$$2^{1.41421} = 2^{\frac{141421}{100000}} = \sqrt[100000]{2^{141421}} \approx 2.66514$$

$$\vdots$$

We can define the value of $2^{\sqrt{2}}$ to be the limit of these approximation values. It looks like the value of $2^{\sqrt{2}}$ is heading to be 2.66514 ….

We can now obtain values for 2^t no matter what type of number t might be: positive or negative; whole, fractional, or irrational.

Each value 2^t is a positive number, as per the biological example, and we have

$$2^m \times 2^n = 2^{m+n}$$

set up to always work.

We've seen along the way that for all numbers t

$$2^{-t} = \frac{1}{2^t}$$

and for each positive whole number n two or bigger

$$2^{\frac{1}{n}} = \sqrt[n]{2}$$

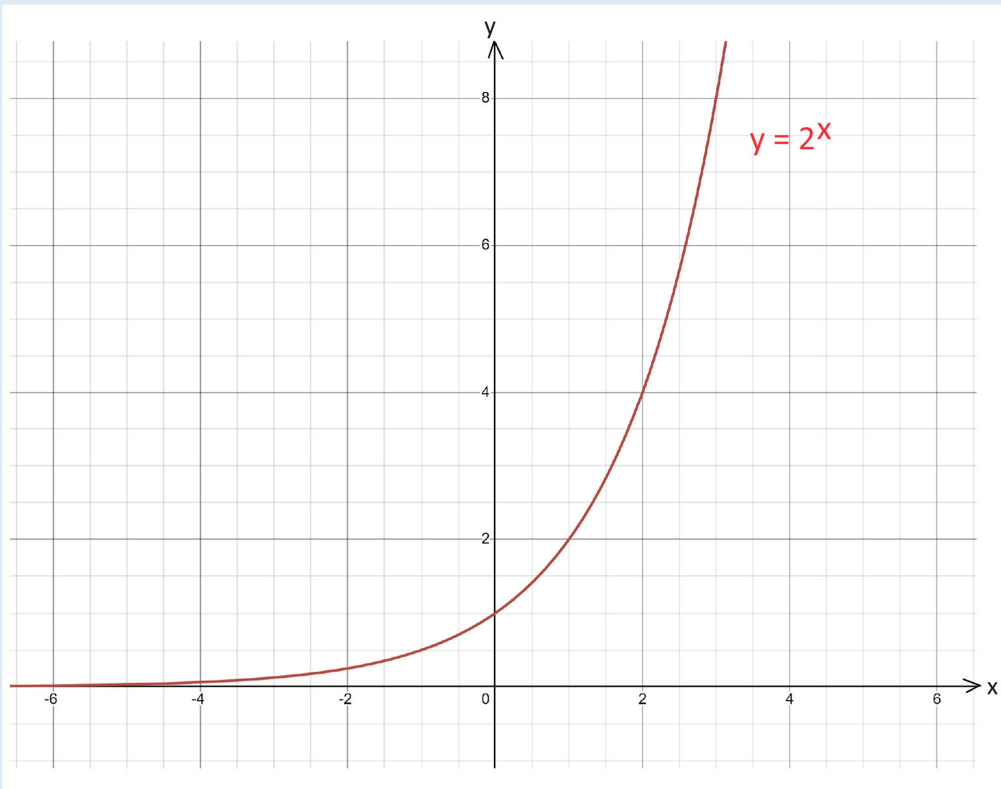
We've also seen that

$$2^0 = 1$$

MUSINGS

Musing 114.10 Using a computer graphing system, sketch a graph of the equation $y = 2^x$.

You should obtain a picture like this:



a) Does your graph show that $2^3 = 8$ and that $2^{-2} = \frac{1}{4}$?

b) What is your graph showing for the value of 2^0?

c) What is your graph showing for the value of $2^{\frac{1}{2}}$? Does it look to be 1.414 ...?

d) What is your graph showing for the value of $2^{\sqrt{2}}$? Does it look to be 2.665...?

MECHANICS PRACTICE

Practice 114.11 Show that $2^{-\frac{3}{2}}$ equals $\frac{1}{4}\sqrt{2}$.

115. Exponents are either Terrible or Exceptionally Good

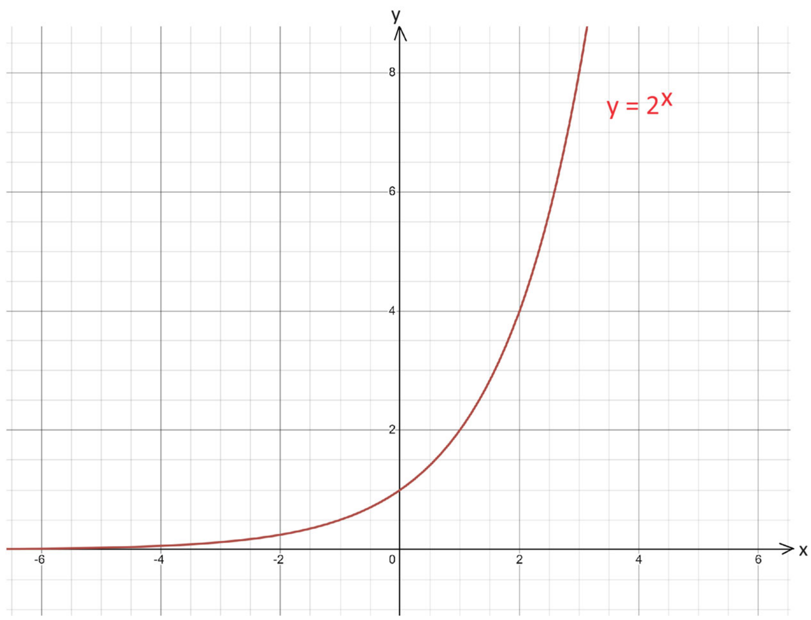
We've had success in playing with the powers of two and using the basic exponent rule

$$2^m \times 2^n = 2^{m+n}$$

to extend the story of exponents from positive whole number exponents (with $2^3 = 2 \times 2 \times 2 = 8$ and $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$, and the like) to quantities of the form 2^x where x need not be a whole number.

We were motivated by paper-folding and a biological example of continuous multiplicative growth which suggested that the basic exponent rule should work for all types of numbers.

Mathematics then told us that $2^0 = 1$ and $2^{-2} = \frac{1}{4}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$ and so on. We learned how to give a meaningful value to 2^x no matter the number x.



Practice 115.1 There is probably nothing special about the number 2 here.

a) Using graphing software, sketch a graph of $y = 3^x$. Does there seem to be a value to 3^x for each real number x? (And is the value 9 for $x = 2$ and 27 for $x = 3$ as you would expect?)

b) Can your program also handle graphs of $y = 10^x$ and $y = (\frac{1}{2})^x$ and $y = (1.013)^x$?

Everything seems hunky-dory right now, for sure.

Yet historically, mathematicians were deeply befuddled and wary of exponents for centuries. It is not clear when exactly they might "turn" on you and cause problems.

For example, given what we've learned, we can compute $(-8)^2$ and $(-8)^{-2}$ without a hitch.

$$(-8)^2 = (-8) \times (-8) = 64$$

$$(-8)^{-2} = \frac{1}{(-8)^2} = \frac{1}{64}$$

We can even compute $(-8)^{\frac{1}{3}}$

$$(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$$

But try graphing $y = (-8)^x$ with a graphing program and you will see it give up and not even try.

Practice 115.2 See for yourself! Ask some graphing software to graph $y = (-8)^x$.

The issue is that your program does not know what value to give to $(-8)^x$ for an infinite number of real values x.

For instance,

$$(-8)^{\frac{1}{2}} = \sqrt{-8}$$

is meaningless as we know that there is no square root of -8. (No number multiplies by itself to be negative.)

The quantities $(-8)^{\frac{1}{4}}$ and $(-8)^{1.41}$ and $(-8)^{1.41459}$, and so many more, cannot exist, at least with the approach we took with exponents in the last section.

Is there a different approach we should take?

That was a significant question in the 1700s and 1800s for mathematics.

Troublesome Zero

We've seen throughout all these writings how confusing and tricky zero can be to work with. Alas, the situation is no different when it comes to zero role in the study of exponents.

Let's ask

What is the appropriate value to give to the quantity 0^0 ?

Some values of exponents with a base of 0 are clear. For example, we know

$$0^2 = 0 \times 0 = 0$$

$$0^3 = 0 \times 0 \times 0 = 0$$

$$0^4 = 0 \times 0 \times 0 \times 0 = 0$$

and so on.

Let me be provocative and absurd and suggest:

The value of 0^0 is $36\frac{1}{2}$.

Here's my argument for this absurdity.

From our basic exponent rule we know

$$0^2 \times 0^0 = 0^2$$

But $0^2 = 0 \times 0 = 0$, *so this reads*

$$0 \times 0^0 = 0$$

Setting 0^0 *as* $36\frac{1}{2}$ *certainly works here:* $0 \times 36\frac{1}{2}$ *does equal* 0*!*

Of course, setting 0^0 as 23 or as 107 or as $-\sqrt{85}$ all would work here too. Every real number could be a possible value of 0^0. (Does this ring a bell from our discussion on dividing by zero in section 17?)

So, what is one meant to make of 0^0?

As I said, the story of exponents is deeply befuddling.

The Breakthrough of the 1800s

It took some giants of mathematics make proper sense of exponents as it required heading into the territory of very advanced mathematics. (Just look at the Wikipedia page on exponentiation to see it evolve from a discussion akin to what I have presented here to some really advanced mathematical thinking.)

But there is a light for us.

As an output of this work, mathematicians established that everything we conducted in the last section about the powers of 2, applies and correctly works for any positive base number, not just the number 2.

To be explicit, they showed:

> If a is a positive real number, then it is possible to make sense of a^x for any real number x.
>
> The value of a^x is a positive number for each value of x.
>
> If x happens to be a positive whole number n, we have
>
> $$a^n = \overbrace{a \times a \times \cdots \times a}^{n \text{ times}}$$
>
> We also have,
>
> $$a^{\frac{1}{n}} = \sqrt[n]{a}$$
>
> and
>
> $$a^{-x} = \frac{1}{a^x}$$
>
> just like we did for the powers of 2.

The message is this:

> *If you are doing work with exponents, try to keep to a positive base number a: everything you intuitively expect to be true will be true.*
>
> *If you must think about an exponent with 0 as a base, or with a negative base, then keep your wits about you and proceed with caution!*

But there is more. We can take the phrase "everything you intuitively expect to be true will be true" to heart when working with a positive base.

Let me explain what I mean by this.

Example: Please rewrite $(5^3)^4$ as a power of five in its own right.
(This is five cubed, which is then raised to the fourth power.)

Answer: We have parentheses in this expression and the order-of-operations rules have us compute the quantity insider the parentheses first. I'll leave it as a product of three fives.

$$(5^3)^4 = (5 \times 5 \times 5)^4$$

This product of fives is being raised to the fourth power.

$$(5^3)^4 \quad = \quad 5 \times 5 \times 5 \quad \times \quad 5 \times 5 \times 5 \quad \times \quad 5 \times 5 \times 5 \quad \times \quad 5 \times 5 \times 5$$

The result is a product of twelve 5s.

$$(5^3)^4 = 5^{12}$$

Practice 115.3

a) Write out $(3^2)^{10}$ in full to see it is the same as 3^{20}.

b) What is $(7^4)^9$ as a power of 7?

c) What is $\left(\left(\frac{2}{9}\right)^4\right)^9$ as a power of $\frac{2}{9}$?

d) What is $(1.3^{204})^{50}$ as a power of 1.3?

We're seeing that for a number a and positive whole numbers m and n,

$$(a^m)^n \quad = \quad \overbrace{\overbrace{a \times a \times \cdots \times a}^{m \text{ times}} \times \overbrace{a \times a \times \cdots \times a}^{m \text{ times}} \times \cdots \times \overbrace{a \times a \times \cdots \times a}^{m \text{ times}}}^{n \text{ times}} \quad = \quad a^{m \times n}$$

The work of mathematics scholars two centuries ago gave this bonus:

$$(a^m)^n \text{ is sure to equal } a^{m \times n}$$

even if m and n are not positive whole numbers.

And there is a second bonus!

Example:
a) Show that $(2 \cdot 3)^5$ equals $2^5 \cdot 3^5$.
a) Show that $\left(\frac{2}{3}\right)^5$ equals $\frac{2^5}{3^5}$

Answer: Since we are working a whole number exponent, we can write out each of these expressions in full.

$$\begin{aligned}(2 \cdot 3)^5 &= 2 \times 3 \ \times \ 2 \times 3 \ \times \ 2 \times 3 \ \times \ 2 \times 3 \ \times \ 2 \times 3 \\ &= 2 \times 2 \times 2 \times 2 \times 2 \ \times \ 3 \times 3 \times 3 \times 3 \times 3 \\ &= 2^5 \times 3^5\end{aligned}$$

$$\left(\frac{2}{3}\right)^5 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3 \times 3} = \frac{2^5}{3^5}$$

Practice 115.4

a) Write out $(a \times b)^4$ in full to see it is the same as $a^4 \times b^4$.

b) Show that $\left(2\sqrt{3}\right)^6 = 64 \times 27$

c) If $\left(\frac{a}{b}\right)^{907} = \frac{a^n}{b^n}$, what is n?

d) What is the value of $\left(2\sqrt[3]{5}\right)^3$?

We're seeing for numbers a and b (with b not zero) and a positive whole number n

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

The mathematical work of two centuries ago came with the second bonus: these two properties of exponents hold even if n is not a whole number.

Next is a summary of all that we can hold on to.

SUMMARY ON THE NICELY BEHAVED EXPONENTS

For a positive real number a, it is possible to make sense of a^x for any real number x.
The value of a^x is a positive number for each value of x.

If x happens to be a positive whole number n, then

$$\boldsymbol{a^n = \overbrace{a \times a \times \cdots \times a}^{n \text{ times}}}$$

with

$$\boldsymbol{a^1 = a}$$

We also have

$$\boldsymbol{a^0 = 1}$$

For a positive whole number

$$\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

For general real numbers x and y and positive base numbers a and b, the following also hold.

$$\boldsymbol{a^x \times a^y = a^{x+y}}$$

Exponents add when multiplying powers with the same base

$$\boldsymbol{a^{-x} = \frac{1}{a^x}}$$

Negative exponents lead to reciprocals

$$\boldsymbol{(a^x)^y = a^{xy}}$$

Exponents multiply when taking a power of a power

$$\boldsymbol{(ab)^x = a^x b^x}$$

The power of a product is the product of the powers

and if b is not zero …

$$\boldsymbol{\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}}$$

The power of a quotient is the quotient of the powers

These exponents rules can often be applied to an example with 0 or a negative number as its base, but one must exercise caution. For example, saying $(-2)^{-n} = \frac{1}{(-2)^n}$ is fine for every whole number n, but $(-2)^{\frac{1}{n}}$ does not exist if n is even.

Since these exponents rules mimic what is true for whole number exponents, one can always think though concrete whole number examples to remember them.

$\mathbf{2^3 \times 2^5 =}$ eight 2s multiplied together $=$ $\mathbf{2^8}$

$\left(5^3\right)^4 = 5^3 \times 5^3 \times 5^3 \times 5^3 = \mathbf{5^{3\times 4}}$

$\mathbf{7^0 = 1}$ because $7^0 \times 7^2 = 7^2$, which is $7^0 \times 49 = 49$

$\mathbf{3^{-2} = \frac{1}{3^2} = \frac{1}{9}}$ because $3^{-2} \times 3^4 = 3^2$, which is $3^{-2} \times 81 = 9$

$\mathbf{2^{\frac{1}{2}} = \sqrt{2}}$ because $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2$

$(\boldsymbol{ab})^3 = \; ab \times ab \times ab = a \times a \times a \times b \times b \times b = \; \boldsymbol{a^3 \times b^3}$

Here are some typical textbook examples that are designed to help students master the mechanics of exponents.

Example: Please make $(4x^2)^3$ look a tad friendlier.

Answer: Whole number exponents are always easier to grasp.

$$(4x^2)^3 = 4x^2 \times 4x^2 \times 4x^2 = 4^3x^6 = 64x^6$$

(I suppose that is friendlier?)

Example: Please make $(4x^2)^{\frac{1}{2}}$ look a tad friendlier.
(Assume that x is a positive number.)

Answer: As we don't have whole number exponents, we must use the exponent rules abstractly.

$$(4x^2)^{\frac{1}{2}} = 4^{\frac{1}{2}} \times (x^2)^{\frac{1}{2}} = \sqrt{4} \times \sqrt{x^2} = 2x$$

Practice 115.5 Can you see that the answer to this problem is "$-2x$" if, instead, we were told that x represents a negative number?

Example: The volume of a sphere is given by $\frac{4}{3}\pi r^3$ where r is the radius of the sphere. The radius is half the diameter d of the sphere. How does this formula appear solely in terms of the sphere's diameter?

Answer: We have $r = \frac{d}{2}$, so this formula appears as

$$\frac{4}{3}\pi\left(\frac{d}{2}\right)^3$$

This is

$$\frac{4}{3}\pi\frac{d^3}{8} = 4 \times \frac{1}{3} \times \pi \times d^3 \times \frac{1}{8}$$

(It never hurts to tease things fully apart.)

And this is the same as

$$\frac{\pi d^3}{6}$$

Example: Find the value of

$$3^5 \cdot (3^{-1})^5$$

Answer: I see a power of a power. So, this expression can be rewritten

$$3^5 \cdot 3^{(-1)\times 5} = 3^5 \cdot 3^{-5}$$

and this is

$$3^{5\,+\,-5} = 3^0 = 1$$

Example: Find the value of the

$$\frac{1}{8^{-\frac{1}{3}}}$$

Answer: A negative exponent leads to a reciprocal, so we have

$$\frac{1}{\frac{1}{8^{\frac{1}{3}}}}$$

To avoid the fraction within a fraction, multiply top and bottom by $8^{\frac{1}{3}}$.

$$\frac{1 \times 8^{\frac{1}{3}}}{\frac{1}{8^{\frac{1}{3}}} \times 8^{\frac{1}{3}}} = 8^{\frac{1}{3}}$$

This is the cube root of 8, which is 2.

Practice 115.6 Show that $\frac{1}{a^{-x}}$ is the same as a^x for a positive number a and real number x.

Example: Find the value of

$$\left(\frac{27}{64}\right)^{-\frac{2}{3}}$$

without the aid of any technology.

Answer: My instinct is to just type this into a calculator. But we have to pretend we're in the 1800s.

Okay.

The numbers 27 and 64 are suspicious. We have that $27 = 3^3$ and $64 = 4^3$.
(How do I know that? I've done textbook algebra problems before!)

Our quantity is

$$\left(\frac{3^3}{4^3}\right)^{-\frac{2}{3}}$$

A power of a quotient is the quotient of the powers. This is

$$\frac{(3^3)^{-\frac{2}{3}}}{(4^3)^{-\frac{2}{3}}}$$

To compute the power of a power, multiply the exponents. This gives

$$\frac{3^{3\times\left(-\frac{2}{3}\right)}}{4^{3\times\left(-\frac{2}{3}\right)}}$$

which is

$$\frac{3^{-2}}{4^{-2}}$$

The fun keeps going!

Negative exponents correspond to reciprocals. We have

$$\frac{\frac{1}{3^2}}{\frac{1}{4^2}} = \frac{\frac{1}{9}}{\frac{1}{16}}$$

Multiplying top and bottom each 9 and by 16 gives

$$\frac{\frac{1}{9} \times 9 \times 16}{\frac{1}{16} \times 9 \times 16} = \frac{16}{9}$$

So, in the end

$$\left(\frac{27}{64}\right)^{-\frac{2}{3}} = \frac{16}{9}$$

Whoa!

Two comments are in order here.

Familiar Powers

School practice problems on this topic often rely on students recognizing some numbers as powers. Typically, these numbers:

$8 = 2^3$ $\quad$ $16 = 2^4$ $\quad$ $32 = 2^5$

$64 = 4^3$ $\quad$ $125 = 5^3$ $\quad$ $216 = 6^3$ $\quad$ $343 = 7^3$ $\quad$ $1000 = 10^3$

All the square numbers: $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, $25 = 5^2$, $36 = 6^2$, $49 = 7^2$, etc.

Order of Operations

Do you recall from sections 8 and 9 that multiplication comes with a hidden vinculum (that is, hidden parentheses)? For instance,

$$2 + 3 \times 4$$

is really

$$2 + \overline{3 \times 4}$$

Since exponents are derived from repeated multiplication, there is a convention to regard any exponential expressions as coming with a hidden vinculum too—and that the hidden vinculum of an exponential term supersedes the hidden vinculum of multiplication. (Got that? Some examples will help.)

We have:

$3 + 2^2 = 3 + \overline{2^2}$ $\quad$ and this equals 7

$3 \cdot 2^2 = \overline{3 \cdot \overline{2^2}}$ $\quad$ and this equals 12

$3^2 \cdot 2^2 = \overline{\overline{3^2} \cdot \overline{2^2}}$ $\quad$ and this equals $9 \times 4 = 36$

Parentheses continue to supersede all grouping conventions.

For example

$$(2+3)^2$$

equals 25 and not 13.

But there is one piece of naughtiness …

It is not clear to what to make of the following with our “order of operation” rules.

$$2^{3+4}+5$$

It is natural to read this as $2^7+5=128+5=133$ (and we have been doing this sort of thing all along already!).

And why does it feel natural to interpret $2^{3+4}+5$ as $(2^7)+5$ and not as $(2^3)+4+5$?

Most likely because “$3+4$” is in its own special small font as part of the superscript, making it feel that it has already been separated out as its own little group.

So, there is an extra caveat to our order of operation rules:

> *Treat exponents in an exponential expression as their own special little group to be evaluated unto themselves.*

Like I said, we’ve been following this caveat without explicitly thinking about it.
I hope pointing it out explicitly hasn’t suddenly made it confusing!

Example: Kindly solve $5 \cdot 25^{2x+1} = 625$.

Answer: We see "$2x + 1$" its own little group here.

Keeping that mind, observe that there is a lot of "fiveness" in this question. We seem to be dealing with powers of 5 and that surely is not incidental.

$$5 = 5^1 \qquad 25 = 5^2 \qquad 125 = 5^3 \qquad 625 = 5^4 \qquad \ldots$$

So, let's unravel things.

Remember that exponential expressions come with their own hidden vinculum. So, our equation really reads

$$5 \cdot \overline{25^{2x+1}} = 625$$

It seems compelling to multiply each side of the equation by $\frac{1}{5}$ (that is, to divide each side of the equation by 5).

$$25^{2x+1} = 125$$

Now it feels compelling to think of 25 as 5^2 and 125 as 5^3.

$$(5^2)^{2x+1} = 5^3$$

We know that a power of a power corresponds to multiplying exponents. So, this equation reads

$$5^{2(2x+1)} = 5^3$$

These powers of five must match. We conclude

$$2(2x + 1) = 3$$

That is,

$$4x + 2 = 3$$

$$4x = 1$$

$$x = \frac{1}{4}$$

Practice 115.7 Please solve

$$5 \cdot 2^{3-x} = 40$$

Example: Kindly solve $7^{x+2} + 7^x = 350$.

Answer: I look at this and again think that none of the numbers are likely incidental.

What has 350 to do with the number 7?

Well,

$$350 = 7 \times 50$$

Hmm.

I look at the expression on the left side, $7^{x+2} + 7^x$, and think there is little I can do here: exponentiation is about multiplication. None of our rules inform us about how it interacts with addition (if it does at all).

Again … Hmm.

What can we possibly do?

We do have an exponentiation rule that tells us something about 7^{x+2}: it's $7^x \cdot 7^2 = 49 \cdot 7^x$.

Ahh! The equation reads

$$49 \cdot 7^x + 7^x = 350$$

There is a common factor of 7^x, and so we can rewrite this as

$$(49 + 1) \cdot 7^x = 350$$

That is,

$$50 \cdot 7^x = 350$$

And now I see why the author of this question (okay, it was me!) chose the number 350.

Multiplying through by $\frac{1}{50}$ we get

$$7^x = 7$$

It must be that

$$x = 1$$

Practice 115.8 Please solve

$$13^{1-x} - 10 \cdot 13^{-x} = 507$$

Here's a truly un-fun high-school textbook problem.

Example: Please make the following expression look friendlier.

$$\frac{(4xy^{-2})^{\frac{1}{2}}x^{-\frac{3}{2}}}{\sqrt{x} \cdot \left(x^{-1}y^{\frac{1}{4}}\right)^{-2}}$$

Why an expression like this would ever come up in one's work as a mathematician or scientist beats me! Such a question is simply designed to test our mettle on implementing the exponential rules.

Just to remind us we have:

$\boldsymbol{a^x \cdot a^y = a^{x+y}}$	Powers add when multiplying exponents with the same base
$\boldsymbol{(a^x)^y = a^{xy}}$	Powers multiply when taking a power of a power
$\boldsymbol{a^{-x} = \frac{1}{a^x}}$	Negative exponents give reciprocals
$\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$	Basic fractions as exponents give roots
$\boldsymbol{(ab)^x = a^x \cdot b^x}$	The power of a product is the product of the powers. (Similarly, for powers of quotients.)

Now to our example.

I guess we are presuming that x and y are positive numbers so that all the quantities in the problem are meaningful.

Here goes.

Answer 1: A Slow Careful Approach

Looking at

$$\frac{(4xy^{-2})^{\frac{1}{2}}x^{-\frac{3}{2}}}{\sqrt{x}\cdot\left(x^{-1}y^{\frac{1}{4}}\right)^{-2}}$$

my first impulse is to write $\sqrt{x}$ as $x^{\frac{1}{2}}$ so that every part of the expression is a power.

I am also seeing $(4xy^{-2})^{\frac{1}{2}}$ and $\left(x^{-1}y^{\frac{1}{4}}\right)^{-2}$, which are powers of a product, and I want to rewrite those too.

Let me do all that to get

$$\frac{4^{\frac{1}{2}}\cdot x^{\frac{1}{2}}\cdot(y^{-2})^{\frac{1}{2}}\cdot x^{-\frac{3}{2}}}{x^{\frac{1}{2}}\cdot(x^{-1})^{-2}\left(y^{\frac{1}{4}}\right)^{-2}}$$

I know $4^{\frac{1}{2}}=\sqrt{4}=2$. We also have some powers of powers. The expression is

$$\frac{2\cdot x^{\frac{1}{2}}\cdot y^{-1}\cdot x^{-\frac{3}{2}}}{x^{\frac{1}{2}}\cdot x^{2}\cdot y^{-\frac{1}{2}}}$$

I see negative exponents. This is

$$\frac{2\cdot x^{\frac{1}{2}}\cdot\frac{1}{y}\cdot\frac{1}{x^{\frac{3}{2}}}}{x^{\frac{1}{2}}\cdot x^{2}\cdot\frac{1}{y^{\frac{1}{2}}}}$$

Let's now multiply top and bottom by y and $x^{\frac{3}{2}}$ and $y^{\frac{1}{2}}$ to get

$$\frac{2 \cdot x^{\frac{1}{2}} \cdot y^{\frac{1}{2}}}{x^{\frac{1}{2}} \cdot x^2 \cdot y \cdot x^{\frac{3}{2}}}$$

I see a common factor of $x^{\frac{1}{2}}$ we can cancel. Doing so gives

$$\frac{2 \cdot y^{\frac{1}{2}}}{x^2 \cdot y \cdot x^{\frac{3}{2}}}$$

What now?

In the denominator we can rewrite $x^2 \cdot x^{\frac{3}{2}}$ as $x^{\frac{7}{2}}$.

$$\frac{2 \cdot y^{\frac{1}{2}}}{y \cdot x^{\frac{7}{2}}}$$

I am not sure what to do here. Let's just try teasing everything apart.

$$\frac{2 \cdot y^{\frac{1}{2}}}{y \cdot x^{\frac{7}{2}}} = 2 \cdot y^{\frac{1}{2}} \cdot \frac{1}{y} \cdot \frac{1}{x^{\frac{7}{2}}}$$

I am going to try rewriting the fractions here as negative exponents. We get

$$2 \cdot y^{\frac{1}{2}} \cdot y^{-1} \cdot x^{-\frac{7}{2}}$$

I know $y^{\frac{1}{2}} \cdot y^{-1} = y^{-\frac{1}{2}}$, so this reads

$$2 \cdot y^{-\frac{1}{2}} \cdot x^{-\frac{7}{2}}$$

The question was to make the original expression look friendlier. That's a subjective command.

But I think we have succeeded. I am going to say that

$$2 \cdot x^{-\frac{7}{2}} \cdot y^{-\frac{1}{2}}$$

is a fine final answer.

Others, however, might prefer not to have negative exponents and will write their final answer instead as

$$\frac{2}{x^{\frac{7}{2}} \cdot y^{\frac{1}{2}}}$$

Some might not prefer to have fractional exponents and will instead present the answer

$$\frac{2}{\sqrt{x^7} \cdot \sqrt{y}}$$

And this can be rewritten

$$\frac{2}{\sqrt{y \cdot x^7}}$$

How "friendly" can we get?

Answer 2: A "Let's See Through It" approach

Looking at

$$\frac{(4xy^{-2})^{\frac{1}{2}} x^{-\frac{3}{2}}}{\sqrt{x} \cdot \left(x^{-1} y^{\frac{1}{4}}\right)^{-2}}$$

we see a power of 4, some powers of x, and some powers of y.

Let's focus on each these sets of powers in turn.

1. Only one power of 4 will appear. After applying the exponent rules, I see it will just be $4^{\frac{1}{2}} = \sqrt{4} = 2$ in the numerator.

2. After applying the exponent rules, the powers of x I'll see will be $x^{\frac{1}{2}}$ and $x^{-\frac{3}{2}}$ in the numerator and $x^{\frac{1}{2}}$ and x^2 in the denominator. The given expression thus contains the factor

$$x^{\frac{1}{2}} \cdot x^{-\frac{3}{2}} \cdot \frac{1}{x^{\frac{1}{2}}} \cdot \frac{1}{x^2}$$

which is the same as

$$x^{\frac{1}{2}} \cdot x^{-\frac{3}{2}} \cdot x^{-\frac{1}{2}} \cdot x^{-2} = x^{-\frac{7}{2}}$$

in the numerator.

3. After applying the exponent rules, the powers of y I'll see will be y^{-1} in the numerator and $y^{-\frac{1}{2}}$ in the denominator. The given expression thus contains the factor

$$y^{-1} \cdot \frac{1}{y^{-\frac{1}{2}}} = y^{-1} \cdot y^{\frac{1}{2}} = y^{-\frac{1}{2}}$$

in the numerator.

This accounts for all terms in the original expression. So, the expression must be equivalent to

$$2x^{-\frac{7}{2}}y^{-\frac{1}{2}}$$

as we had before.

One becomes quite swift at this "see through" approach as one practices ghastly examples like these. Feel free to work through all the practice problems in a typical high-school algebra book if this is a skill you'd like to conduct at speed.

MUSINGS

Musing 115.9 We are in the weeds of typical textbook algebra: practicing exponent rules for what seems like just the sake of practicing exponent rules.

I am curious: How does this strike you? How are you feeling about this?

Musing 115.10 There are **two** ways to interpret the expression

$$2^{3^4}$$

either as $2^{(3^4)} = 2^{81}$ or as $(2^3)^4 = 2^{3\times 4} = 2^{12}$.

a) How many different ways are there to interpret the expression this tower of exponents?

$$2^{3^{4^5}}$$

(Does this question feel like déjà vu? Do you remember the vinculum numbers of Musing 8.2?)

b) Insert parentheses into the expression

$$m^{a^{b^{c^d}}}$$

so that it evaluates as m^{abcd}.

Comment: As we have just seen that an expression given as a "tower of exponents" is ambiguous: it can be interpreted and evaluated multiple ways. If ever presented with such an expression, the convention has become to interpret it with nested parentheses that have you evaluate the powers from the top downwards. For example,

$$2^{3^4} = 2^{(3^4)}$$

$$2^{3^{4^5}} = 2^{\left(3^{(4^5)}\right)}$$

Musing 115.11 What Is the value of $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$?

I mention this example as there is a mathematical curiosity to note about it. It is related to this question:

Is it possible to raise an irrational number to an irrational power and obtain a value that is rational?

We can answer this question as follows:

We know from section 59 that $\sqrt{2}$ is an irrational number.
Thus $\sqrt{2}^{\sqrt{2}}$ is an example of an irrational number raised to an irrational power.

If the value of $\sqrt{2}^{\sqrt{2}}$ happens to be a rational number, that is, a fraction, then it is an example of what we seek.

If, on the other hand, $\sqrt{2}^{\sqrt{2}}$ happens have an irrational value, then $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is also an example of an irrational number raised to an irrational power.

But

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\times\sqrt{2}} = \left(\sqrt{2}\right)^{2} = 2$$

This is a rational value.

So, one way or the other, either $\sqrt{2}^{\sqrt{2}}$ or $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is an example of an irrational number raised to an irrational power giving a rational answer. We just don't know which one is the relevant example!

We've answered YES to our question without being able to pin down a choice of two options. Curious!

Musing 115.12 Recall that a googol is 10^{100} and that a googolplex is 10^{googol} .
What is the googolth root of a googolplex?

MECHANICS PRACTICE

Practice 115.13 Use the exponent rules to explain why

$$\sqrt{a} \cdot \sqrt{b}$$

equals the square root of ab. (Here a and b are positive numbers.)

More generally, show that $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$. (That is, that $a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$.)

Practice 115.14 Please evaluate each of these expressions.

a) $(3^2)^3$ b) $(5^{-2})^{-1}$ c) $7^6 \cdot 7^{-4}$ d) $(8^2)^2 \cdot (8^4)^{-1}$ e) $\frac{9^0 \cdot 9^{\frac{1}{2}}}{9^0}$

f) $\left(\frac{(5^2)^4 \cdot (-5)^3}{(25^{-1})^3}\right)^0$ g) $\frac{(2^{-3})^2}{2}$ h) $\sqrt[3]{9^6}$ i) $\frac{2^{-4} \cdot 2^7 \cdot 2^{-4} \cdot 2^{-2}}{2^{-3}}$ j) $\sqrt{2} \cdot \sqrt{4} \cdot \sqrt{8} \cdot \sqrt{16}$

k) $\frac{\left(10^{\frac{2}{3}}\right)^6 \cdot \sqrt{10^3}}{100{,}000}$ l) $\sqrt[9]{0.000000001}$

Practice 115.15 Please make each of these expressions look friendlier.

a) $4x^2 \cdot x^{-3}$ b) $2\sqrt{a} \cdot 2\sqrt[3]{a} \cdot 2\sqrt[6]{a}$ c) $\frac{(2^x \cdot 3^x)^2}{6^x}$ d) $\frac{a^{-2}}{3a^{-3}b^{-2}}$ e) $\frac{m^{-1}}{(m\sqrt{m})^2}$

Practice 115.16 In each equation, kindly find the value(s) of the unknown that makes the equation true.

a) $\frac{1}{4} \cdot 4^x = 64$ b) $3^{2m+1} = 81$ c) $\frac{1}{3} \cdot \left(\frac{1}{3}\right)^w = \frac{1}{27}$ d) $\sqrt{9^x} = \frac{1}{3}$

e) $5^{9x-1} + 5 = 630$ f) $49^{x-1} = 343^{x-1}$ g) $4^{x+3} - 4^x = 252$

h) $(2^x)^2 - 12(2^x) + 32 = 0$

i) $3^{2x} - 4 \cdot 3^x + 5 = 2$

Practice 115.17 Dana and Dayle are each trying to make sense of

$$7^{\frac{3}{4}}$$

Dana says that it is the fourth root of the number seven cubed.

$$7^{\frac{3}{4}} = 7^{3\times\frac{1}{4}} = (7^3)^{\frac{1}{4}} = \sqrt[4]{7^3}$$

Dayle says it is the fourth root of seven, which is then cubed.

$$7^{\frac{3}{4}} = 7^{\frac{1}{4}\times 3} = \left(7^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{7}\right)^3$$

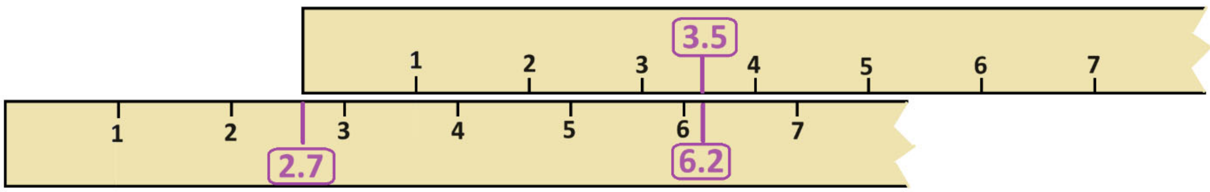
Who is correct?

116. Addition Rulers; Multiplication Rulers

Did you know that you can use a pair of ordinary pair of rulers as an addition calculator?

To work out $2.7 + 3.5$, say, place two rules on top of each other, with their left ends (the zero positions) aligned. Slide the top ruler so that its left end is aligned with first number of the sum, 2.7, on the bottom ruler.

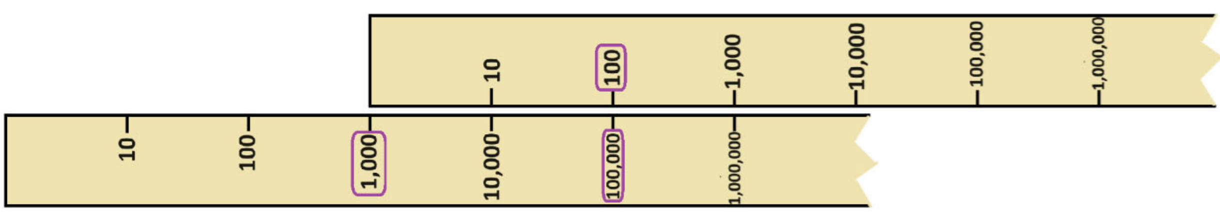
Now look for the second number of the sum, 3.5 on the top ruler and see which number it aligns with on the bottom ruler. That third number is the desired sum.



Practice 116.1 Compute $6.3 + 1.7$ this way with a pair of rulers.

Of course, in computing a sum $a + b$, these rulers are adding a length b from the top ruler to a length a on the bottom ruler to give a total length of $a + b$ on that bottom ruler.

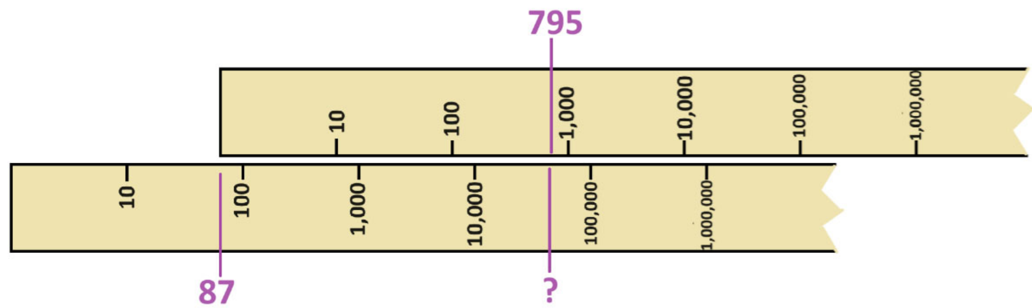
We can turn these addition rulers into multiplication rulers instead. Think of the numbers $1, 2, 3, \ldots$ marked along the ruler as the exponents for the powers of ten. In fact, make two rulers with strips of paper and label them explicitly with the power of ten, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, … . (The left end of each ruler corresponds to $10^0 = 1$.)



The picture is showing that the answer to $1{,}000 \times 100$ is $100{,}000$.

This is working because $1{,}000 = 10^3$ and $100 = 10^2$ and their product, $10^3 \times 10^2$, is the power of ten with exponent $2 + 3 = 5$. And the fifth spot on the bottom ruler is indeed labeled $10^5 = 100{,}000$.

Practice 116.2 Can you imagine computing 87×795 with these rulers? (Maybe we need to insert marks and labels between the powers of ten?) Does the answer $69{,}165$ seem feasible from the (not well-drawn) picture?



By the way, according to my calculator:

$10^{1.94}$ is really close to 87

$10^{2.90}$ is close to 795

and

$10^{1.94\,+\,2.90} = 10^{4.84}$ is close to $69{,}165$

Does it look like the rulers just physically added together the lengths 1.94 and 2.90?

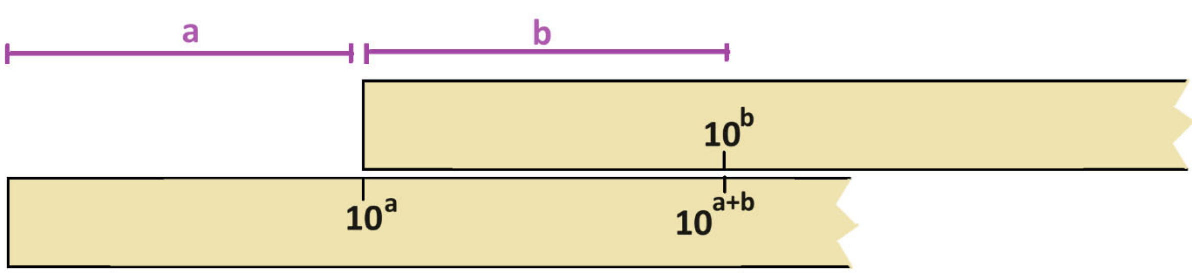
Practice 116.3

a) By experimenting on a calculator, find a number a so that 10^a is close to the value 3.

b) Find a number b so that 10^b is close to the value 4.

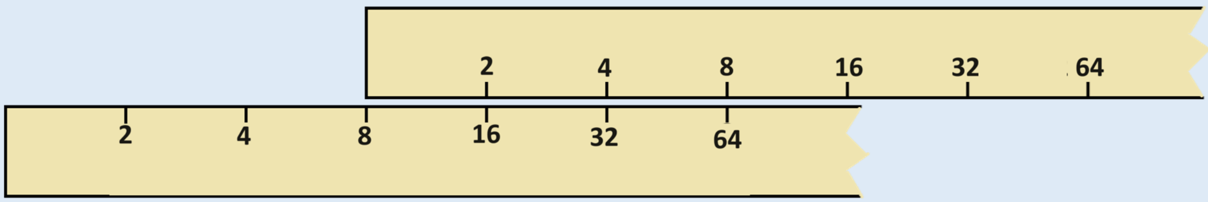
c) What is the value of $a + b$?

d) Make a guess as to the value of 10^{a+b}. Check it on a calculator. Was your guess close?



MUSINGS

Musing 116.4 Make some "powers of two" rulers with two strips of paper.



a) The picture illustrates that 8×4 equals 32.
Can you see this?
Could explain to a friend why the rulers are working this way?

b) Use your rulers to determine 16×4.

c) Use your rulers to estimate 7×9. Do they show a value just shy of 64?

117. A Scientific Crisis – and its Resolution

There was a scientific crisis in Europe during the Renaissance, brought on simply by the task of having to conduct arithmetic by hand.

The 1400s and 1500s saw Western scholars make new advances in the arts and sciences, leading to new understandings of the natural world. The invention of Galileo's telescope, which he called a *perspicillum*, opened up the workings of the heavens too to fuel extraordinarily rapid progress in astronomy.

As data gathering methods became more precise, astronomers found themselves burdened by the simple process of arithmetic when performing calculations. While addition was manageable with chalk on slates or with quill on paper, conducting multiplication was onerous.

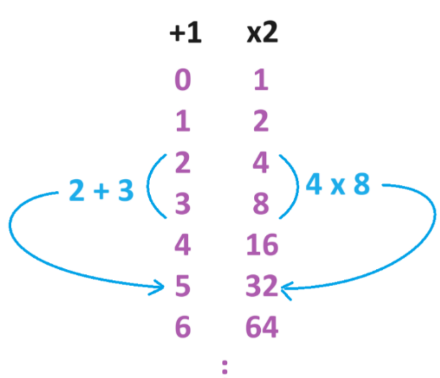
$$\begin{array}{r} 278{,}862{,}108 \\ +\ 305{,}721{,}661 \\ \hline = \phantom{000{,}000{,}000} \end{array} \qquad \begin{array}{r} 278{,}862{,}108 \\ \times\ 305{,}721{,}661 \\ \hline = \phantom{000{,}000{,}000} \end{array}$$

Not Too Bad **ICK!**

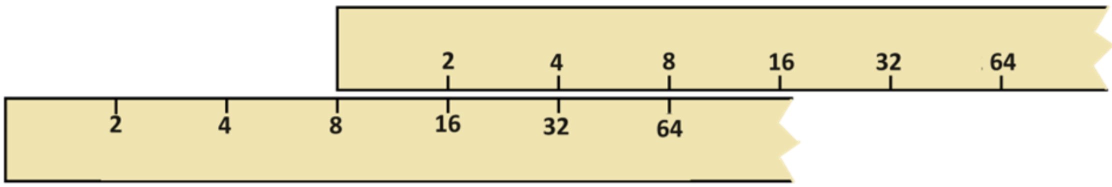
Astronomers used extensive tables of angular measures in their work with numbers adjusted by a factor of 10,000,000 to avoid dealing with fractions (decimals and decimal notation was not yet in use). This left them regularly conducting calculations with seven-, eight-, nine-digit numbers and they were severely hampered when it came to multiplying and dividing numbers or finding their square and cube roots.

Progress in science was severely held back.

Yet mathematicians had observed an interplay between multiplication and simpler addition. For example, in 1544 German mathematician Michael Stifel illustrated such a connection with the counting numbers (constructed by repeatedly adding 1 to zero) and the doubling numbers (constructed by repeatedly multiplying the number 1 by two). It is the same interaction we just observed with our sliding rulers.



The trouble was that scholars did not have an understanding of exponents beyond whole-number exponents. This meant that they could not, for instance, provide values between the lines in Stifel's table, or properly place markings on a pair of rulers between the whole number positions for conducting multiplications with accuracy.



In the 1590s, Scottish mathematician John Napier decided to tackle this very problem. He later wrote in reflection in 1614:

> *Seeing there is nothing that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers ... I began therefore to consider in my mind by what certain and ready art I might remove those hindrances.*

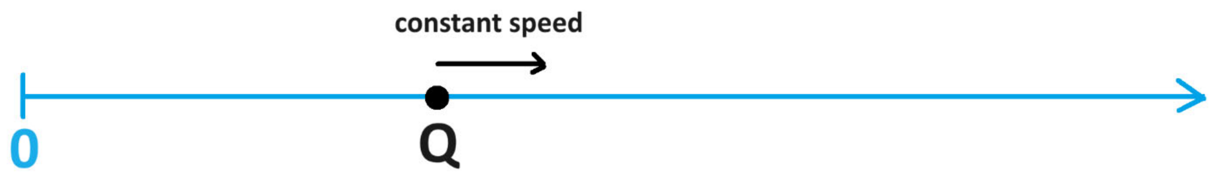
He succeeded. Napier's approach might seem strange and curious from our modern perspective. But we must remember that an understanding of exponents (yet alone a notation for them) was not available to him. He had to be innovative.

Napier took a kinematic approach. He imagined a particle P moving along a line segment of length $r = 10{,}000{,}000$, starting at 0 on the left and moving to the right.

The particle starts out at a speed of r units per second, but it gets slower and slower as it moves to the right. In fact, the speed of particle is given by how much further it needs to travel: if the particle has x more units to cover, it now moves at a speed of x units per second.

Practice 117.1 Will the particle ever reach the rightmost point of the segment?

Next, Napier imagined a second particle Q moving along a line, starting at 0 on the left, but always moving the constant speed of r units per second. This particle will move infinitely far to the right.



Napier was able to show that distance particle P has yet to travel and the distance Q has already traveled have the same sort of multiplicative/additive relationship akin the one Stifel had pointed out with the doubling numbers and the counting numbers.

Moreover, Napier had figured out a computational method that allowed him to approximate the distances covered by the particles every thousandth of a second, which gave him a table like Stifel's, but with the "in-between" numbers.

And it was this table that saved science! He showed the world how to reduce complicated arithmetical operations to ones of manageable addition and subtraction.

Napier coined the name **logarithm** for the numbers in his table, coming from the Greek words *logos* (ratio) and *arithmos* (number) to represent how he was comparing the values of the distances in the motions of the two particles.

Napier published his 90-page table of logarithm values in 1614.

It wasn't until more than a century that mathematicians properly understood exponents and could see Napier's work as theory of general exponents. But by then the name *logarithm* was firmly entrenched in scientific community's vernacular, and the name stays with us to this day.

And that's a fine thing. We should continue to honor Napier's magnificent achievement that saved scientific progress at a crucial time.

Logarithm Tables

Using my modern understanding of exponents (and my calculator!), I am able to do what Stifel and Napier could not initially do, namely, accurately fill in some values between the lines of Stifel's table.

I will also call these values *logarithms*, but they are not the logarithmic values Napier computed and what is shown here is not part of the table he presented to the world. But we can use this table to illustrate, nonetheless, how scholars converted multiplication problems into addition problems via Napier's work.

number	Stifel logarithm
1	0
2	1
3	1.585
4	2
5	2.322
6	2.585
7	2.807
8	3
9	3.170
10	3.322
11	3.459
12	3.585
13	3.700
14	3.807
15	3.907
16	4

Do you see the doubling numbers (this time to the left) and the matching counting numbers to the right, as per Stifel's table?

The decimal values you see have all been rounded to three decimal places. (So, any discrepancies we might encounter will be because of this rounding in the thousandth place. But this should not be too problematic for us.)

Also, I've only given logarithm values for the first sixteen numbers, so the multiplication problems we'll illustrate here will be small!

Suppose we wanted to work our 3×5 using the table. (Of course, we know the answer is 15.)

Here's the method:

1. The logarithmic value for $\mathbf{3}$ is 1.585.
2. The logarithmic value for $\mathbf{5}$ is 2.322.
3. The sum of these two values is 3.907.
4. Looking back at the table, the number with logarithmic value 3.907 is $\mathbf{15}$.

$$\begin{array}{r} 3 \\ \times\ 5 \\ \hline = \end{array} \qquad\qquad \begin{array}{r} 1.585 \\ +\ 2.322 \\ \hline =\ 3.907 \end{array}$$

In summary …

To multiply two numbers: Look up their logarithmic values. Add those values and then see which number has logarithmic value equal to the sum. That number is the product of the original two numbers.

Practice 117.2 Use the table to compute each of these products.

3×4 $\qquad$ 7×2 $\qquad$ 3×3

Practice 117.3

a) Knowing that $18 = 3 \times 6$, what is the logarithmic value that goes with the number 18?

b) Do you get the same value you think of 18 as 2×9 instead?

c) Make a guess as to logarithmic value that goes with the number 17.

MUSINGS

Musing 117.4 Might you care to research the details of Napier's "ratio of distances" approach to create logarithmic tables that solved the multiplication problem for science?